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THE OCTOBER MEETING IN NEW YORK

The two hundred sixty-third regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, October 27, 1928, extending through the usual morning and afternoon sessions. The attendance included the following sixty-seven members of the Society:

R. G. Archibald, A. A. Bennett, Benton, B. L. Brown, Carrić, Dadourian, Darkow, Dehn, Demos, Douglas, Dresden, Eisenhart, Feld, Fiske, Fite, Fort, Gehman, Gill, Gronwall, L. S. Hill, Hille, Hofmann, Hollcroft, Huber, Dunham Jackson, M. I. Johnson, R. A. Johnson, Kaplan, Kenny, Kholodovsky, Kline, Koopman, Mark Kormes, Lefschetz, McGiffert, Marden, Meder, Richard Morris, Mullins, C. A. Nelson, K. E. O'Brien, Ore, Pell-Wheeler, Pfeiffer, Pierpont, Post, Raudenbush, Ritt, Rutledge, Schoonmaker, Seely, Serghiesco, Siceloff, Simons, Smail, P. A. Smith, J. M. Thomas, Trjitzinsky, Weida, Weisner, M. E. Wells, D. E. Whitford, Whittemore, Wiener, W. A. Wilson, Margaret M. Young, Zippin.

At the meeting of the Council, the following persons were elected to membership in the Society:

Mr. Andrew Campbell Berry, Harvard University;
Miss Olive Margaret Hughes, Bryn Mawr College;
Mr. William Thomas Reid, University of Texas;
Professor Allen A. Shaw, University of Arizona;
Dr. Warren Jennison Willis, patent attorney, New York City.

The following members of the London Mathematical Society have entered the American Mathematical Society under the reciprocity agreement since the last meeting:

Mr. Francesco Tavani, London;
Professor T. P. Trivedi, Karachi, India.

Associate Secretary Dresden reported the following elections by mail vote of the Council:

To sustaining membership: Members of the Department of Mathematics, University of Wisconsin;

To ordinary membership:

Dr. Frederick Richard Bamforth, University of Chicago;
Professor David Francis Barrow, University of Georgia;
Mr. Leon Battig, Oberlin College;
Dr. Clifford Bell, University of California at Los Angeles;
Professor Arthur H. Blue, Western Union College;
Mr. Arthur Barton Brown, Harvard University;
Professor Helen Calkins, Sweet Briar College;
Mr. Paul T. Copp, Purdue University;
Mr. Paul Cramer, Purdue University;
Professor Louis Antoine Victor De Cleene, St. Norbert College;
Dr. Miltiades Stavros Demos, Harvard University;
Professor Carl Christian Engberg, University of Nebraska;
Mr. William Irvin Foster, Rochester Junior College;
Mr. Harold Sinclair Grant, University of Pennsylvania;
Professor Samuel Oliver Grimm, Lebanon Valley College;
Captain Elmer Ellsworth Hagler, Jr., United States Army;
Professor Marie Mathilda Johnson, Oberlin College;
Dr. Pierce Waddell Ketchum, University of Illinois;
Mr. Edward August Knobelauch, University of Pennsylvania;
Mr. Trueman Lester Koehler, Lehigh University;
Mr. Semen Arsenijevitch Lepeshkin, Brown University;
Professor Anna Marm, Bethany College;
Sister Mary Bertrand (Walton), Marywood College;
Mr. John Ellsworth Merrill, Case School of Applied Science;
Professor Elmer Beneken Mode, Boston University;
Mr. Joseph Kimbark Peterson, Harvard University;
Mr. Thurman Stewart Peterson, Ohio State University;
Miss Mina S. Rees, Hunter College;
Professor Charles Edward Schroeder, Boston College;
Professor Charles Louis Searey, University of Nevada;
Mr. James Singer, Princeton University;
Professor Carlton W. Smith, State Teachers College, Superior, Wis.;
Professor Atherton Hall Sprague, Amherst College;
Mr. Carl Walther Strom, Luther College;
Professor Emory Earl Walden, Lambuth College;
Professor Charles Ernest Weatherburn, Canterbury University College, Christchurch, New Zealand;
Professor Thomas Payne West, University of Idaho, Southern Branch;
Miss Jean Winston, University of Cincinnati;

Nominees of the Penn Mutual Life Insurance Company, Philadelphia: Messrs. J. E. G. Bryant, M. L. Johnson, Adolph F. Schwartz, Lawrence L. Stevens, Clarence E. Willows.

The Council accepted with thanks the invitation of Lehigh University to hold the Eastern Christmas meeting of 1929 in Bethlehem, Pa.

The Council accepted the recommendation of the com-

mittee on colloquia that Professor S. Lefschetz be invited to deliver the colloquium lectures at the summer meeting in 1930 at Brown University.

It was voted to approve the use of the name of the Society in the announcements of the American Yearbook; and the President was authorized and requested to appoint, each time for a period of three years, a representative of the Society on the Editorial Board of the Yearbook.

Professor Bennett presided at the morning session, and Professor Jackson at the afternoon session.

Titles and abstracts of the papers read at this session follow below. The papers of Franklin, Garver, Kasner, Stone, Suschkewitsch, Whyburn, and Zippin were read by title. Dr. Arnold was introduced by Associate Secretary Arnold Dresden, Mr. Rashevsky by Professor J. I. Taylor, and Professor Suschkewitsch by Professor H. H. Mitchell.

1. Professor T. R. Hollcroft: *Invariant postulation.*

The invariant postulation of a manifold, simple or multiple, on a variety in i dimensions is the number of invariants among the coefficients of the variety that are necessary and sufficient for the variety to contain a manifold of given nature. The invariant postulation of a certain manifold is obtained readily from the ordinary postulation in which both the nature and position of the manifold are taken into account. Invariant postulation, although a concept not heretofore used except in the case of points in the plane, is of importance mainly because of the geometric relations revealed by it. Some of these relations have been proved before by other methods, and some are new.

2. Dr. H. E. Arnold: *The rational space quintic curve of the second species and its relation to the rational plane quartic curve.*

The rational space quintic curve of the second species, R_3^5 (II), possesses ∞^1 quadrisecant lines, which are given by a pencil of binary quartics. The first part of the present paper concerns itself with the determination of conditions for special quadrisecants, obtained, in general, in terms of the Morley invariants. (See R. M. Winger, American Journal, vol. 36.) By means of these conditions a correspondence is set up between certain types of R_3^5 (II) and those of R_2^4 (the rational plane quartic curve). The R_3^5 (II) for which I_2 vanishes is determined by a binary quintic form whose roots represent coplanar points. In the second part of the paper several facts concerning this plane are obtained. Finally, some properties of R_3^5 (II) are derived from the Jonquière quartic, and a geometric meaning is given to the unique sextic apolar to the pencil of line sections of R_2^4 which pass through a fixed point of the plane.

3. Dr. Louis Weisner: *Invariants of a plane 5-point.*

From the complete system of invariants of a binary quintic, consisting of I_4, I_8, I_{12}, I_{18}, expressed in terms of the roots, invariants J_6, J_{12}, J_{18}, A_{27} of the associated plane 5-point are found by the method of complementary determinants explained by Coble. But these four invariants of the plane 5-point do not form a complete system because A_{27} turns out to be an alternating function. In the present paper a complete system is found to consist of J_6, J_{12}, J_{18}, J_{33}, the subscripts denoting the degrees of the invariants. The results are applied to a redetermination of the invariants of E. H. Moore's cross-ratio group of order 120.

4. Professor R. G. Archibald: *An arithmetical function analogous to Euler's ϕ-function.*

An arithmetical function $\omega(n)$ is defined as the number of integers relatively prime to n in the sequence $1\cdot 2\cdot 3/6,\ 2\cdot 3\cdot 4/6,\ \cdots,\ n(n+1)\cdot(n+2)/6$. E. Lucas (*Théorie des Nombres*, vol. I, 1891, p. 403) and R. D. Carmichael (*The Theory of Numbers*, 1914, p. 36) suggested the problem of finding the number of terms in the foregoing sequence which are relatively prime to n. As far as the author knows, a formula giving the number of such terms has not yet been published. In the present paper, by the use of Lucas' ψ-function, an explicit formula for $\omega(n)$ is obtained in terms of n if, where m is an integer relatively prime to 6, n is different from $2^k3m (k \geqq 0)$. In these cases, solutions of a set of linear congruences (always solvable) are to be obtained. If $\omega_6(30\ m)$ denotes the number of terms in the given sequence relatively prime to 6 m when $n = 30\ m$, and if m is any positive integer relatively prime to 6, $\omega(36\ m) = 2\omega(18\ m) = \omega(6\ m) + \omega_6(30\ m) = \omega(12\ m) + \omega(24\ m)$. Moreover, if m_1 and m_2 are two relatively prime numbers each relatively prime to 6, and if $k \geqq 2$ and $l \geqq 1$, $\omega(m_1m_2) = \omega(m_1)\omega(m_2)$, $\omega(2^km) = \omega(2^k)\omega(m)$, $\omega(3^km) = \omega(3^k)\omega(m)$, $\omega(2^l3^km) = \omega(2^l3^k)\omega(m) = \omega(2^l)\omega(3^km) = \omega(3^k)\omega(2^lm)$.

5. Dr. Jesse Douglas (National Research Fellow): *On the inverse problem of the calculus of variations.*

The problem is as follows: given a system of paths, under what conditions can it be identified with the totality of extremals of a calculus of variations problem? We represent the system of paths by the differential equations $d^2x^i/dt^2 = H_2^i(x,\ p)$, where $p^i = dx^i/dt$ (Annals of Mathematics, vol. 29 (1928), pp. 143–168), and the calculus of variations problem by $\int F(x,\ p)dt = \text{minimum}$. H_2^i is homogeneous of the second degree and F of the first degree in the variables p. Necessary and sufficient conditions are found to be represented by the system of covariant partial differential equations (1) $F_{|i},_j - F_{|j},_i = 0$, where the stroke denotes $\partial/\partial p$ and the comma covariant differentiation based on the affine connection $\Gamma^i_{j,k} = \frac{1}{2}\partial^2 H_2^1/\partial p^j \partial p^k$. There are three cases as to (1): (a) it may have no solutions F (the general case); (b) it may have a class of solutions depending linearly on a finite number of arbitrary constants; (c) it may have a class of solutions involving

arbitrary functions. In the present paper, sufficient conditions are established for the case (b). The only operations required for the application of these conditions are (covariant) differentiation and the algebraic discussion of a certain finite system of linear equations.

6. Dr. Jesse Douglas: *A generalization of homogeneous functions and of Euler's equations.*

The results here are a by-product of the writer's work on spaces of K-spreads (see this Bulletin, Jan.-Feb., 1928). Let $H_{\alpha\beta\cdots}^{\lambda\mu\cdots\nu}{}_{\cdots\gamma}$, where there are r lower and s upper indices each varying from 1 to K, be a system of functions of the variables p_α^i; $i=1, 2, \cdots, N$; $\alpha=1, 2, \cdots, K$; we imagine the system p_α^i to be distributed into N sets of K variables, with i fixed for each set. Let these sets undergo cogredient linear transformation: $q_\alpha^i = A_\alpha^\beta p_\beta^i$ reciprocally $p_\alpha^i = B_\alpha^\beta q_\beta^i$, the coefficients A_α^β, B_α^β being therefore the same for every value of i. Suppose that, identically,

$$(1) \qquad H_{\alpha\beta\cdots}^{\lambda\mu\cdots\nu}{}_{\cdots\gamma}(q) = A_\alpha^\rho A_\beta^\sigma \cdots A_\gamma^\tau B_\phi^\lambda B_\psi^\mu \cdots B_\omega^\nu H_{\lambda\mu\cdots\nu}^{\phi\psi\cdots\omega}{}_{\cdots\rho}(p).$$

The property expressed by this equation reduces to homogeneity of degree $r-s$ in case $K=1$. Generalizations of Euler's equations of the first and higher orders are established. Generalization is further made to the case where (1) involves as factor in its second member a power of the determinant $|A_\alpha^\beta|$ (relative tensor); also to the case where the independent variables form N systems ${}^i p_{\alpha\beta\cdots}^{\lambda\mu\cdots\nu}{}_{\cdots\gamma}$, which undergo cogredient tensor transformation.

7. Mr. Nicholas Rashevsky: *An electrostatic problem.*

The solution of the electrostatic problem for a space limited by two infinite parallel planes, one of which is provided with a set of equidistant parallel blades, at right angles to the planes, is given in the form of an infinite product. The use of the first factor of the product as the first approximation leads to elliptic integrals. By a logarithmic transformation, the above solution gives also the solution of the corresponding problem for two coaxial cylinders, one of which is provided with equidistant radial blades. The solution is applied to the calculation of voltage factors in certain special types of three-electrode tubes.

8. Dr. Lulu Hofmann: *Remarks on a certain aspect of plane projective transformations.*

In two projectively related planes, the metric properties of corresponding segments on corresponding lines and corresponding angles at corresponding points are known. This paper investigates the metric properties of corresponding segments radiating out from corresponding points. The principal result is as follows. With every fixed pair of corresponding points P_0, P_0' in two projective planes π, π', and every value of a positive quantity k, is associated a pair of elliptic quartics Q_k, Q_k', corresponding under the projectivity, which are the loci of all pairs of corresponding points P, P', such that the distances of any two such points from the

respective fixed points have the same ratio: $\overline{PP}_0 = k\overline{P'P}_0'$. These quartics $Q_k[Q_k']$ have one double point at the infinite point of the vanishing line of $\pi[\pi']$ and the other at the fixed points $P_0[P_0']$. They consist of two odd branches lying on different sides of the respective vanishing lines and having the equi-segmental axes for asymptotes. They are tangent to the circles about $P_0[P_0']$ as centers at the cyclic points. In particular for $k=1$, $Q_1[Q_1']$ represents in $\pi[\pi']$ the locus of the double points determined by the projectivity as $\pi'[\pi]$ is successively laid on $\pi[\pi']$ in the ∞^1 ways possible such that P_0 and P_0' coincide.

9. Professor Edward Kasner: *Polygenic functions of two complex variables.*

In a paper read at the Bologna Congress, the author began the study of polygenic functions of two variables by showing that the partial derivatives are represented by a pair of uniformly parametrized circles or *clocks*. (For the case of one variable, see Science, Dec. 16, 1927.) He now studies the three partial derivatives of second order, which give rise to quite complicated configurations of interest. (For the case of one variable, see the Transactions of this Society, Oct., 1928.)

10. Professor Raymond Garver: *A solution of the quartic equation.*

By the use of two theorems in the author's paper recently published in the Messenger of Mathematics, the quartic equation is solved in a manner that seems to present some interest. The present paper will appear in the American Mathematical Monthly.

11. Professor M. H. Stone: *Hermitian symmetric operators in abstract Hilbert space.*

The present paper resolves the characteristic value problem for an operator T with the following properties: (a) T orders to each element of a set $\mathfrak{F}$ everywhere dense in an abstract complex Hilbert space $\mathfrak{H}$ an element of $\mathfrak{H}$; (b) if f and g are elements of $\mathfrak{F}$, $Q(f, Tg) = Q(Tf, g)$, where Q is a Hermitian bilinear form determining the metric of $\mathfrak{H}$; (c) if two elements f and f^* exist in $\mathfrak{H}$ such that, for every g in $\mathfrak{F}$, $Q(f, Tg) = Q(f^*, g)$, then f belongs to $\mathfrak{F}$. The theory includes the known theory of operators for which $\mathfrak{F} \equiv \mathfrak{H}$, developed by Hilbert and others, and leads to analogous results. It includes also the results obtained by J. von Neumann for "real" operators, as stated by him in the Göttinger Nachrichten, 1927, Heft 1, pp. 1–57.

12. Mr. P. M. Swingle: *End sets of bounded continua irreducible between two points.*

A study is made of continua, irreducible between two points, based particularly upon various types of their subsets called "end sets." Among other results, necessary and sufficient conditions for the indecomposability of such continua are obtained.

13. Professor Philip Franklin: *The operators of quantum mechanics.*

In this note we study some properties of the transition from the Hamiltonian function to the Schrödinger operator. In particular we show that any attempt to define the correspondence by linear operators must fail to yield a result independent of coordinates.

14. Professor Philip Franklin: *Algebraic matric equations.*

The equations here studied are of the form

$$P(X) = A,$$

where the left member is a polynomial, and A and X are matrices. For the special case

$$X^n = A,$$

we determine a general form for X giving all the roots. With some restrictions on X, the results are extended to the more general case. In particular, after further specializing X to be expressible as a polynomial in A, we obtain from our results those of Roth (Transactions of this Society, vol. 30, p. 579).

15. Professor Anton Suschkewitsch: *On a generalization of the associative law.*

In this paper it is pointed out that the theorem of Lagrange to the effect that the order of a group is divisible by the order of a subgroup may be proved without the complete use of the associative law. A generalization of this law is proposed, and some theorems regarding it are established.

16. Professor G. T. Whyburn: *Local separating points of continua.*

The point P of a continuum M will be called a *local separating point* of M if there exists a compact open set R containing P and such that if C is the component of $M \cdot \overline{R}$ which contains P, then $M \cdot \overline{R} - P$ is separated between some two points of $C - P$. The point P of an E-set M is called an *indispensable point* of M if $M - P$ is not an E-set. In this paper a study is made of the local separating points of continua M lying in any locally compact, metric, and separable space. These points are characterized in various ways, some in terms of the indispensable points of certain cuttings of M, and theorems are obtained elucidating their properties. Perhaps the most interesting ones are the following: (1) all save possibly a countable number of the local separating points of M are points of order two of M; (2) every uncountable set of local separating points of M contains an uncountable subset every pair of points of which cuts M. The notion of a *local separating continuum* is introduced, and many important properties of local separating points are shown to hold for local separating continua.

17. Professor Edward Kasner: *Higher partial derivatives of polygenic functions.*

The higher partial derivatives of polygenic functions depend upon the curves of approach and also upon the correspondence between the curves. The order of differentiation must be taken into account. Thus in general

$$\frac{\partial^2 w}{\partial z_1 \partial z_2} \neq \frac{\partial^2 w}{\partial z_2 \partial z_1}$$

In fact equality, for all approaches, holds only when w is an analytic function. Thus there are in general four distinct derivatives of second order. For rectilinear approach these reduce to three. Higher orders are also discussed.

18. Professor Norbert Wiener: *The summability of Fourier series.*

The author applies his new method in Tauberian theorems to the demonstration of the Hardy-Littlewood theorems on the convergence and summability of Fourier series.

19. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem.*

The author proves that in a one-dimensional continuous curve containing at least one simple closed curve the Janiszewski-Mullikin theorem is not satisfied. In an acyclic continuous curve it is vacuously satisfied. For the proof, he requires a theorem of Gehman's on subsets of a plane continuous curve which lie on an acyclic subcontinuous curve which he generalizes to n dimensions.

20. Professor W. A. Wilson: *Certain problems related to the cutting of a simply connected plane region by a continuum.*

Let R be a simply connected plane region with a bounded frontier F and C be a bounded continuum contained in $\overline{R}$ such that $C \cdot F$ is the sum of two closed sets α and β between which C is irreducible and C is neither indecomposable nor the union of two indecomposable continua. A study is made of the frontiers of the regions into which C divides R for various types of frontiers F. When C is a simple arc and F is a regular frontier, the nature of the frontiers depends upon whether F contains 0, 1, or 2 continua irreducible between α and β, and the various possibilities are catalogued. If C is an arc, F is the irreducible sum of two continua H and K neither of which disconnects F, and the ends of C lie on different components of $H \cdot K$, the frontiers are $C+H$ and $C+K$. Various sufficient conditions for the extension of this last result to the case that C is not a simple arc are given. In this connection it is also shown that, if F is the union of two continua H and K and is disconnected by neither of them precisely two components of $H \cdot K$ are accessible by a continuum from R and can therefore be joined by an irreducible continuum contained in $\overline{R}$ and connected im kleinen at all of its points save possibly those on $H \cdot K$.

21. Professor W. A. Wilson: *Some properties of multi-coherent continua.*

This paper appears in full in the present issue of this Bulletin.

22. Professor J. F. Ritt: *On the zeros of exponential polynomials.*

By an exponential polynomial is meant a function $a_0e^{\alpha_0 z}+\cdots+a_ne^{\alpha_n z}$, with constant a's and α's. The distribution of the zeros of such functions has been studied by Tamarkin and by Pólya. In the present paper it is proved that if the quotient of two exponential polynomials is an integral function, then the quotient is an exponential polynomial.

23. Professor B. O. Koopman: *On the singularities of linear partial differential equations.*

We consider the system of m equations of any order in m unknown functions, in the independent variables $(x, \cdots, x_n)$, and having analytic coefficients. A slight modification of the Cauchy-Kovalevsky calculus of limits establishes the following: If R_{2n} is a $2n$-dimensional region of the complex $x_1 \cdots x_n$-space containing no singular point of the system, and S_{2n-2} a regular portion of the analytic surface

$$S(x_1, \cdots, x_n)=0,$$

in R_{2n}, not tangent to a characteristic, there exists a region P_{2n} of R_{2n}, containing S_{2n-2}, throughout which every solution of the Cauchy problem is analytic, whatever be the (analytic) Cauchy data on S_{2n-2}. From this it follows that if

$$f(x_1, \cdots, x_n) = 0$$

is an isolated analytic singularity of an analytic solution of the equations, and if it contains only ordinary points, it is a characteristic. These results have been obtained in the case of two independent variables by LeRoux and Delassus by methods which do not admit of extension; and by Delassus and Hadamard for n variables under very restrictive assumptions.

24. Dr. Morris Marden (National Research Fellow): *Zero-free regions of a certain partial fractions.*

This paper tries to find simple regions of the plane which do not contain any zeros of the partial fraction

$$\sum_{i=1}^{n} \alpha_i/(z-a_i),$$

when α_i is a complex constant. It presents a few such regions where the points a_i are collinear, or where the numbers a_i are all real or appear in conjugate imaginary pairs, as well as in cases where the points a_i are arbitrarily distributed in the plane.

25. Dr. T. C. Benton: *On continuous curves which are homogeneous except for a finite number of points.*

A continuous curve is said to be homogeneous if for every pair of points x and y belonging to the curve there exists a (1, 1) continuous correspondence of the *curve* into itself which makes x correspond to y. This paper classifies the plane continuous curves which are homogeneous except for a finite number of points. For one non-homogeneous point c, the set consists of a finite ($\neq 2$) or countably infinite set of simple closed curves which have only c in common. For two such points, the set consists of a finite number ($\neq 2$) of arcs joining the two points. For three or more points a skeleton set is formed by replacing by a single arc the set of arcs which join two non-homogeneous points but contain no other non-homogeneous points. The skeleton set may be either a set of arcs having only one point in common or a simple closed curve, or a set which is in (1, 1) correspondence with the projection of one of the regular polyhedra or of one of the semi-regular polyhedra on one of its faces in such a way that none of the projections of edges have any intersections except projections of vertices. The case of unbounded plane homogeneous continuous curves is also taken up and it is found that except for the one-point case where the set corresponds to a finite number of rays leaving the non-homogeneous point, there is no such set other than the whole plane.

ARNOLD DRESDEN,
Associate Secretary

THE FIFTY-SIXTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-sixth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 27, 1928. The Chairman, Professor E. T. Bell, presided. The total attendance was 37, including the following 23 members of the Society:

Alderton, A. D. B. Andrews, Basoco, E. T. Bell, Bernstein, Buck, Cajori, Corbin, Garver, W. L. Hart, M. W. Haskell, E. R. Hedrick, Hotelling, Frank Irwin, Jerbert, D. N. Lehmer, S. H. Levy, W. A. Manning, W. E. Milne, Noble, T. M. Putnam, A. R. Williams, B. C. Wong.

The following officers were elected for the coming year:

Chairman, Professor M. W. Haskell; Secretary, Professor B. A. Bernstein; Program Committee, Professors E. T. Bell, H. F. Blichfeldt, R. M. Winger, and B. A. Bernstein (ex officio).

It was resolved that the members of the Section living in the Northwest be requested to change the place of the next Summer meeting from the Oregon Agricultural College to the University of California, in order that the Section may participate in the meetings of the Pacific Division of the American Association for the Advancement of Science to be held in June.

It was resolved unanimously: That the Society be requested to make the meetings of the Section Meetings of the Society.

It was decided to hold the next Fall Meeting at the University of California on October 19, 1929.

Titles and abstracts of the papers presented at the meeting follow below. Professor Bell's first paper was given at the request of the Program Committee. The fourth and fifth papers of Professor Bell, Professor Garver's second and third papers, and Professor Whyburn's paper were read by title.

1. Professor E. T. Bell: *The theory of algebraic numbers in the light of Kronecker's program.*

It is suggested as at least plausible, and as a unique piece of scientific irony, that the present unrest concerning the foundations of analysis which was inaugurated by Kronecker half a century ago, reacts destructively upon his own theory of algebraic magnitudes in which it originated.

2. Professor E. T. Bell: *Invariant sequences of polynomials.*

There are determined all sequences of uniform functions of one variable such that the derivative of each term of a given sequence is equal to the preceding term, and each term is changed by a linear transformation of the variable into a multiple of itself, the multiplier being a function of the rank of the term alone, and the linear transformation and the multiplier function being the same for all terms. The terms are necessarily polynomials. By linear transformations on the rank and the variable, any number of distinct sequences having the stated properties for any given linear transformations on their variables, can be replaced by new sequences, all of which are transformed alike by the same linear transformation of the variable, and hence are instances of a single sequence of the original kind. In this respect the theory of any number of distinct sequences of the kind described can be unified. The number of invariant sequences of polynomials is infinite, to the same degree as the set of all uniform functions of one variable.

3. Professor E. T. Bell: *On certain finitely solvable equations between arithmetical functions.*

The number of equations between arithmetical functions for which we can prove either that they have no solution, or only a finite number of solutions, and in the latter case state the solutions, is negligible. Fourteen new equations of this type, involving the binary quadratic class number, are completely solved or are shown to be insolvable. As a point of interest, the solution is achieved by utilizing the conjecture of Descartes in 1638, which was proved only in 1911 by Dubois, (see Dickson's *History*, vol. II, pp. 276, 302).

4. Professor E. T. Bell: *Outline of a theory of arithmetical functions in their algebraic aspects.*

In the Journal of the Indian Mathematical Society for October, 1927, Dr. Vaidyanathswamy calls attention to the existence of inverses of multiplicative arithmetical functions. He remarks that they do not seem to have been utilized before, as they are not explicitly mentioned in Dickson's *History*. The reference in the History which covers them implicitly is vol. I, p. 323, item 174. The present paper is a restatement in more general form of a complete algebraic theory of arithmetical functions, not necessarily multiplicative, which has been constructed by the writer in several papers during the past sixteen years. This restatement generalizes and simplifies all of the preceding work.

5. Professor E. T. Bell: *An interpretation of certain decomposable algebraic forms as functions of divisors.*

The forms are in n indeterminates, where n is an arbitrary constant integer and the coefficients of the n linear factors into which the forms are decomposable are in any given algebraic number field. The decomposition is double, in that it has a dual interpretation, first when the indeterminates are elements of an abstract field, second, when they are elements of the irregular field of all extended numerical functions. When the relevant algebraic number field is defined by a primitive root of unity, the forms degenerate to circulants in each interpretation; another degeneration is that to the resultant of any pair of algebraic equations.

6. Professor Florian Cajori: *Early determinations of the heights of mountains.*

The author makes a detailed examination of the processes and instruments most probably used by the Greeks in determining the heights of mountains. He sets forth the modifications introduced in the Middle Ages and the Renaissance, as well as the improvements in geodetic procedure and in the construction of instruments during the seventeenth and eighteenth centuries. He deals next with the early development of the barometric method, also of the hypsometric and pendulum methods.

7. Professor Florian Cajori: *Hobbes, Wallis, and Barrow on principles of mathematics.*

The author points our that Wallis founded mathematics on arithmetic, that Hobbes and Barrow founded it on geometry. With Wallis arithmetic was an abstract science, to which geometry was subordinate. According to Hobbes, Wallis mistook the study of symbols for the study of geometry. According to Barrow, numbers themselves could not even be added to or subtracted from one another. The discussions carried on involved also the definitions and postulates of Euclid, the consideration of angles of contact and the concept of limits. In particular, Barrow and Wallis defended Euclid's definition of proportion against the attacks of Hobbes who pronounced the definition unworkable because it demanded an infinite number of trials. Hobbes criticises Wallis's reasoning by induction. Wallis anticipated some nineteenth century ideas on arithmetization.

8. Professor Florian Cajori: *Isaac Newton's edition of Varen's geography.*

The author finds that Newton made numerous corrections in Varens *Geographia Generalis* and added thirty geometric figures. Of particular interest is Newton's correction of a numerical error in Varen's account of the measurement of a degree of the earth's meridian due to Snell, and the use of Snell's value in correcting Varen's table for the distance a mountain of known height can be seen at sea (disregarding refraction). These corrections show conclusively that in 1672 Newton was familiar with the result of Snell's measurement. On no other occasion,

previous to writing the *Principia*, is it known definitely what meridian value Newton actually used.

9. Professor Raymond Garver: *Quartic equations with certain groups.*

A general method, employing the Tschirnhaus transformation, is devised which may be of some value in the problem of determining all equations of a certain degree with a prescribed group. The method can be applied easily to quartic equations; the results are equivalent to those given by Seidelmann in volume 78 of the Mathematische Annalen.

10. Professor Raymond Garver: *Linear fractional trans formations on quartic equations.*

The reduction of quartic equations to normal forms by means of linear fractional transformations is usually carried out with the aid of either algebraic geometry or invariant theory. This paper considers the problem from a purely algebraic point of view, and treats a number of normal forms.

11. Professor Raymond Garver: *Two notes on cyclic cubics.*

This paper gives (1) a simple proof that every cyclic cubic is of Seidelmann's form $x^3-3(p^2+3q^2)x+2p(p^2+3q^2)=0$, and (2) a Tschirnhaus transformation that serves to reduce every such cubic to the normal form $y^3-3(P^2+3)y+2P(P^2+3)=0$.

12. Dr. A. R. Jerbert: *Quintuples of curves in four-space.* Preliminary report.

A system of five differential equations of the form $dx_i/dt=a_{ij}x_j(i, j=1, 2, 3, 4, 5)$ serves as analytic basis. A fundamental system of solutions is interpreted as defining, to within a projectivity, 5 points in 4-space. As the parameter t varies, each of the five points describes a curve. The resulting configuration consists, therefore, of five curves in one-to-one correspondence. The projective properties of the configuration are readily expressed in terms of the invariants and covariants of the defining system.

13. Professor W. M. Whyburn: *On the distribution of the characteristic values of differential and integro-differential systems.*

The general system of linear differential equations $\sum_{j=1}^{n}\sum_{s=0}^{k}{}_sP_{ij}(x)y_j^{(s)}=0$, $(i=1, 2, \cdots n)$, where the functions ${}_sP_{ij}(x)$ are summable on X: $a\leqq x\leqq b$, is studied. The matrix calculus is used to develop adjoint relations and the results are shown to coincide with the classical results for the special cases of a single equation of the kth order $(i=j=1)$ and a system of n equations of the first order $(k=1)$. A canonical form is derived for the system when the determinant of the matrix $({}_kP_{ij}(x))$, k fixed, is different from zero on X.

The above differential system is considered when one of the coefficients ${}_rP_{mt}$, depends upon a parameter λ in such a way that ${}_rP_{mt}(x, \lambda)$ is sum-

mable on X for each fixed λ on L: $L_1<\lambda<L_2$, continuous in λ on L uniformly with respect to x on X, and bounded numerically on XL by a summable function of x. Necessary and sufficient conditions that there exist continua of characteristic values of the differential system and a set of homogeneous boundary conditions are derived. As a corollary of this result it is shown that a self-adjoint system cannot have a continuum of characteristic values and furthermore, the values of λ for which such a system is incompatible form intervals that are everywhere dense on L. Corresponding systems of integro-differential equations are considered.

14. Professor W. A. Manning: *The class of triply and quadruply transitive groups.*

It is known that if a triply transitive group of degree n and class $u(>3)$ contains a permutation of even order on u letters, $n\leqq 2u$; and that for quadruply transitive groups of class $u(>3)$ in which some permutation of degree u is of even order, $n\leqq 2u-1$. The author now announces that if one of the permutations of degree u of a triply transitive group of class $u(>3)$ is of odd prime order p, $n<2pu/(p-1)+3$. For all quadruply transitive groups of class $u(>3)$, $n\leqq 2u+1$. This last result betters Bochert's limit by unity.

15. Mr. M. A. Basoco: *On the Fourier expansions of doubly periodic functions of the third kind.*

The doubly periodic functions of the third kind may be classified into two groups according as the excess m of the number of zeros over the number of poles of the function is positive or negative. In a series of papers in the Annales de l'Ecole Normale, (1884–1888), Appell has developed a theory for obtaining the Fourier expansion of these functions. It is found that the theory is applicable in a practical manner to functions for which $m<0$, while for functions such that $m>0$, the theory, while complete from a function theoretic point of view, does not lead, in general, to arithmetically useful results, since it leaves certain constants expressed in the form of definite integrals, the actual evaluation of which is quite impracticable.

Using other methods, numerous expansions of functions exhibited as quotients of theta products with $m>0$ have been obtained in a form suitable for use in the theory of numbers.

B. A. Bernstein,
Secretary of the Section.

THE AMHERST COLLOQUIUM

A short program, five lectures on one subject instead of the usual two series by different speakers, proved nevertheless decidedly attractive. The list of those registering for Professor A. B. Coble's Colloquium numbered seventy-seven, nearly all being present on Tuesday, September 4th, at the opening lecture. The college opened the large lecture-room in Appleton Hall, with ample blackboard space and excellent acoustic properties. Professor H. L. Rietz, Vice-president of the Society, presided and introduced the lecturer. As it is expected that the lectures will be published promptly in the Society's Colloquium Series, no full account of the contents is here necessary. Excerpts from the quite full syllabus and from a supplementary sketch kindly furnished by Dr. Coble will suggest the extensive résumé and new researches, which sum up a line of the lecturer's publications extending back at least to 1915. Other authors cited were principally Stahl, Schottky, Picard, Cayley, Frobenius, and the Encyklopädie.

The title of the series was *The determination of the tritangent planes of the space sextic of genus four.* The standard definition of a function theta of p variables is given, by the aid of a square array which determines its (quasi) periods. The zero points of the variables are shifted by half-periods, leading to definitions of $2^{2p}-1$ odd and even thetas and their characteristics. This is ground-work for the proper subject of the lectures, as follows.

If the original periods are replaced by new periods (integer combinations of the old and vice versa) the $2^{2p}-1$ half periods are permuted, and the 2^{2p} odd and even functions also, according to a finite group G_{NC}. A geometric representation of this group is found by taking the half periods as points in a finite space S_{2p-1}, modulo 2. The collineation

group G_{NC} has then an invariant null system N and the 2^{2p} theta characteristics behave under it like quadrics whose polar systems coincide with N. The finite geometry indicates a *basis notation* in which the points (or half periods) and quadrics (or odd and even thetas) are expressed by means of $2p+2$ subscripts (Annals, 1916, p. 101).....

Products and powers of these first-order thetas aid in defining thetas of the second and higher orders, whose relations form a bridge to the algebraic and abelian functions. Not these, however, but the groups concerned and their invariants were the central object, and a variety of geometric pictures were invoked to make the theory vivid.

The geometric theory of the theta functions of genus two is in the main related to figures determined by six points. We use the three six-points: P_6^1, six points on a line; P_6^2, six points in a plane (usually on a conic); and P_6^3, six points in space on a norm cubic curve C^3. The P_6^1 and P_6^3 are *associated sets* of points,..... The P_6^2, if on a conic, is self-associated; otherwise it determines projectively an associated Q_6^2.

With this geometric background there is a projective invariant theory.....A first object is to show that the theta-relations are satisfied by virtue of the projective relations among the invariants and covariants. The invariant relations and the modular relations ($\breve{u}=0$) are identical; the same is true of the covariant relations and the theta-relations ($\breve{u}\neq 0$). A second object is to show that the period transformations of the functions appear in the geometry as the passage from one set P_6 to a congruent set P_6' under Cremona transformation.....

The functions of genus three are intimately connected with the figure of seven points P_7^2 of the plane, and the eight base points P_8^3 of a net of quadrics in space. These eight base points P_8^3 are congruent under Cremona transformation to 36 such 8-points and give rise to a Cremona group of order $8!\cdot 36$. Also the 8-point can be expressed parametrically in terms of modular functions, and period

transformations (mod 2) produce the same Cremona group. In this case both modes of approach are unified by the actual existence of a curve of genus three which defines the functions: namely, the locus of nodes of the quadrics of the net. Similarly in the case $p=4$ the set of 10 nodes of a Cayley symmetroid are congruent to $2^8 \cdot 51$ such 10-points and give rise to a Cremona group of order $2^8 \cdot 51 \cdot 10!$. Also Schottky has found that these ten nodes can be expressed parametrically in terms of modular functions of genus four. Again the period transformations (mod 2) produce the Cremona group. Thus far however a curve of genus four which defines the functions has not been discovered. The isolation of the tritangent planes of the normal space sextic curve of genus four is then dependent upon the detection of a curve of genus four attached to a symmetroid which will degenerate to a curve of genus three if either two nodes of the symmetroid coincide or four nodes are coplanar.

With two sessions on each of the first two days, the course was finished on Thursday. On Tuesday afternoon many of the participants motored to Mount Holyoke College, where the local mathematical faculty served tea and escorted the visitors to points of interest. Wednesday afternoon, Smith College colleagues were similarly hospitable. For Amherst, Professors Olds and Esty, with President Pease, were indefatigable in arrangements and personal courtesies; and the reception by President and Mrs. Pease at the Lord Jeffery Inn Tuesday evening created a nucleus for the memories of natural beauty, serene intellectuality, and pervasive friendliness which formed the setting of this twelfth Colloquium.

H. S. White

THE HEROIC AGE OF GEOMETRY*

BY J. L. COOLIDGE

The remarks which I shall have the honor to make today are addressed primarily, if not exclusively, to the older generation. The prophet Joel describes the golden age to come by saying "Your old men shall dream dreams, and your young men shall see visions." That distinction is eternally valid. Youth has visions of the future. Age dreams of the past. Our younger members are facing forward, looking towards the splendid mathematical discoveries that will be made during the coming decades, among which will be included those which they themselves will have made. It stirs the blood and fortifies the courage to feel that one is called upon to contribute in this way to the advancement of science, and that a grateful posterity will recognize what one has done. If the present passion for materialism continues, the number of contributions to mathematical science in the next hundred years may not greatly exceed half a million. Go on, oh younger generation, happy in the thought that the mathematicians of a century hence will, by an unfailing instinct, pick out your own particular contributions from all the rest!

For you who, like myself, belong to the generation that is passing, who have reached the age to dream dreams, I propose a humbler task. I propose that we look backward. The most notable epoch in all the long history of geometry, the heroic age, was almost exactly a hundred years ago. It was the moment when geometric supremacy was passing from France to Germany, a fact which the French geometers were slow to recognize, for Chasles, in his *Aperçu historique des méthodes en géométrie*, regretted that he could not

* An address read by invitation of the program committee before the Society, and the Mathematical Association of America, December 28, 1928.

speak of the work of the German geometers because he could not read their language. I shall try to show you some of the problems which interested the geometers of that time, and the methods chosen for their solution. I shall try to point out what was most significant in their work, and what was their influence on the progress of science.

Karl Wilhelm Feuerbach was born in 1800, contemporary with Macaulay and von Moltke, and died in 1834, professor at the gymnasium in Erlangen. In 1822, he published in Nürnberg a pamphlet entitled *Eigenschaften einiger merkwurdigen Punkte des geradlinigen Dreiecks*. The original of this, which may have been his doctor's dissertation, is hard to find, but a reprint appeared in 1908. The writer is concerned with the geometry of the triangle. When he finds two notable points he determines their distance, when there are three he determines the center and radius of the circumscribed circle. The methods are entirely elementary. The inscribed and escribed circles seem to attract him particularly, and finally he proves, quite casually, on page 48, that the circle through the feet of the altitudes passes through the middle points of the sides, and is tangent to the inscribed and escribed circles. This is the first known proof of the nine-point or Feuerbach circle theorem, though the author himself missed three of the nine points. Previous writers had found the center and radius of the circle, Bevan giving their determination as a problem in 1804.* The existence of nine points thereon was discovered in 1821 by Brianchon and Poncelet.† It is no disparagement to Feuerbach that he did not see Bevan's article which appeared in an obscure English journal, while the work of the French writers may well have appeared after his own had gone to press.

The fate of his theorem was that of many another;‡

* Leybourn's Mathematical Repository, vol. 1, p. 18.

† Gergonne's Annales de Mathématiques, vol. 11, p. 215.

‡ See Macaulay, *History of the nine-point circle*, Proceedings of the Edinburgh Mathematical Society, vol. 11 (1892).

first it was overlooked, then rediscovered. Steiner announced it without proof in 1828* and found the remaining three points in 1833.† Next Terquem proved it completely in 1842.‡ Salmon proved the tangency part in 1860 as did Casey,§ although they called the circle merely a six-point circle. Since that time there has been a perfect cloudburst of demonstrations. It has been said that the study of the philosophy of Kant has risen almost to the rank of one of the liberal professions; the same is measurably true of proving Feuerbach's theorem. The last adept would seem to be Sawayama, who gave nine demonstrations, all presumably new, in 1911.‖ Perhaps there is an element of absurdity in all this, but can we deny that we owe to Feuerbach the most beautiful theorem in elementary geometry that has been discovered since the time of Euclid?

Steiner's important contributions to elementary geometry during the period that we are considering were by no means limited to proving the Feuerbach theorem. He brought out and developed several of the most important principles connected with the modern geometry of the circle, notably¶ the idea of the power of a point with regard to a circle, the centers of similitude and radical axes. These ideas, though perhaps new to Steiner were not strictly speaking new to geometry, as they had already been presented by Gaultier,** Durande†† and Poncelet.‡‡ But Steiner handled his methods with consummate skill; witness his beautiful solution of Malfatti's problem to construct three circles, each tangent to two sides of a triangle, and to the other two circles, which he gave

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* See his Collected Works, Berlin, 1881, pp. 195, 196.

† Ibid., p. 491, Note.

‡ Nouvelles Annales de Mathématiques, vol. 1, p. 197.

§ Quarterly Journal of Mathematics, vol. 4, p. 153.

‖ L'Enseignement Mathématique, vol. 13 (1911).

¶ *Einige geometrische Betrachtungen*, Journal für Mathematik, vol. 1 (1826).

** Journal de l'École Polytecnique, vol. 9 (1813).

†† Annales de Mathématiques, vol. 11 (1820).

‡‡ *Propriétés Projectives des Figures*, Paris, 1822.

without proof, and for which no simple proof was published till that of Hart thirty years later.* Moreover, his proof of the Feuerbach theorem was contained in a paper which was important in another way, for it deals with constructions possible with the aid of a ruler and one circle fully drawn and with given center. This is not, of course, the first attempt to replace the Platonic instruments, ruler and compass. There were, for a long time, two opposing tendencies, one to extend the list of permissible instruments, the other to restrict it. In the latter category we might mention the idea of using a compass with fixed opening, which goes back to Abul Wafa in the 10th century† or the really astonishing work of Mascheroni in proving that every determination of a point that is possible with ruler and compass, is also possible with the compass alone.‡ The idea of using a ruler and given circle was first broached by Poncelet,§ but was carried through independently and in far greater detail by Steiner.

While in France and Germany distinguished mathematicians were doing their best to build up the ancient science of geometry, further East others were equally zealous in what must have seemed uncommonly like an effort to tear it down. The period we are considering included the birth of the non-euclidean geometry, at present a perfectly respectable branch of mathematics, but a heretical doctrine one hundred years ago. The "fons et origo" was skepticism about Euclid's twelfth postulate, that dealing with parallels. Candid persons had felt for a long time that this was not really as self evident as a reputable axiom ought to be, and countless attempts had been made to prove it. Now one of the classical methods of procedure in mathematics is the reductio ad absurdum, and quite naturally it occurred to those who were trying to prove this axiom, to see what would

* Quarterly Journal of Mathematics, vol. 1 (1855).

† See Tropfke, Journal Asiatique, (5), vol. 5 (1855), p. 226.

‡ *Geometria del Compaso*, 1797.

§ *Propriétés Projectives des Figures*, 1822, pp. 187–189.

happen if they replaced it by some other assumption about parallels. This was first tried by Saccheri at the middle of the 18th century, and many others after him. In each case things went well until the writer slid into some unwarrantable assumption in order to show that he was in difficulties when he wasn't. It took six decades from the time of Saccheri, and the genius of Gauss, to see that the parallel axiom was really independent of the others. But Gauss was busy with other matters and did not take the trouble to publish his results, so that the credit for first publicly announcing the non-euclidean geometry goes to a Russian, Lobachevski, who, on February 12, 1826, read before the Physico-Mathematical Society of Kasan a paper entitled *Exposition succincte des principes de la géométrie.** No trace of the manuscript of this address has ever been found, but we know from Lobachevski's later writings what must have been included therein. It is also permissible to doubt that it deserved the adjective succinct. His Russian article on the *Principles of geometry*, which appeared in the Kasanskij Wjestnik, the Kasan News in 1829–30, purports to be an extract from the previous paper, and covers 66 pages. It is very incomplete, and we only get to the root of Lobachevski's ideas by studying his subsequent publications.

Let us hasten to bracket with Lobachevski, the independent discoverer Johann Bolyai, whose *Appendix, Scientiam Absolute Veram Spatii Exhibens* was attached to his father Wolfgang's *Tentamen Juventutem Studiosam in Elementa Matheseos . . . . Introducendi*, which was published at Maros Vasarhely in 1832.

The truly startling thing about these two works is their similarity. It is not to be wondered at that, when Bolyai first saw Lobachevski's work in 1835, he made the natural mistake of believing that it was copied from his own.† We have learnt in a hundred years that there are a good many

* For an account of Lobachevski, see Engel, *Lobatschefskij*, Leipzig, 1898.

† See Stäckel, *Wolfgang and Johann Bolyai*, Leipzig, 1913, pp. 140 ff.

different ways of setting to work to establish the non-euclidean geometries. Kipling has written somewhere:

"There are nine and sixty ways, of constructing tribal lays,
And every single one of them is right."

It is much the same in constructing non-euclidean geometries. It is therefore really astonishing, that Lobachevski and Bolyai showed the following similarities:

1. Both defined parallel lines as the limiting positions of intersecting ones.

2. Both reached at a bound the hyperbolic geometry, overlooking completely the possibility of an elliptic geometry, where there were no parallels.

3. Both introduced at an early stage the horosphere, which is the surface orthogonal to a bundle of parallel lines, bringing out the remarkable fact that on this surface we have the euclidean geometry.

4. Both pointed out that the formulas of spherical trigonometry could be established without the parallel axiom.

5. Both studied the ratio of the non-parallel sides of a quadrilateral, two of whose sides are equal and parallel.

Of course there are divergences as well as similarities in the writings of the two. Lobachevski made continual use of the parallel angle associated with a given distance, that is, the angle which the parallel to a given line, through a point at a given distance from that line, makes with the normal. Bolyai manipulated skillfully his theorem that the ratio of the sines of two angles of a triangle is that of the circumferences on the opposite sides as diameters. Lobachevski is more prolix, and especially interested in trigonometric formulas, Bolyai is brief, but he brings out the function of the space constant in admirable fashion. My own final impression is that the points of similarity are far more remarkable than those of difference.

It was many years before the significance of non-euclidean geometries was fully understood, and developed beyond the point reached by the discoverers. It was only in 1854 that the third classical geometry, the elliptic geometry of Riemann

was first exhibited.* Skepticism lingered; in order to convince the average man, or the average mathematician, it was necessary to see the non-euclidean geometry at work. A sphere, or rather a hemisphere where opposite points of the equator count as identical, gives a very good elliptic plane; a beautiful specimen of a Lobachevskian plane, or rather of a part of one, was found by Beltrami in a surface of constant negative curvature.† These examples did not go beyond two dimensions. Klein seized on the theory of projective measurement, casually thrown out by Cayley‡ to give a perfect example of Lobachevskian or Riemannian geometry in as many dimensions as may be desired. Lie's theory of continuous groups threw a new and valuable light on the questions involved; contributions came in from various quarters. Since the beginning of the present century the subject has been somewhat transcended, owing to the newer methods of differential geometry, and the theory of relativity. Of the various methods of attack, that of Riemann based on the study of quadratic differential forms has shown itself best able to meet the new demands. But we owe an enormous debt to Lobachevski and Bolyai, not only for enriching geometry, but for initiating a movement which has been of incalculable importance to philosophy. Our whole modern conception of mathematics as a logical system based on arbitrary axioms may be traced back to their pioneer work.

We have had occasion already to speak of Steiner once or twice; we have not yet said a word about his most important work, *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*,§ one of the corner stones on

* *Ueber die Hypothesen welche der Geometrie zur Grunde liegen.* See his Collected Works. 2d ed., Leipzig, 1892, pp. 272 ff.

† *Saggio di interpretazione della geometria non-euclidea*, Giornale di Matematiche, vol. 6 (1868).

‡ Klein, *Ueber die sogenannte Nicht-Euklidische Geometrie*, Mathematische Annalen, vol. 4 (1871); Cayley, *Sixth Memoir on Quantics*, Philosophical Transactions, vol. 149 (1859).

§ Berlin, 1832. See his Collected Works, vol. 1.

which the whole subject of synthetic projective geometry rests. Who were the predecessors, from whom, presumably, Steiner drew inspiration? They were certainly French. Among the earlier writers there were, of course, Pascal, Desargues and Brianchon. Then Carnot, although preceding the modern movement, exercised considerable influence through his *Géométrie de Position*,* especially through his theory of transversals. More important by far was Poncelet, whose *Propriétés Projectives des Figures*, composed in a Russian prison in 1813, first saw the light in 1822. Poncelet's root idea may be stated about as follows. A geometric figure may be wonderfully simplified by a central projection from plane to plane, or by a homology in plane or in space. What are the properties of figures which are invariant under such a transformation, how can theorems about the simplified figures be stated in invariant terms so as to give properties of the original figures? We see at once that the essentials consist in concurrence, collinearity, coplanarity and cross ratios. Poncelet also introduced a rather cumbrous system of what he called "ideal chords" determined not by the given curve, but by another, to get rid of imaginary intersections. Finally he laid stress on the method of polar reciprocation with regard to a conic or quadric to establish the principle of duality. This last principle was a good deal talked about at the time; a long and not very edifying dispute between Gergonne and Poncelet as to priority of discovery is spread over the pages of the Annales de Mathématiques.† The former conceived duality in a broader spirit as an inherent characteristic of geometric figures, but did not follow this important idea very far; the latter did not much believe in any duality he did not see, so he created it by polarization.

All of these threads were drawn together by Steiner in his development of projective geometry, which is nothing if not "systematisch." He starts with a point, the points

* Paris, 1803.

† Volume 18 (1827–28).

SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: $1/$. Similarly, O means "zero"; to indicate "cap O." backspace and overprint period; thus: $O/$.
15. **Avoid a dash over a letter, except for those shown below.**
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma \cdots$ (all) A B $\Gamma \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma \cdots}$ (all except ^{o}).

★ Light Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma \cdots}$ (all except $_{o}$).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \gamma\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \kappa\ \lambda\ \mu\ \nu\ \xi\ \pi\ \varrho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ f\ p\ q}\ \mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ K\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b}\ \mathfrak{A\ B\ D}$.

Script (special font) $\mathcal{A\ B\ C} \cdots$ (all). No lower case manufactured.

* Hebrew—בג א ת troublesome to handle.

★ Dashed Italics—$\overline{A}\ \overline{a}\ \overline{B}\ \overline{b}\ \overline{C}\ \overline{c}\ \overline{E}\ \overline{e}\ \overline{F}\ \overline{f}\ \overline{G}\ \overline{g}\ \overline{H}\ \overline{I}\ \overline{\imath}\ \overline{j}\ \overline{K}\ \overline{k}\ \overline{M}\ \overline{m}\ \overline{N}\ \overline{n}\ \overline{o}\ \overline{P}\ \overline{p}\ \overline{q}\ \overline{R}\ \overline{r}\ \overline{s}\ \overline{t}\ \overline{u}\ \overline{v}\ \overline{X}\ \overline{x}\ \overline{Y}\ \overline{y}\ \overline{Z}\ \overline{z}$.

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$.

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$.

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \overline{\Gamma}$.

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \dot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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* Additional characters **readily available at small cost.**

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

of a line, and those of a plane. The plane is the simplest two-dimensional locus of points, the point the simplest two-dimensional envelop of planes, the line is the simplest one-dimensional figure of points or planes, the pencil of lines is a self-dual figure. The principle of duality appears at the very start, and a good part of the work is printed in double columns. Cross ratios of points, lines and planes are then defined, and it is shown how they are unaltered by projections and intersections. Steiner talks a good deal about "projective forms" but the definition is rather mixed with theorems about them. The underlying thought is that they are one-dimensional forms whose elements are in one-to-one correspondence, with equality of corresponding cross ratios. Steiner shows easily enough that two forms which are connected by a finite number of projections and intersections are projective in this sense; his proof of the converse is faulty owing to his inability to handle the question of the continuity of the projective relation. Starting with these data, Steiner obtains all the other figures he wants to study by construction; a conic is given by the intersections of corresponding lines in two projective pencils, and enveloped by lines connecting corresponding points in two projective ranges.

Every modern student of projective geometry will see that the fundamental ideas of his science are here set out in order. Steiner laid a very solid foundation whereon his successors might build. It is true that the Latin school of geometers, including such mathematicians as Chasles and Cremona, drew their inspiration more from Poncelet than from Steiner, but the latter had a notable lineal descendant in von Staudt. This profound thinker perceived two flaws in the perfection of the Steinerian structure. In the first place, although cross ratios are projectively invariant, their original definition is metrical; secondly, there is no satisfactory treatment of imaginary elements. These defects he set out to remedy in heroic fashion.* Each set of four ele-

* See his *Geometrie der Lage*, 1847, and *Beiträge zur Geometrie der Lage*, 1856–1860.

ments of a fundamental one-dimensional form he calls a "throw" and associates with a number; four harmonic elements, in right order, are associated with the number -1. He then defines the sum and the difference of two throws, and shows how a throw can be found corresponding to every rational number. At that point he was blocked as he had no means of handling irrational throws, no Dedekind cut. The lack was later filled by Klein.* Von Staudt's other highly original idea was to define an imaginary point as an elliptic involution of points, to which a sense of description has been attached. He showed how projective geometry, enlarged by these new elements, followed the old laws and permitted the old constructions. His culmination is the treatment of a complex throw.

The synthetic geometry of Poncelet and Steiner was vigorously pursued for a century, the last heroic figure being that of Reye. In recent years we have tended to question the wisdom of too sharp a separation between synthetic and analytic methods, and to confine our researches either to tinkering with the fundamental assumptions, or to the field of projective differential geometry. But the synthetic methods have a compelling charm when rightly presented, and afford a most admirable training for every geometer, wherever his specialty may lie.

The great progress of synthetic geometry in the years we have been considering was, fortunately, not at the expense of analytic geometry; on the contrary, the rivalry between the two was of great value to mathematical science. The progress of algebraic methods kept pace with that of the synthetic ones; strangely enough the first writer we must mention in this new connection is our previous acquaintance, Karl Wilhelm Feuerbach. In 1827 we find him publishing his *Grundriss zur analytischen Untersuchung der dreieckigen Pyramide*. His object here is analogous to that previously pursued in the study of the triangle, to establish relations

* Mathematische Annalen, vol. 7.

between the distances between the notable points of the pyramid. But he now passes from synthetic to analytic methods, and consequently arrives at very different theorems from those reached before. To begin with, and most important, he sets up a novel set of coordinates. He has the idea of linear dependence firmly fixed in his mind. Suppose that the coordinates of a variable point are expressed as linear combinations of those of four fixed non-coplanar points

$$X = ax + bx' + cx'' + dx''',$$
$$Y = ay + by' + cy'' + dy''',$$
$$Z = az + bz' + cz'' + dz''',$$
$$1 = a + b + c + d.$$

The quantities a, b, c, d connected by the last identity may be taken as a new system of coordinates. They may be interpreted as the distances to the four faces of a tetraedron, with right algebraic signs, divided by the corresponding altitudes. A linear equation among them will give a plane. Feuerbach has an additional tool in an identity which he attributes to Lagrange, but which I am ashamed to say I have not been able to find in his writings. If A, B, C, D, and E be five points in space, their distances are connected by the relation

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 & AE^2 & 1 \\ BA^2 & 0 & BC^2 & BD^2 & BE^2 & 1 \\ CA^2 & CB^2 & 0 & CD^2 & CE^2 & 1 \\ DA^2 & DB^2 & DC^2 & 0 & DE^2 & 1 \\ EA^2 & EB^2 & EC^2 & ED^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

He proves a good many theorems with the aid of his new tools; some are interesting, others unimportant. Perhaps his most notable result is to give in very simple form the equation of a sphere tangent to four given spheres. It must not be forgotten that Feuerbach never heard of a determinant.

If we must give Feuerbach full credit for the independent discovery of one form of tetrahedral coordinates, we must hasten to point out that he was not the only discoverer. A more important mathematician than he made the same discovery in the same year, August Ferdinand Möbius.* The idea of this able geometer was to base geometric theorems on the properties of the center of gravity of a system of points. If two points be given, any point of their line will be the center of gravity of the two, provided they be endowed with proper masses, positive or negative. These masses are taken as homogeneous coordinates for the point. Two masses whose sum is zero are made to correspond to the infinite point of the line. A similar definition, when three non-collinear points are given, will give the homogeneous coordinates of a point in their place while four non-coplanar points will yield the homogeneous barycentric coordinates of a point in space, exactly proportional to the coordinates of Feuerbach. In representing a straight line, Möbius uses linear dependence on two points thereof, while a plane is given by linear dependence on three. This parametric method yields, of course, the simplest means of handling linear problems. When it comes to studying a curve, in plano or in spatio, Möbius represents the coordinates of a point thereof as rational functions of a given parameter, while two parameters are used for a surface. As the coordinates used are homogeneous, he replaces rational functions by polynomials, whose degree gives the degree of the curve or surface. He knew too much to imagine that all algebraic curves and surfaces could be expressed in this way, but he confined himself to rational varieties, and in particular to quadratic ones. A notable feature is that he has a clear grasp on the fundamental idea of an invariant, that is to say, a property of a geometric figure which is unaltered by the transformations of a certain group. He studies successively motions which keep distances unaltered, affine transformations where

* *Der barycentrische Calcul*, Leipzig, 1827.

areas or volumes are altered in a fixed ratio, and collineations which he defines as transformations which carry collinear or coplanar points into collinear or coplanar ones. His fundamental relations here are the cross ratios of four collinear points, triangle ratios involving six coplanar ones, and tetrahedral ratios involving eight non-coplanar ones. If four points of a plane, no three collinear, be invariant for a collineation, all points of their net of rationality are invariant. Möbius assumes that the transformation must therefore be the identity. He expresses a collineation by a linear transformation of his homogeneous barycentric coordinates; he grasps the principle of duality both in the form of Gergonne, and that of Poncelet.

It would be a mistake to imagine that Möbius' contributions to geometry were limited to the publications of the *Barycentrischer Calcul*, and contemporary articles dealing with the same subject. We are indebted to him for the discovery of the null system in space, and a geometric theory of circle transformations in the plane which is useful in the study of the simplest functions of a complex variable. The form of the *Barycentrischer Calcul* is not exactly that which we should choose today, but the wealth of fruitful ideas is remarkable.

It would certainly seem that Feuerbach and Möbius were enough to share the credit for discovering tetrahedral or trilinear coordinates; but such is not the case. Omitting independent discoveries of a somewhat later date, we must now render full credit to Julius Plücker for his work of 1828.* He started out with the deliberate intention of showing that all of the beautiful results which Poncelet and Steiner had reached by synthetic methods were easily obtained by algebraic analysis. His interest in trilinear coordinates was much less than that of the previous writers mentioned; they appeared to him merely as a sort of abridged notation. Thus if A and A' be the expressions for the distances of a point

* *Analytisch-geometrische Entwickelungen*, vol. 1, 1828; vol. 2, 1832.

from two given lines, the equations $A=0$, $A'=0$ and $A+\nu A'=0$ will represent respectively the first line, the second line or an arbitrary line through their intersection. An almost identical point of view was contemporarily, though doubtless independently, expressed by Bobilier,* to whom we owe also the equation of the first polar of a point with regard to a given curve.† This idea of using a single letter to replace a whole polynomial gave Plücker an easy method of proving not only straight line theorems, but a number of beautiful properties of circles that had been found synthetically by the French geometers. When it comes to conic sections, he uses Cartesian coordinates, in his first volume, giving a good deal of attention to change of axes, and polar reciprocation.

Between writing the first and the second volume, Plücker became very much impressed with the idea of duality, so that Volume 2 is entirely given to line geometry in a plane. The coefficients of a line are the ratios of the coefficients in its equation when expressed in terms of homogeneous Cartesian point coordinates. A large part of the book is given to curves of the second class. The domain is real, so that an ellipse can be defined as a curve of the second class which has a tangent pointing in every direction. The volume, like its pre ecessor, ends with polar reciprocation.

One has the impression, on the whole, that Plücker's work at this moment was less original than that of Möbius, though easier to read, and yielding more results. It is also to be remembered that his greatest contributions to algebraic geometry came later. We owe to him the coordinates of a line in space, which, in the perfected form devised by Klein, are highly important in that branch of science vaguely called "Higher Geometry." We are also indebted to Plücker for the beautiful identities connecting the numbers of point and line singularities of an algebraic plane curve. Let us mention in passing, that we do not surely know to this day

* Annales de Mathématiques, vol. 18 (1827–28).

† Ibid., vol. 19 (1828–29), p. 108.

when a set of solutions of these equations in positive integers necessarily corresponds to an existing curve.

It would certainly seem that with the names of such geometers as Poncelet, Steiner, Möbius, and Plücker, we might close the list of those who contributed to geometric science in the second period just one hundred years ago; it would seem so, and we should be justified, were it not for the the fact that there lived in the small but important town of Göttingen, a man by the name of Johann Carl Friedrich Gauss who published his *Disquisitiones Generales circa Superficies Curvas* in 1827.* This work is not only the firm basis on which all the mass of subsequently discovered differential geometry rests, but is also the most important individual contribution ever made to that branch of mathematical science.

What were the most novel and fruitful ideas contained in Gauss' geometric work? To begin with, and most important, Gauss was the first writer to make consistent use of the method of parametric representation of a surface, that is to say, the method of expressing the Cartesian coordinates of the points of a surface as functions of two independent variables. The idea was not absolutely new. Euler used it in approaching the general problem of the applicability of one surface to another, his exact words being:†

"Et quia per naturam superficiarum quaelibet coordinata debet esse functio binarum variabilium a se invicem non pendentes." Unfortunately this important idea did not bear immediate fruit. There was much interest in differential geometry at the beginning of the nineteenth century, especially on the part of the French mathematicians. Writers like Monge and Dupin who made really important contributions, regularly expressed z as a function of x and y, Cauchy equated a function of the three variables to zero.

* Commentationes Societatis regiae Gottingensis, vol. 6 (1828); vol his Collected Works, vol. 4, Göttingen, 1873. An English transiation with notes by Thompson was published in Princeton, N. J., 1902.

† Leonardi Euleri Opera Posthuma, vol. 1, St. Petersburg, 1862, p. 494.

The superiority of the Euler-Gauss method is so evident that it "jumps to the eyes." The modern developments of differential geometry would never have been possible without it. Gauss first states it in his *Theoria Attractionis* of 1813.* He uses it also in his prize memoir presented to the Scientific Society of Copenhagen in 1822,† but it is in the *Disquisitiones* that it is first convincingly set forth.

A second important feature of Gauss' treatment is the use made of spherical representation. Gauss considers not one surface but two. Previous writers paid attention to the direction cosines of the normals; Gauss treats these as the Cartesian coordinates of a point on an auxiliary sphere. The beauty of the conception can be seen in the following way. In the plane we may associate with each curve a circle of unit radius whose points correspond to the oriented normals of unit length of the given curve. The curvature is the ratio of the lengths of corresponding infinitesimal arcs of circle and curve. In the same way the total curvature of a surface is the ratio of corresponding infinitesimal surface elements on sphere and surface. It was in this fashion that Gauss presented his idea; it had been given in substance, if not in so many words, by Olinde Rodrigues,‡ who wrote the equation

$$\frac{1}{RR'} = \left(\frac{\partial X}{\partial x}\right)\left(\frac{\partial Y}{\partial y}\right) - \left(\frac{\partial X}{\partial y}\right)\left(\frac{\partial Y}{\partial x}\right).$$

Still the credit for actual statement is Gauss', the year 1816.§

The important part of Gauss' work on the measure of curvature is not in the statement given, but in the demonstration that this expression is invariant under every transformation of the surface which leaves distances invariant, so that mutually applicable surfaces have, at corresponding points, the same measure of curvature. Moreover, this

* Collected Works, vol. 5, 1867, p. 14.
† Ibid., vol. 4, pp. 189 ff.
‡ Correspondence de l'École Royale Polytechnique, vol. 3, 1816, p. 168.
§ Works, vol. 8 (1900), p. 367.

particular invariant, and its extension to higher spaces, has proved to be fundamental in non-euclidean geometry, as pointed out by Riemann and Beltrami and many others.

A good proportion of the *Disquisitiones* is devoted to the study of geodesic or shortest lines. Gauss reaches the differential equations for such lines by straightforward methods of the calculus of variations. His equations show at once the characteristic property that the principal normal to the curve is normal to the surface, a fact first discovered by Johann Bernoulli.* What is rather curious is that Gauss does not point out that geodesic lines may be characterized by the fact that geodesic curvature is everywhere zero, even though Gauss himself appears to have been the first to study this sort of curvature.† He presently proves the existence of geodesically parallel curves. The latter part of the essay is largely devoted to the study of geodesic triangles.

If the ultimate influence of the *Disquisitiones* was incalculable, we can not affirm that its importance was immediately felt.‡ He had an immediate and devoted follower in Minding, who published an entirely Gaussian article on the development of curves on surfaces in 1830.§ But in 1831, we find, Mlle. de St. Germain writing¶

"Si par rapport aux surfaces on avait besoin de connaître la mesure de courbure . . . le mémoire d'Euler contient tout ce que l'on sait d'important à cet égard." The surprising thing is not that a French woman should write this, but that Crelle's Journal should publish it.

It is fair to say that Mlle. de St. Germain will have nothing to do with any measure of curvature but mean curvature. It seems unlikely that the ideas of Gauss were understood or appreciated at all in France till the indefatigable

* See his letter to Leibnitz in Gott. Gul. Leibnitii et Johann Bernoullii Commercium, vol. 1, Lausanne, 1745, p. 393 near bottom.

† See his fragment on *Seitenkrummung*, Works, vol. 8, pp. 386 ff.

‡ In this matter I cannot agree with Stäckel, *Gauss als Geometer*, Gauss' Collected Works, vol. 10, Part II, Section IV.

§ Journal für Mathematik, vol. 6.

¶ Ibid., vol. 7.

Liouville lectured on them in the autumn of 1850.* Perhaps we may say that the final consecration only came in 1867 when Bonnet† proved that every set of solutions of the Codazzi equations determines a single surface completely, except for motions of space.

Such was the work of the leading geometers of one hundred years ago. What shall we say of them today? Well, for my part, I am willing to quote Scripture and say "There were giants in the land in those days." Think of Poncelet in a Russian military prison at Saratoff laying down the fundamental principles of projective geometry, Lobachevski and Bolyai breaking the chains of servitude to Euclid that had lasted for twenty-one centuries, Gauss laying an absolutely secure foundation for the geometry of any manifold in any space. Such work calls for ability of a high order. Moreover, there is one common element in all this work to which I should like to call your attention, the very nice balance between specific results and general theory. A distinguished Scots philosopher, whose lectures I once had the privilege of attending, was forever insisting that it was the great task of philosophy to "Combine the univairsal and the parteecular in a higherrr uneetie." It seems to me that geometry is called upon to do much the same thing. No individual theorem, be it that of Feuerbach, or even that of Pythagoras, can rise to the dignity of a general mathematical principle. Every mathematical proposition or system must be given the greatest possible breadth, in reason. But I wonder whether we are not today in some danger of extinguishing the vital spark in geometry by the excessive abstraction and generality of the results which we seek and publish. It is my personal credo that geometry is a branch of art. I find more emotional appeal in a Raphael Madonna or a Gothic cathedral than I could ever get from a picture of "Things in general" or a building intended for all possible purposes.

Perhaps the difficulty is inevitable, and there remain no

* See Liouville, Comptes Rendus, vol. 32 (1851), p. 533.

† Journal de l'École Polytechnique, vol. 25, Cahier 42, pp. 31 ff.

geometric results to be found which are at once so definite and yet so general as what has been found in the past. This may be so, but I like to think otherwise. I do not so far despair of the Republic. If anyone had asked Lagrange, who died but fifteen years before the beginning of the epoch we have considered, whether he believed there was much yet to do in geometry, he might well have given a hesitant answer. Let us have confidence that our five-thousand-year old science of Earth-measurement has still some beautiful secrets which she will yield when challenged by anyone who is sufficiently able and sufficiently fearless. Let us believe that

> "Some noble deed of note may yet be done
> Not ill-befitting men that strove with gods."

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THE CRITICAL POINTS OF FUNCTIONS AND THE CALCULUS OF VARIATIONS IN THE LARGE*

BY MARSTON MORSE

1. *Introduction.* The conventional object in a paper on the calculus of variations is the investigation of the conditions under which a maximum or minimum† of a given integral occurs. Writers have accordingly done little with extremal segments that have contained more than one point conjugate to a given point. An extended theory is needed for several reasons.

One reason is that in applying the calculus of variations to that very general class of dynamical systems or differential equations which may be put in the form of the Euler equations, it is by no means a minimum or a maximum that is always sought. For example, if in the problem of two bodies we make use of the corresponding Jacobi principle of least action the ellipses which thereby appear as extremals always have pairs of conjugate points on them, and do not accordingly give a minimum to the integral relative to neighboring closed curves, so that no example of periodic motion would be found by a search for a minimum of the Jacobi integral. In general if one is looking for extremals joining two points, or periodic extremals deformable into a given closed curve, the a priori expectation, as justified by the results of this paper, in general problems, would seem to be that many more solutions would fail to give a minimum than would give a minimum. Even if the ultimate object is

* An address presented before the Society at the request of the program committee, April 6, 1928.

† For work on the absolute minimum see Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 419–437, and Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2. Further references will be found in these works.

narrowed down to the search for minima the existence of such minima is tied up in the large with the existence of extremals which do not furnish minima.

However, the real feeling of the author is that the classical calculus of variations is both logically and esthetically incomplete. The object of the present paper is to show by selected theorems and examples something of the more extended results now within reach of the science.* The author wishes to say, however, that in view of the tremendous technical difficulties involved, and the scope of the investigations now being carried on it is bound to be several years before a final idea of the possibilities of the new theory can be realized.

No theory in the large can well escape analysis situs. It is not to be expected, however, that pure analysis situs should contain all the results necessary for all the applications. For the present purposes a fundamental paper in analysis and analysis situs is the author's paper cited below.†

To reduce the study of the existence and nature of extremals to the study of critical points, admissible curves in any problem in the calculus of variations are replaced by nearby curves composed of a succession of short extremals.‡ The value of the given integral along the resulting broken extremal g is regarded as a function f of the coordinates of the end points of the component short extremals. Critical points of f occur when and only when g has no corners.

* Of the earlier work the most significant from the points of view of this paper is the "minimax principle." See Birkhoff, Colloquium Publications of this Society, vol. 9, *Dynamical Systems*, pp. 133–139.

† Marston Morse, *Relations between the critical points of a real function of n independent variables*, Transactions of this Society, vol. 27 (1925), pp. 345–396. This paper will be referred to as Morse I.

A *critical point* of a function is a point at which all of its first partial derivatives vanish.

‡ This idea is doubtless rather old. The manner in which it has been applied is variable. See Birkhoff, loc. cit. In proving his own theorems in his work not yet published the author has had occasion to go into many questions connected with the application of this idea that have never previously been considered.

Thus the search for extremals is reduced to the search for critical points of f.

Up to the present, a study has been made of three fields, first of extremals joining two fixed points, second of closed extremals, and third of extremals from a point to a manifold cut transversally by the manifold.

The first of these fields occupies the principal place in the classical calculus of variations. The second is of central importance for the theory of dynamical systems. The third leads to a geometry suggestive of real algebraic geometry without the algebra. Particularly interesting are the relations in the large that may be found between straight lines, hyperplanes, hyperspheres, and closed manifolds.

2. *The Hypotheses on the Integrand.** Let S be a closed region in the space of the variables $(x_1, \cdots, x_m) = (x)$. Let

$$F(x_1, \cdots, x_m, r_1, \cdots, r_m) = F(x, r) \tag{1}$$

be a positive, analytic function of its arguments, for (x) on S, and (r) any set not (0). Suppose further that F is positively homogeneous of order one in the variables (r). We employ the parametric form, taking $F(x, x')$ as our integrand, where (x') stands for the set of derivatives of (x) with respect to the parameter t. We suppose that the problem is *positively regular*, that is, that

$$\sum_{ij} F_{r_i r_j}(x, r)\eta_i\eta_j > 0, \quad (i, j = 1, 2, \cdots, m), \tag{2}$$

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* For that which concerns the classical theory of the calculus of variations in space the reader may refer to the following articles:

Mason and Bliss, *The properties of curves in space which minimize a definite integral*, Transactions of this Society, vol. 9 (1908), pp. 440–466.

Bliss, *The Weierstrass E-function for problems of the calculus of variations in space*, Transactions of this Society, vol. 15 (1914), pp. 369–378.

Bliss, *Jacobi's condition for problems of the calculus of variations in parametric form*, Transactions of this Society, vol. 17 (1916), pp. 195–206.

Hadamard, *Leçons sur le Calcul des Variations*, vol. 1, Paris, 1910.

Carathéodory, *Die Methode der geodätischen äquidistanten und das Problem von Lagrange*, Acta Mathematica, vol. 47 (1926), pp. 199–236.

for (x) and (r) as before, and (η) any set not (0) nor proportional to (r).

In general, for the theorems in the small in this paper, the hypothesis $F>0$ may be omitted, and the hypothesis that F be analytic replaced by the requirement that F be of class C'''. Further details will not be given here.

3. *Conjugate Points and their Orders.* Our future work depends not only upon a suitable definition of conjugate points but also of their orders.

Let there be given an extremal g which is an ordinary analytic curve joining a point P to a point Q. Let λ be the direction of g at P. Let the direction cosines of the directions neighboring λ be admissibly* represented as functions of $(m-1)$ parameters (α). Let s be the arc length of the extremal through P measured from P in the direction determined by (α). On the extremals issuing from P with the directions determined by (α) the coordinates (x) will be analytic functions of s and (α). Let $\Delta(s)$ be the jacobian of the coordinates (x) with respect to s and (α), evaluated for the parameters (α) that determine λ.

Each point on g, not P, at which $\Delta(s)$ vanishes, is called a conjugate point of P, and the order of the vanishing of $\Delta(s)$ the order of that conjugate point.

The order of a conjugate point is shown to be invariant† under admissible space transformations, and to be at most $m-1$. A conjugate point of the rth order will be counted as if it were r distinct conjugate points.

4. *The Type Number of an Extremal.* We shall first consider extremals joining two fixed points P and Q. Before turning to a theory of all such extremals it is necessary to consider one such extremal alone. Suppose g is an ordinary

* That is, the direction cosines are to be analytic functions of the parameters (α) of such sort that not all of the jacobians of $m-1$ of the direction cosines with respect to the parameters (α) are zero.

† Although conjugate points have been defined in various ways the notion of the order of a conjugate point does not seem to have received the attention it requires.

analytic curve without multiple points. The author has shown how g and a neighborhood of g can be mapped analytically, on a straight line segment h and a neighborhood of h. For present purposes then we can suppose that g is a segment of the x_1 axis.

Let us cut across g with n hyperplanes perpendicular to g. These hyperplanes divide g into $n+1$ successive segments. Suppose the hyperplanes are placed so near together that no one of these $n+1$ segments contains a conjugate point of its initial end point. Let us number these hyperplanes in the order of increasing x_1. Let P_i be any point near g on the ith one of these hyperplanes. Let the points

$$P,\ P_1,\ P_2, \cdots,\ P_n,\ Q$$

be successively joined by extremal segments neighboring g, and let the resulting broken extremal be denoted by E. Let $[u]$ be the set of the coordinates, other than x_1, of the points P_i. The value of the given integral J taken along E will be an analytic function of the variables $[u]$, and will be denoted by $J(u)$.

The function $J(u)$ will have a critical point when $[u]=[0]$. Let H be the quadratic form making up the terms of second order in $J(u)$ in a Taylor's expansion about $[u]=[0]$. We have the following fundamental theorem.*

If Q is a conjugate point of P of the rth order, the rank of H is $p-r$. If Q is not conjugate to P the rank of H is p, and its type number† is the sum of the orders of the conjugate points of P preceding Q, where $p=(m-1)n$.

Note particularly that the type number of H will be unchanged if we increase n to any arbitrary integer. The pre-

* For the case $m=2$, the proof of this theorem has already appeared See Marston Morse, *The foundations of a theory in the calculus of variations in the large*, Transactions of this Society, vol. 30 (1928), pp. 213–274. This paper will be referred to as Morse II.

† By the type number of H is meant the number of negative coefficients appearing in H when H has been reduced by a real, non-singular, linear transformation to squared terms only.

ceding theorem was developed primarily for use in the following theory in the large. Incidentally, it may be used to give what is probably the definitive answer as to the generalization of the famous Sturm separation theorem, for the case of systems of m ordinary, second order, self-adjoint, linear, differential equations in m dependent variables.

5. *Conjugate Points characterized by Deformations.** The entities deformed are families of curves joining P to Q and depending on μ parameters. They are called *μ-families.* The μ parameters are supposed represented by a point P on a closed μ-dimensional complex† C_μ. Each point P on C_μ corresponds to a curve of the family. The curves are supposed to vary continuously with the position of P on C_μ.

Let g be the given extremal, and let J_0 be the value of J along g. Suppose Q is not conjugate to P, but that there are k points $(k>0)$ conjugate to P and preceding Q. We are going to consider continuous deformations of μ-families subject to the condition,

$$J \leqq J_0 - e^2, \qquad e > 0, \tag{3}$$

imposed on each member of the μ-family. We have the following characterization of the number of conjugate points in terms of deformations.

Corresponding to any sufficiently small neighborhood N of g, there exists within N an arbitrarily small neighborhood N_1 of g, and an arbitrarily small positive constant e, with the following properties. The type number k is the least integer k for which the $0, 1, \cdots, (k-2)$-families on N_1, satisfying (3), can be deformed into a single curve on N, subject to (3), and for which there exists at least one $(k-1)$-family which cannot be so deformed into a single curve.

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* The theory of these deformations in the small has already been developed in the paper Morse II for the case $m=2$. The work appearing there for $m=2$ applies with hardly any change for a general m.

† Terms referring to pure analysis situs are used in the senses defined by Veblen in The Cambridge Colloquium, 1916, Part II, *Analysis Situs.*

A theorem similar to this but without use of the genera notion of μ-families is given by Birkhoff* for the case of closed trajectories of dynamics when $k=1$. For $k=1$ our theorem refers to a 0-family of curves joining P to Q, that is to a pair of curves joining P to Q. The theorem states that there exists a pair of such curves in N_1, which satisfy (3), but which cannot be deformed into each other subject to (3) without passing out of N.

A simple example of the theorem is obtained by considering the integral of the arc length of an n-sphere. Let the extremal g be an arc of a great circle slightly longer than a semi-circle. If $n=2$, then $k=1$, and the statement of the preceding paragraph is clearly true. For a general n, $k=n-1$, and one can apply the theorem as stated.

If $k=0$, the theorem says nothing. However, the theorem of the calculus of variations commonly known as Osgood's Theorem† is seen to fill the gap in a most natural way.

6. *Theorems in the Large.* The preceding theorem is a typical deformation theorem in the small. One can also consider the mutual relations of μ-families in the large. In particular one can introduce the notion of "bounding" or "non-bounding" families. It is by means of such considerations that one arrives at the following theorems in the large.

Suppose the region S homeomorphic with the interior and boundary of an $(m-1)$-sphere, and that S's boundary is *extremal convex.* That is, suppose there exists a positive constant e so small that any extremal segment on which $J<e^2$ and which joins two boundary points will lie interior to S except at most for its end points.

If P_0 and Q_0 are any two points of S there exist in their respective neighborhoods points P and Q which are joined by no extremals on which P is conjugate to Q.

Such a pair of points will be termed *non-specialized.* For such a pair there will be at most a finite set of extremals

* Birkhoff, *Dynamical systems with two degrees of freedom*, Transactions of this Society, vol. 18 (1917), p. 249.

† Osgood, Transactions of this Society, vol. 2 (1901), p. 273.

joining P to Q with lengths less than a given constant, although altogether there may be an infinite number of extremals joining P to Q among the extremals of all lengths. For a non-specialized pair of points the number of conjugate points of P prior to Q will be called the *type number* of g. We have the following theorem.

If on the extremals joining two non-specialized points P and Q, J is bounded, then there will be at most a finite set of extremals joining P to Q, and between the integers M_k which give the total number of extremals of type k joining P to Q, the following relations will hold true:

$$
\begin{aligned}
&1 \leqq M_0, \\
&1 \geqq M_0 - M_1, \\
&1 \leqq M_0 - M_1 + M_2, \\
(4)\quad &1 \geqq M_0 - M_1 + M_2 - M_3, \\
&\cdots\cdots\cdots\cdots\cdots, \\
&1 \leqq [M_0 - M_1 + M_2 - M_3 + \cdots + (-1)^{r-1} M_{r-1}](-1)^{r-1}, \\
&1 = M_0 - M_1 + M_2 - M_3 + \cdots + (-1)^r M_r.
\end{aligned}
$$

Simple examples will show that there may perfectly well be an infinite number of extremals joining two non-specialized points P and Q. Here the following more general theorem holds.

Corresponding to any positive constant a there exists a positive constant b with the following properties. If P and Q are two non-specialized points the relations (4) *will still be true if M_k be taken as the number of extremals of type k for which $J < a$, plus the number of extremals of type k in a suitably chosen subset of the extremals on which $a \leqq J < b$.*

Let P and Q be any pair of points, specialized or not. Let N_k be the number of extremals joining P to Q on which there are k points conjugate to P. If there are an infinity of such extremals we replace N_k by ∞. The sequence $N_0, N_1, N_2, \cdots$ will be called the *conjugate number sequence* for P and Q. If P and Q are non-specialized, we can readily prove the following.

If there are N_k extremals of type $k>1$, there are at least N_k extremals of the two adjacent types.

If there are N_1 extremals of type 1, there are at least N_1+1 extremals of the two adjacent types.

If there are N_0 extremals of type 0, there are at least N_0-1 extremals of the adjacent type 1.

These statements still hold if any of the N's are ∞, understanding $\infty-1$ or $\infty+1$ as standing merely for an infinity. Many other facts can be proved about the conjugate number sequence. One of the most curious as well as important facts is the following.

If P and Q are non-specialized and if any one of the symbols N_k is zero, then either at least two of the preceding symbols are ∞, or else the set of preceding symbols satisfies (4).

Suppose P and Q are any two points on S, specialized or non-specialized. The extremals joining P to Q may very well be infinite in number. In any case the extremals which joint P to Q, and for which J is less than a constant J_0, will either be finite in number, or else can be grouped into a finite set of r-parameter, connected, and in general analytic families, for which $r<m-1$, and on which J is a constant. Each isolated extremal joining P to Q to which a type number has not already been assigned, and each family of extremals joining P to Q, can be counted as if it were a finite set of extremals of definite types, and so counted all the preceding relations between the integers M_k still hold.

A strong aid in proving the existence of extremals joining any two points P and Q, whether specialized or not, is the following principle.

If in the respective neighborhoods of two points P_0 and Q_0, no matter how small the neighborhoods, there are two non-specialized points P and Q which are joined by at least one extremal of type k on which $J<J_0$, then there will be at least one extremal joining P_0 to Q_0 on which there are at least k points and at most $k+m-1$ points conjugate to P_0.

With the aid of this principle we have been able, in the examples which are to follow, to give strong existence

theorems which hold not only for non-specialized points, but also for specialized points as well.

7. *An Example.* Suppose we have an analytic surface S, with boundary b, and homeomorphic to a circular disc. On S consider the integral of arc length. Suppose S is extremal convex. Suppose S possesses a knob-shaped protuberance. More exactly suppose there is a portion of S homeomorphic to a disc, and bounded by a closed geodesic g that is shorter than nearby closed curves.

Any two points P and Q near g, or between g and b, will be connected by an infinite number of geodesics of type zero. By the aid of the preceding theorems one can then prove the following.

There will be an infinite number of geodesics joining P and Q upon each of which there is at least one point conjugate to P.

The later geodesics will include geodesics of arbitrarily great length which pass out of the region between g and b, and out of the neighborhood of g, thereby necessarily crossing the knob in some way.

This example suggests the more general question as to what effect the existence of a closed extremal in S will have upon the nature of extremals joining two points in S. If one admits closed extremals with conjugate points on them in varying numbers, and considers the problem in the different dimensions, one finds both curious and varied effects. The results are intimately connected with the so-called Poincaré rotation number.

8. *Extremals on General Regions.* The study of extremals in regions of general connectivities has been undertaken as yet only in a preliminary way. However, one case, namely the case of a region Σ homeomorphic with a portion of m-space between two concentric $(m-1)$-spheres has been studied, and has yielded much light. As in the case of the earlier region S so here we suppose Σ extremal-convex. A first result is the following.

If P and Q are any two non-specialized points on Σ, there exists

at least one extremal joining P to Q of every type k congruent to zero mod $m-2$.

It is a curious fact that, while this theorem affirms the existence of an infinite number of extremals when $m>2$, it affirms the existence of but one extremal when $m=2$. However, the gap is filled by the affirmation that when $m=2$ there is always an infinite number of extremals joining P to Q of the one type zero.

Concerning any pair of points P and Q, whether specialized or not, we can prove the following.

In the conjugate number sequence for any two points P and Q whatsoever there can nowhere be more than $2m-4$ consecutive zeros $(m>2)$.

In proving this theorem we have proved the a priori existence of an infinite set of extremals. This theorem shows that both the region and the dimension notably affect the results. It affords a contrast between the search for extremals that give minima and the search for extremals in general. Of the infinite set of extremals hereby affirmed to exist at most the first can give a minimum.

A simple example of an integral and extremal convex region such as Σ is the following. As the integral we take

$$\int \frac{1+(r-1)^2}{r} ds,$$

where ds is the differential of the arc length in m-space, and r the distance to the origin. The region between the two $(m-1)$-spheres, $r=a$ and $r=b$, is extremal-convex if we take $0<a<1<b$. It is easy to verify our theorems for any two points on the $(m-1)$-sphere, $r=1$.

9. *Extremals on Closed Manifolds.* Here the simplest and most important type of manifold has been investigated, namely an analytic manifold Σ homeomorphic to an m-sphere. Among others, the following two existence theorems have been proved.

If P and Q are any two non-specialized points on Σ, there

exists at least one extremal joining P to Q of every type k congruent to zero, mod $m-1$.

In the conjugate number sequence for any two points P and Q whatsoever there can nowhere be more than $2m-3$ consecutive zeros ($m>1$).

In proving the first of these theorems there is introduced a new method which might be called *the method of continuation of extremals by homeomorphisms*. First the existence of non-bounding r-families is gone into on the m-sphere. The greatest aids here are apparently the fundamental theorems on the characterization of conjugate points by deformations, and the knowledge of all the facts about geodesics on m-spheres. Once the existence of the non-bounding r-families is proved for the m-sphere it is of course proved for Σ. The existence of the required extremals then follows on Σ.

If Σ were any analytic closed manifold whatsoever, the method would require the selection of some manifold Σ_1 homeomorphic to Σ, upon which the study of extremals was as simple as possible. From facts about Σ_1 essentially belonging to the calculus of variations one would pass to facts about r-families, that is, facts essentially of analysis situs. From these last facts one would finally pass to the theorems about extremals on Σ.

This method may be contrasted with the much-used method of analytic continuation of extremals with respect to a parameter. Here one may be kept from the ultimate facts by the impossibility or difficulty of continuation beyond a certain point, or, in case one deals with an infinite set of extremals, with difficult questions of uniformity. All such difficulties as these are foreign to the above methods of continuation by homeomorphisms.

10. *The Case $m=2$.** The case $m=2$ is essentially general in so far as deformations in the small are concerned so that the deformation theory given in the paper on the case $m=2$ needs practically no change to hold for the general case. A

* Morse II, loc. cit.

peculiarity of the case $m=2$ is that in an extremal convex region there can never be more than a finite number of extremals joining two points among the extremals with lengths less than a finite constant.

In the case $m=2$ it appeared desirable in a first paper, in the part that dealt with the theory in the large, to assume that the region was covered by a proper field of extremals, and to assume the extremals reversible. This had as a consequence that there were at most a finite set of extremals in S joining two points, but what was more important it reduced the essential passage from the small to the large to a matter of about three pages.* The problem of the extremals joining two fixed points when $m=2$ under this special field hypothesis has been taken up at the author's suggestion by D. E. Richmond† with a view to finding how complete were the relations already found. By methods peculiar to the special case treated he has established the completeness of the relations up to a certain point, and has found certain additional restrictive relations of a very elegant nature.

It is of interest to state in this connection that the author has proved his own relations between the numbers of critical points of a function and the connectivities of a general region are complete in general, and turning to the calculus of variations has proved his relations complete for the theory of extremals from a point to a manifold, and this under general hypotheses and for a general m. The method of proving a set of relations complete has been to take any set of integers M_k which satisfy the author's relations, and to set up an example to which these integers M_k belong.

11. *Closed Extremals.*‡ Closed extremals have been treated by the author up to the present time only for the case $m=2$.

* Morse II, loc, cit., pp. 227–231.

† Richmond, *Number relations between types of extremals joining a pair of points*, American Journal of Mathematics, vol. 50 (1928), pp. 370–388; and *A new proof of certain relations of Morse in the calculus of variations in the large*, which will appear in this Bulletin, Mch.-Apr., 1929.

‡ Morse II, loc. cit. Also Hadamard, loc. cit.

Here we start by cutting across the given closed extremal g with n perpendiculars whose intersections with g follow each other cyclically, and are so numerous and so near together that the resulting n segments of g have no pairs of conjugate points on them. We let $(u_1, \cdots, u_n) = (u)$ be distances from g measured respectively along these n perpendiculars, and we set up the closed broken extremal E joining successively the points on these perpendiculars at the respective distances (u) from g. Let $J(u)$ be the value of J along E.

As previously $J(u)$ will have a critical point when $(u) = (0)$. But here the quadratic form H of the terms of second order of $J(u)$ in an expansion about $(u) = (0)$ has marked differences from the form associated with an extremal joining two fixed points. For the purposes of the following theorems suppose g mapped conformally on a segment g_1 of the x axis of length ω, and this by a transformation which has the period ω in x.

The rank of H is n, $n-1$, or $n-2$, according as the Jacobi differential equation corresponding to g_1 has respectively no solutions of period ω not identically zero, has a one-parameter family of solutions of period ω not identically zero, or has no solutions other than those of period ω.

When one comes to the type number of H one finds another difference from the earlier results in that this type number does not depend solely on the distribution of conjugate points. In case the rank of H is n the type number depends upon a further classification of closed extremals as follows.

If $x = x_0$ is not conjugate to $x_0 + \omega$, then in the xy plane any point (x_0, a), $a > 0$, neighboring $(x_0, 0)$ can be joined to $(x_0 + \omega, a)$ by an extremal segment g'. The segment of g from x_0 to $x_0 + \omega$ will be said to be relatively *convex* or *concave* according as the slope of g' at $x_0 + \omega$ is greater or less than the slope of g' at x_0. In case the rank of H is n these slopes will not be equal, and in this case the type number of H is now determined as follows.

Let m be the number of points conjugate to x_0 preceding

$x_0+\omega$. *When* x_0 *is not conjugate to* $x_0+\omega$, *the type number of* H *is* m *if* g *is convex, and* $m+1$ *if* g *is concave. When* x_0 *is conjugate to* $x_0+\omega$, *the type number is* m.

In case the rank of H is $n-1$ or $n-2$, and the given closed extremal is isolated, it can be shown that for all subsequent purposes g may be counted as equivalent to a finite set of closed extremals for which the rank of H is n.

Closed extremals were considered in the large in a ring-shaped region bounded by two simple closed curves. For such a region the theory in the large is not essentially different from that previously given and will not be elaborated here.

There remains the important study of closed extremals in the higher spaces,* and on closed, as well as open, manifolds. Here even the most casual inspection discloses a wide and varied store.

12. *Extremals through a Point* O *cut Transversally*† *by a Closed Manifold* Σ. We here make the same hypotheses with respect to the integral as in §1. In the domain of definition of F we suppose we have a closed, analytic, regular,‡ $(m-1)$-dimensional manifold Σ. We suppose also that Σ is so placed in the field of extremals through O that there are no conjugate points to O on the extremals from O to Σ until after their intersections with Σ.

Let g be an extremal through O cut transversally by Σ at P. Let (v) be a set of $m-1$ parameters in an admissible‡

* For a first remarkable difference see Carathéodory, *Über geschlossene Extremalen und periodische Variationsprobleme in der Ebene und im Raume*, Annali di Matematica, (4), vol. 2 (1925), pp. 297–320. Here among other things it is shown that the Poincaré Law does not hold in the higher spaces.

† An interesting paper in this connection is that by White, M. B., *The dependence of focal points upon curvature for problems of the calculus of variations in space*, Transactions of this Society, vol. 13 (1912), pp. 175–198. This paper is primarily concerned with the differential and geometric nature of focal points in the small, in three-space, in particular in connection with the search for minima. No use is made of these results by the author.

‡ An analytic manifold will be called regular if on it the coordinates (x) of points neighboring any given point admit a representation in terms of $m-1$ parameters in which not all of the jacobians of $m-1$ of the co-

representation of Σ near P. On the extremals near g cut transversally by Σ at points (v), the coordinates (x) will be analytic functions of the parameters (v), and of the arc length s measured on these extremals from Σ. Let $\Delta(s)$ be the jacobian of these m coordinates (x) with respect to (v) and s, evaluated for the set $(v)=(v)_0$ that corresponds to P.

The points on g different from P at which $\Delta(s)=0$ *are called the focal points of P on g, and the order of vanishing of* $\Delta(s)$ *at these points the order of the focal points.*

The value of J taken from O to the points (v) will be indicated by $J(v)$. It appears first that $J(v)$ has a critical point when $(v)=(v)_0$. Let H be the quadratic form of the terms of second order in the expansion of $J(v)$ about $(v)=(v)_0$. Set $m-1=n$. We have the following theorem.

If O is a focal point of P of the rth order, the rank of the matrix of the form H is $n-r$. *If O is not a focal point of P, the type number of H equals the number of focal points of P between P and O, always counting focal points according to their orders.*

From the author's relations* between the critical points of a function we then obtain the following theorem in the large.

Suppose O is not a focal point of Σ, *nor on* Σ. *Of the extremals from O to* Σ *which are cut transversally by* Σ *at their ends P on* Σ, *let* M_i *be the number upon which there are i focal points of P between O and P. Then between the numbers* M_i *and the connectivities* R_i *of* Σ *the following relations hold:*

$$
\begin{aligned}
&M_0 \geqq 1 + (R_0 - 1),\\
&M_0 - M_1 \leqq 1 + (R_0 - 1) - (R_1 - 1),\\
&M_0 - M_1 + M_2 \geqq 1 + (R_0 - 1) - (R_1 - 1) + (R_2 - 1),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&M_0 - M_1 + \cdots + (-1)^n M_n\\
&\qquad = 1 + (R_0 - 1) - + \cdots + (-1)^n (R_n - 1).
\end{aligned}
\tag{5}
$$

ordinates with respect to the $m-1$ parameters are zero at once. Such a representation is called admissible.

* See Morse I, and Morse, *The analysis and analysis situs of regular n-spreads in* $(n+r)$*-space*, Proceedings of the National Academy, vol. 13 (1927), pp. 813–817.

If O is on Σ the above relations still hold if we count O as a nul-extremal of type zero.

The following corollary furnishes a strong existence theorem.

If O is not a focal point of Σ, the number of extremals from O cut transversally by Σ on which there are k focal points of Σ, is at least $R_k - 1$. The total number of extremals is at least

$$1 + (R_0 - 1) + \cdots + (R_n - 1). \tag{6}$$

The theorem and corollary apply at once to the problem of the number of normals from O to Σ, as follows.

If O is not a center of principal normal curvature of Σ, and if M_i is the number of normals from O to Σ on which there are i centers of principal normal curvature of Σ between O and the feet of the normals on Σ, then the relations (5) *hold.*

For example, if Σ is an admissible surface homeomorphic with a torus it follows from (6) there will be at least four normals to Σ from any point not a center of principal normal curvature. If Σ is a surface in 4-space homeomorphic with the complex obtained by identifying the opposite faces of an ordinary cube, one sees from (6) that there will be at least eight normals from a point not a center of principal normal curvature. It is possible to multiply these examples in any space. Moreover, even when O is a center of principal normal curvature, it is still possible to get strong existence theorems by a limiting process.

The question of the number of $(m-1)$-planes tangent to Σ through a given $(m-2)$-plane is closely related to the preceding theory on normals, and it is easy to proceed to existence theorems on the tangency of this $(m-1)$-plane. In fact there is a whole theory of the relations of straight lines, hyperplanes and hyperspheres to closed manifolds that can be treated after the spirit of this section.

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THE FORMS $ax^2+by^2+cz^2$ WHICH REPRESENT ALL INTEGERS

BY L. E. DICKSON

THEOREM. *$f=ax^2+by^2+cz^2$ represents all integers, positive, negative, or zero, if and only if*: I. *a, b, c are not all of like sign and no one is zero*; II. *no two of a, b, c have a common odd prime factor*; III. *either a, b, c are all odd, or two are odd and one is double an odd*; IV. *$-bc$, $-ac$, $-ab$ are quadratic residues of a, b, c, respectively.*

We shall first prove that I-IV are necessary conditions. Let therefore f represent all integers. It is well known that I follows readily.

If a and b are divisible by the odd prime p, f represents only $1+\frac{1}{2}(p-1)$ incongruent residues cz^2 modulo p. This proves II.

Next, *no one of a, b, c is divisible by* 8. Let $a\equiv 0 \pmod 8$. Every square is $\equiv 0$, 1, or 4 (mod 8). First, let $b=2B$. Since f represents odd integers, c is odd. Since $by^2\equiv 0$ or $2B$ (mod 8) and $cz^2\equiv 0$, c, or $4c$, f has at most six residues modulo 8. If m is a missing residue, f represents no $m+pn$. Second let b and c be odd. Then $4b\equiv 4c\equiv 4 \pmod 8$. Thus the residues of f modulo 8 are obtained by adding each of 0, 4, b to each of 0, 4, c; we get only seven residues 0, 4, b, c, $4+b$, $4+c$, $b+c$.

No one of a, b, c is divisible by 4. Let a be divisible by 4. Since a is not divisible by 8, $a\equiv 4 \pmod 8$. Evidently $f\equiv 0$, b, c, or $b+c \pmod 4$. No two of these are congruent modulo 4. If $b\equiv \pm 1 \pmod 4$, they are 0, ± 1, c, $c\pm 1$. Evidently c is not congruent to 0, ± 1, or ∓ 1. Hence $c\equiv 2 \pmod 4$. Since $b\not\equiv 0$, this proves that one of b and c is $\equiv 2 \pmod 4$. By symmetry, we may take $b\equiv 2 \pmod 4$. If $b\equiv 6 \pmod 8$, we apply our discussion to $-f$ instead of

f. Hence take $b \equiv 2 \pmod 8$. Thus $a \equiv 8n+4$, $b = 8m+2$, and c is odd. Since $x^2 \equiv 0$ or $1 \pmod 4$, $ax^2 \equiv 0$ or $8n+4 \pmod{16}$. Since $y^2 \equiv 0$, 1 or 4 (mod 8), $by^2 \equiv 0$, $8m+2$, or 8 (mod 16). We employ only even residues of f modulo 16. Then z is even, and $cz^2 \equiv 0$ or $4c \pmod{16}$. But $c \equiv \pm 1 \pmod 4$, $4c \equiv \pm 4 \pmod{16}$. Evidently ax^2+by^2 has at most 2×3 residues modulo 16. The missing two even residues are seen to be s and $s+4$, where $s = 10$ if n and m are both even, $s = 2$ if n is even and m odd, $s = 6$ if n is odd and m even, $s = 14$ if n and m are both odd. According as $4c \equiv 4$ or -4, f is not congruent to $s+4$ or s modulo 16.

No two of a, b, c are even. Let us set $a = 2A$, $b = 2B$. By the preceding result, A and B are odd. Also, c is odd. If $A = 4n-1$, we use $-f$ in place of f. Hence let $A = 4n+1$. Then $f \equiv 2x^2+2By^2+cz^2 \pmod 8$. Consider only odd residues of f. Then $cz^2 \equiv c \pmod 8$. The residues of $2x^2+2By^2$ are 0, 2, $2B$, $2B+2$. When these are increased by c, the sums must give the four odd residues modulo 8. Hence no two are congruent. Thus no two of 0, 1, B, $B+1$ are congruent modulo 4. Since B is odd and $\not\equiv 1 \pmod 4$, $B \equiv 3$, $B+1 \equiv 0$, a contradiction.

This completes the proof of property III. Properties II and III imply the following property.

V. *a, b, c are relatively prime in pairs.*

Thus $cd \equiv -b \pmod a$ has a solution d which is prime to a. Suppose that d were a quadratic non-residue of an odd prime factor p of a. Write $a = pA$. Consider values of x, y, z for which f is divisible by p. Then $z^2 \equiv dy^2 \pmod p$, whence y and z are divisible by p. Hence $f = pF$, where $F \equiv Ax^2 \pmod p$. Evidently Ax^2 takes at most $1+\frac{1}{2}(p-1)$ values incongruent modulo p. Hence there is an integer N that is not congruent to one of them. Thus f fails to represent $p(N+pw)$ for any value of w. This contradiction proves that $v^2 \equiv d \pmod p$ is solvable. The usual induction shows that it is solvable modulo p^n. Also, $d^2 \equiv d \pmod 2$. By means of the Chinese remainder theorem, we see that $w^2 \equiv d \pmod a$, is solvable whether a is odd or double an odd integer,

Then w is prime to a since d is. Since $(cw)^2 \equiv -bc \pmod{a}$ this proves IV.

We shall now prove that I–IV imply that f represents every integer g. It is known* that I, IV and V imply that $f=0$ has solutions x', y', z' which are relatively prime in pairs. Then the greatest common divisor of the three numbers $\alpha=ax'$, $\beta=by'$, $\gamma=cz'$ is 1. For, if they are all divisible by a prime p, one of x', y', z' is divisible by p (otherwise a, b, c would all be divisble by p). By symmetry, let x' be divisble by p. Then neither y' nor z' is divisible by p. Hence b and c would be divisble by p, contrary to V. Hence† if D is any given integer, ξ, η, ζ may be chosen so that

$$\alpha\xi + \beta\eta + \gamma\zeta = D. \tag{1}$$

We seek a solution of $f=g$ of the form

$$x = nx' + \xi, \qquad y = ny' + \eta, \qquad z = nz' + \zeta. \tag{2}$$

Since $ax'^2 + \cdots = 0$, $f=g$ is satisfied if

$$2Dn = g - e, \tag{3}$$

where

$$e = a\xi^2 + b\eta^2 + c\zeta^2. \tag{4}$$

If ξ', η', ζ' is a second set of solutions of (1), write

$$X = \xi - \xi', \qquad Y = \eta - \eta', \qquad Z = \zeta - \zeta'.$$

Then

$$\alpha X + \beta Y + \gamma Z = 0. \tag{5}$$

We seek the general solution of (5). Let δ be the greatest common divisor of $\alpha=\delta A$ and $\beta=\delta B$. Then δ is prime to γ, whence $Z=-\delta w$. Hence

$$AX + BY = \gamma w, \quad A, B \text{ relatively prime.} \tag{6}$$

There exist integers r, s satisfying

$$Ar + Bs = 1. \tag{7}$$

* Dirichlet-Dedekind, *Zahlentheorie*, ed. 4, §157, p. 432 (Supplement X).

† Since the g. c. d. 1 of α, β, γ, is a linear function of them. Multiply the relation by D.

Multiply the second member of (6) by (7). Thus

$$A(X - \gamma rw) + B(Y - \gamma sw) = 0.$$

The quantities in parenthesis are equal to Bm and $-Am$, where m is an integer. The resulting values of X and Y, together with $Z = -\delta w$, give the general solution of (6). Hence if ξ', η', ζ' is one solution of (1), the general solution is

$$(8)\quad \xi = \xi' + \gamma rw + Bm,\ \eta = \eta' + \gamma sw - Am,\ \zeta = \zeta' - \delta w,$$

where w and m are arbitrary, while r, s satisfy (7).

First, let a, b, c be all odd. Then $x'+y'+z' \equiv 0 \pmod 2$. But x', y', z' are not all even. Hence just one of them is even. By symmetry, we may take x' even, y' and z' odd. Then α is even, β and γ are odd, δ is odd, A is even, B is odd. Write

$$(9)\qquad e' = a\xi'^2 + b\eta'^2 + c\zeta'^2.$$

When working modulo 2, we may discard the exponents 2 in (4) and (9). Take $w=0$, $m=1$. Then, by (8),

$$\xi \equiv \xi' + 1,\quad \eta \equiv \eta',\quad \zeta \equiv \zeta', \qquad \pmod 2,$$
$$e \equiv \xi + \eta + \zeta \equiv e' + 1.$$

For $w=m=0$, evidently $e=e'$. Hence we may take $e \equiv g \pmod 2$. We may take $D=1$. Then (3) yields an integral value of n. Hence $f=g$ is solvable.

Second, let a and b be odd, but c the double of an odd integer, whence $c \equiv 2 \pmod 4$. Since $f \equiv x+y \pmod 2$, $x'+y'$ is even. But x' and y' are relatively prime. Hence x' and y' are odd. Thus α and β are odd, γ is even, δ is odd, A and B are odd. By (1) and (4),

$$(10)\qquad D \equiv \xi + \eta \equiv e, \qquad \pmod 2.$$

If g is odd, we take $D=1$. (See footnote on p. 59.) By (10), $g-e$ is even and (3) yields an integer n.

But if g is even, we take* $D=2$. By (10), $\xi+\eta$ and e are even. In (8), take $w=0$, $m=1$. Then

$$\xi = \xi' + B, \quad \eta = \eta' - A, \quad \zeta = \zeta'. \tag{11}$$

In case ξ' and η' are odd, we replace ξ', η', ζ' by the preceding solution having ξ and η even. Hence we may choose the initial solution ξ', η', ζ' so that ξ' and η' are even. Then (11) gives

$$e \equiv e' + aB^2 + bA^2 \equiv e' + a + b \pmod 4.$$

Hence if $a+b\equiv 2 \pmod 4$, we may choose e so that $e\equiv g \pmod 4$. Then (3) yields an integral value of n. But if $a+b\equiv 0 \pmod 4$, we take $w=1$, $m=0$ in (8) and see that ξ and η are even since γ is even. Then since δ is odd and $c\equiv 2 \pmod 4$,

$$e \equiv 2\zeta^2 = 2(\zeta' - \delta)^2 \equiv 2\zeta'^2 + 2 = e' + 2 \pmod 4.$$

As before, $f=g$ is solvable.

COROLLARY. *If $ax^2+by^2+cz^2$ is not a Null form, it does not represent all integers.*

Examples with $a=1$, $c=-C$, $C>0$.

(i) $b=1$. Then C must be odd or double an odd integer and -1 must be a quadratic residue of C. Then every odd prime factor of C is $\equiv 1 \pmod 4$. A necessary and sufficient condition on C is that it be a sum of two relatively prime squares.

(ii) $b=2$. Then C must be odd and -2 a quadratic residue of C. Then its prime factors are $\equiv 1$ or $3 \pmod 8$. A necessary and sufficient condition on C is that it be of the form r^2+2s^2, r odd, r and s relatively prime.

(iii) $b=3$. Then C must be odd or double an odd integer, C prime to 3, while C and -3 must be quadratic residues of each other. Hence every prime factor of C is $\equiv 1 \pmod 6$. Necessary and sufficient conditions on C are that C be odd and of the form r^2+3s^2, where r and s are relatively prime.

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* Elimination of ξ, η, ζ between (1) and (2) gives $\alpha x+\beta y+\gamma z=D$. Here $D\equiv x+y\equiv f \pmod 2$. Hence $D\equiv g \pmod 2$.

THE THEORY OF MULTIPLE IMPLICATION AND ITS APPLICATION TO THE GENERALIZED PROBLEM OF EPIMENIDES

BY H. B. SMITH

Let us allow $p, q, r, \cdots$ to stand for propositional functions, that is, for variables whose truth-values are in general dependent on the meaning of the terms that enter into them, and let the expression $(p\ q\ r\ \ldots)$ have the following verbal interpretation:

> "p, q, r, $\cdots$ are simultaneously true for some meanings of the terms that enter into them."

We shall speak of this expression, $(p\ q\ r \cdots)$, or its negative, as the *existential* of the *n elements*, p, q, r, $\cdots$, and we shall say that it is of the first *order* and *n*th *degree*. The function containing existentials of the first order or their negatives as elements, will be of the second order and so on. For the particular case of one variable we should write:

(p) $=p$ is true for some meanings of the terms,
$(p)'$ $=p$ is true for no meanings of the terms,
$(p')'=p$ is true for all meanings of the terms,
(p') $=p$ is true for not all meanings of the terms,

the prime being used as the symbol of denial or negation.

The effect of the symbol ( ), regarded as an operator, will be in general to weaken the expression on which it operates, unless this expression be zero (o) or one (i). Thus,

$$p \text{ implies } (p),$$
$$(p')' \text{ implies } p,$$

but not conversely. This simple provision is important, for it will lead at once to the result that certain generally recognized principles are untrue in a logic of complete generality. Thus it will turn out that the equality,

$$p \text{ implies } q = p' \text{ or } q$$

does not hold true without limitation, namely, the limitation that the variables stand only for "propositions," for zero or one values, and not for propositional functions. This limitation is one that reduces the generality of our science and is directly related to an application of our theory which we shall make before we end. If we write,

$$p < q = p \quad \text{implies} \quad q,$$

we shall be prepared to set down the following definition,

$$(pq)' = p < q',$$

and from this follows the law of expansion for the general case,

$$(pqr\cdots)' = pq < r' + \cdots = p < q' + r' + \cdots,$$

that is, the elements are conceived to be conjoined as in a product, so that the consequent becomes the sum of the contradictories of the separate parts.

We might, if we liked, derive the fundamental laws of the calculus of propositions by assuming certain very simple properties of the existential, together with the fundamental equations,

$$(pq) = pq + p'q'(p)(q) + (p)(q')' + (q)(p')',$$
$$(pq)' = (p)' + (q)' + p'q(q') + q'p(p'),$$
$$(\overline{pq}) = (p') + (q'),$$
$$(\overline{pq})' = (p')'(q'),$$

the bar over pq standing for the negation of the product, or $p'+q'$. Or, we can derive the simple properties of the existential from the laws of the calculus of propositions. Obviously we may raise the degree of the function without changing its value, by adding to it as many one-elements as we choose. This fact is important, because it will enable us to provide that the symbol of implication can always be made to appear in our development. Thus $(o'=i)$,

$$(p)' = (ip)' = i < p',$$
$$(p)' = (pi)' = p < o.$$

And it is clear, too, that the elements may be permuted in any order, for

$$(p\,q)' = p < q' = q < p' = (qp)' ,$$

and that no change will result if we add to the existential as many elements already contained in it as we like. Similarly, the appearance of a pair of contradictory elements is a condition for the vanishing of the existential, if it be not negated; otherwise a condition for its becoming unity.

We approach now the consideration of a matter of the first importance, involving as it does, the creation of an infinite series of meanings of the "true" and the "false," together with an indication of how each one of these distinctions can be uniquely defined. We assume as true, for *all* meanings of the terms and for all meanings of p,

$$p < (p) ;$$

and we assert, moreover, that the converse,

$$(p) < p,$$

is true for *not all* meanings of the terms.

Now by allowing p to take on the values p', (p), $(p)'$, etc. and by the contradiction and interchange of antecedents and consequents, we should obtain a series of results that express the relation of the existentials of first and second order and beyond to one another and to the free variable within the propositional universe. These results are represented diagrammatically in the following table, the longer space being understood to "contain" the smaller spaces which it extends beyond.

	$((p))\,\vert\,((p))'$
	$(p)\,\vert\,(p)'$
	$((p)')'\,\vert\,((p)')$
	$p\,\vert\,p'$
	$((p')')\,\vert\,((p')')'$
	$(p')'\,\vert\,(p')$
$((p'))'\,\vert\,((p'))$	

We are now in a position to define each one of these meanings of "true" and "false." If we write

$$f(p) = p \text{ is false},$$
$$t(p) = p \text{ is true},$$

it will be clear that $t = f'$, no matter what value f may have, and that, consequently, t will be unambiguous as soon as f has been defined.

Suppose, for example, we wish $f(p)$ to mean p'. It will be enough to require f to satisfy the equation

$$f(f) = f',$$

for while there is an infinite series of meanings of "false" that satisfy the implication

$$f(f) < f',$$

for example,

$$(p)', \quad ((p))', \quad ((p)'), \cdots,$$

and which do not satisfy the converse, and while there is an infinite series of distinct meanings of "false" that satisfy the implication

$$f' < f(f),$$

for example,

$$(p'), \quad ((p')), \quad ((p')')', \cdots,$$

and which do not satisfy the converse, there is only one meaning of "false," given by

$$f(p) = p',$$

which satisfies both implications at once. Accordingly $f = p'$ is unique and $t = p$.

Any meaning of "false" from among the members of the infinite series of meanings that this term can have, can be uniquely determined by requiring f to satisfy a given equation whose form can be found at once by inspection. Thus, if

$$f(p) = p < o, \qquad t(p) = (p < o)',$$

is the meaning of "true" and "false" desired, we assume f and t to satisfy respectively the equations

$$f(f) = f < o, \qquad t(t) = (t < o)',$$

for these solutions will prove to be unique. If

$$f(p) = (p' < o)', \qquad t(p) = p' < o,$$

are the meanings we choose, we require f and t to satisfy, respectively,

$$f(f) = (f' < o)', \qquad t(t) = t' < o,$$

and so on.

The unique solution of the equation $f(p) = o$ is $p = i$ for all meanings of f, as will appear as the result of a complete induction,

$$f(p) = p', \quad (p)', \quad (p'), \quad \text{etc.}$$

Hence p is true absolutely, that is, for all meanings of t,

$$t(p) = p, \quad (p), \quad (p')', \quad \text{etc.},$$

if f satisfy the implication

$$f < f(f).$$

The unique solution of the equation $f(p) = i$ is $p = o$ for all meanings of f. Hence p is false absolutely, if f satisfy the implication*

$$f(f) < f.$$

An interesting application of our theory will result if we should now go on to specialize the meaning of p and under-

* For further results that concern the theory of multiple implication see the writer's *Symbolic Logic*, (N. Y., Crofts, 1927). For a brief history of the problem of Epimenides as of other insolubilia see Professor E. R. Guthrie's admirable little monograph entitled *The Paradoxes of Mr. Russell*. This work, little known because privately printed, has not attracted the attention it deserves.

stand p to mean "this proposition." We should then have

$$f(p) = \text{this proposition is false;}$$

or, if we like,

$$1.f(1).$$

It is this situation that is supposed to give rise to a paradox, for it is supposed to lead to the identity of f with its own contradictory. Thus if 1. is o, $f(1)$ is i, if 1. is i, $f(1)$ is o and 1. and $f(1)$ are taken to be the same thing. The difficulty here depends on the assumption that every meaningful assertion must be specifically either zero or one, that is, must be a proposition, within Russell's understanding of that term. The solution to the difficulty will appear as soon as this limitation has been removed. Let us begin by defining the parts that enter into our assertion. The meaning of "false" has been defined already. It remains to define "this proposition."

We have here, perhaps, a range of choice in definition but let us require p to satisfy the equation

$$\{f < f(f)\} + \{f(f) < f\} = i.$$

If this condition be understood to stand for the meaning of p, it will be seen to have two unique solutions, $p = o$ and $p = i$. This, then, is the solution of the difficulty involved in the assertion

"this proposition is false,"

or, as we might indicate it,

$$1.f(1).$$

That is, if 1. is two-valued, being ambiguously zero or one, then $f(1)$ is similarly two-valued and no contradiction can arise.*

* We must not, of course, on this account assume the "identity" of p and $f(p)$. To do so would be to fall into the familiar fallacy of confusing the use of "all" in the collective and distributive sense. For all meanings (collectively) p and $f(p)$ are the same, but they are not equal for all meanings, where "all" has the meaning of "every."

The same solution can be shown to hold in the more general case. Consider a "cycle" of three assertions of the form

$$1.f(2) \qquad 2.f(3) \qquad 3.f(1)$$

We may now attach to these too a determinate and consistent meaning. From the symmetry of the situation we may take it for granted that each one has the same general character and that this general character is the one that we assumed above. That is, these three assertions are to be conceived as two-valued in the sense defined. Since 1, 2 and 3, then, are two-valued, so will the following be two-valued in the same sense, viz.

$$f(1) \qquad f(2) \qquad f(3)$$

and for two reasons. In the first place, because if p is two-valued so is $f(p)$, (for the condition which p is required to satisfy, may be regarded equally as the condition which $f(p)$ is required to satisfy, for all meanings of f, and there are two and only two such values, viz., $f(p) = o$ and $f(p) = i$), and in the second place, because they are repetitions of 1, 2 and 3. Consequently, no contradiction can arise from the simultaneous assertion of 1, 2 and 3. The same remarks will evidently hold of a "cycle" of any number of assertions of the same form and the generality of the method is evident.

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CONCERNING COVARIANTS OF THE RATIONAL PLANE QUARTIC CURVE WITH COMPOUND SINGULARITIES*

BY J. H. NEELLEY

1. *Introduction.* After investigating the invariants of the rational plane quartic curve R_2^4 with compound singularities† it is of interest to see what effect such singularities have on covariants of the curve. This paper observes the effect of the tacnode and the ramphoid cusp upon associated curves and sets of covariant parameters.

2. *Covariant Forms of the Curve with a Tacnode.* To examine these forms we let $t = \pm a$ be the tacnodal parameters and t_1 and t_2 the contacts of the distinct double line of the curve with an acnode. This may be done if 0 and ∞ are the contacts of the two tangents to the curve from the tacnode.‡ These tangents and the line through their contacts give the very neat representation

$$(1) \qquad \begin{cases} x_0 = t^4 - a^2t^2, \\ x_1 = t^2 - a^2, \\ x_2 = t^3 + st^2 + \sigma_2 t, \end{cases}$$

where $\sigma_1 = t_1 + t_2$, $\sigma_2 = t_1 t_2$, and $s = [a^2\sigma_1^2 + (a^2+\sigma_2)^2]/(2a^2\sigma_1)$.

The pencil of conics§

$$(2) \qquad g_2 - \lambda K = 0,$$

where g_2 is the envelope of lines which cut the curve in self-apolar sets of points and K is the locus of the vertices

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* Presented to the Society, April 6, 1928.

† Neelley, *Compound singularities of the rational plane quartic curve*, American Journal of Mathematics, vol. 49 (1927), pp. 389–400.

‡ Neelley, loc. cit., p. 394.

§ J. E. Rowe, *Covariants and invariants of the rational plane quartic*, Transactions of this Society, vol. 12 (1911), pp. 298–299.

of the flex-triangles of the first osculants of R_2^4, is not affected by the singularity. It has one member which passes through the tacnode and that is the conic on the contacts of the double lines. This conic has the tacnodal line

$$x_0 - a^2 x_1 = 0 \tag{3}$$

as its tangent at the tacnode which is to be expected as several double line contacts coincide at a tacnode.* The tacnode does not affect the covariant line on the points q_i† nor the line which is the locus of a point such that tangents from it to R_2^4 are apolar to the flexes.†

Next we examine the combinants of two line sections (ξx) and (ηx) of the curve (1).‡ These combinants also seem to be independent of the tacnode except C which has $x_0 - a^2 x_1$ as a squared factor. This was expected as C gives the double lines of R_2^4. The fundamental involution§ determined by two binary quartics does not give any special information as none of the forms which make up the complete system‖ of these two quartics vanish identically.

3. *Covariant Forms of the Curve with a Ramphoid Cusp.* This singularity is a special form of the tacnode but it affects the covariants very strikingly. Suppose $t=0$ at the cusp and $t=\infty$ at a distinct real flex. There is always such a flex as three flexes coincide at the cusp.¶ Then the cuspidal line, the flex line and the join of cusp and flex give

$$\left\{ \begin{aligned} x_0 &= at^4, \\ x_1 &= 4bt^3 + 6ct^2, \\ x_2 &= 4dt + e. \end{aligned} \right. \tag{4}$$

Two lines (ξx) and (ηx) cut out the line sections

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* Neelley, loc. cit., p. 395.
† Rowe, Transactions of this Society, vol. 13, p. 390.
‡ Salmon, *Modern Higher Algebra*, 4th ed., *p.* 296.
§ Meyer, *Apolarität und rationale Kurven*, p. 9.
‖ Salmon, loc. cit., p. 224.
¶ Neelley, loc. cit., p. 390.

$$(5)\qquad \begin{cases} U \equiv a\xi_0 t^4 + 4b\xi_1 t^3 + 6c\xi_1 t^2 + 4d\xi_2 t + e\xi_2 = 0, \\ V \equiv a\eta_0 t^4 + 4b\eta_1 t^3 + 6c\eta_1 t^2 + 4d\eta_2 t + e\eta_2 = 0. \end{cases}$$

The pencil of conics considered in §2 is derived from combinants of $U+\mu V$ by means of the rate of exchange

$$\begin{vmatrix} (a\xi)\,(b\xi) \\ (a\eta)\,(b\eta) \end{vmatrix} = \begin{vmatrix} a & & \\ & b & \\ & & x \end{vmatrix}^*,$$

and so on. Here g_2 has the equation

$$(6)\qquad 16b^2d^2x_0^2 + a^2e^2x_1^2 + 12ac^2ex_0x_2 + 8abdex_0x_1 = 0,$$

and K the equation

$$(7)\qquad b^2d^2x_0^2 + acd^2x_0x_1 + ac^2ex_0x_2 = 0.$$

Obviously g_2 is on the cusp (0, 0, 1) and has the line $x_0=0$ as its tangent at that point. Also K is degenerate with $x_0=0$ as one of its lines. So g_2 and K have contact at the cusp and the pencil is composed of conics having contact with R_2^4 at the cusp, all of the conics cutting out at least four coincident points of the quartic at that point. So the octavics cut out by the pencil of conics have four-fold roots. Then if $\lambda=12$ we have another degenerate member of the pencil. This is the Stahl conic N† which is the envelope of the flex lines of the cubic osculants of R_2^4. Also since the conic on the flexes and that on the contacts of the double lines are members of this pencil, we have means of proving the following known theorem.‡

THEOREM 1. *Three flexes and one of the points q_i coincide at a ramphoid cusp and two double lines coincide there.*

The pencil of covariant octavics cut out by the pencil is

$$(8)\qquad \begin{aligned} &(16-\lambda)b^2d^2t^8 + 4bd(8be - cd\lambda)t^7 \\ &\quad + 2(8b^2e^2 + 24bcde - 3c^2d^2\lambda)t^6 \\ &\quad + 4ce(12be + 3cd - cd\lambda)t^5 + c^2e^2(48-\lambda)t^4 = 0, \end{aligned}$$

* Rowe, loc. cit., p. 296.

† Stahl, *Ueber die rationale ebene Curve vierter Ordnung*, Journal für Mathematik, vol. 101 (1886), pp. 300–325.

‡ Neelley, loc. cit., p. 390 and p. 395; Wieleitner, *Theorie der ebenen algebraischen Kurven höherer Ordnung*, 1905, pp. 280–281.

which has several covariant forms that vanish identically due to the four-fold root.* When $\lambda = 48$, we have the conic on the contacts of the double lines and the octavic it cuts out has a five-fold root, and so a $C_{2,10}$† vanishes also.

The line on the points q_i is

$$8(bde + 3cd^2)x_0 - ae^2x_1 = 0, \tag{9}$$

and the locus of a point such that tangents from it to the curve are apolar to the flexes is the line

$$2(bde - 6cd^2)x_0 - ae^2x_1 = 0. \tag{10}$$

So these lines cut out a pencil of covariant quartics

$$[bde(1 - 5\lambda) - 6cd^2(1 + 3\lambda)]t^4 - 2be^2t^3 - 3ce^2t^2 = 0, \tag{11}$$

with at least a double root, and again we have covariant forms which vanish. One member is the square of a quadratic and so it has a $C_{3,6}$‡ which vanishes. This quartic is cut out by the distinct double line on the cusp. The cuspidal line gives a member of the pencil with a four-fold root and so several covariant forms vanish.‡ The bicombinant conic on the nodes and the points q_i also has contact with R_2^4 at the cusp and so cuts out another octavic with a four-fold root.

The Salmon combinant C§ of U and V gives the double lines of R_2^4 with the equation

$$\begin{aligned} &8b^4d^4x_0^4 + 4(6ab^2cd^4 - ab^3d^3e)x_0^3x_1 \\ &+ 2(ab^4e^3 + 9ab^2c^2d^2e - 6ab^3cde^2)x_0^3x_2 \\ &+ 6(3a^2c^2d^4 - a^2bcd^3e)x_0^2x_1^2 \\ &+ 3(a^2b^2ce^3 - 6a^2bc^2de^2 + 9a^2c^3d^2e)x_0^2x_1x_2 = 0, \end{aligned} \tag{12}$$

which gives the following theorem.

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* Tracey and Moore, *Covariant conditions for multiple roots of binary forms*, soon to appear in the American Journal of Mathematics.

† Cayley, Collected Mathematical Papers, vol. II, p. 316.

‡ Cayley, loc. cit., p. 555.

§ Salmon, loc. cit., p. 296.

THEOREM 2. *Two double lines coincide at a ramphoid cusp and a third double line is on the cusp, or five contacts of double lines coincide at a ramphoid cusp.*

The combinant D gives the flex lines, and their equation is of the form

$$acx_0^3 x_2 \cdot Q = 0, \tag{13}$$

where Q is a quadratic in x_0, x_1, x_2. This form proves the following theorem.

THEOREM 3. *Three flex lines unite to form the cuspidal line at a ramphoid cusp.*

We also find x_0 to be a factor of the combinant E which is the envelope of lines which cut R_2^4 in catalectic quartics.

This close connection between the cusp and covariant forms is very nicely brought out by considering the invariants of a pencil of line sections on the cusp. Suppose $(\xi x) = 0$ is the cusp line at $t = 0$ and $(\eta x) = 0$ is on the cusp and also has contact with R_2^4 at $t = \infty$. Then

$$(a\xi)(\alpha t)^4 \equiv at^4, \quad (b\eta)(\beta t)^4 \equiv 6ct^2,$$

and so*

$$I_2 = I_2' = I_4 = I_6 = 0.$$

But I_2 is the conic K and I_2' is the conic N so these meet at the cusp. Also I_4 is R_2^4 itself and I_6 gives the flex lines. So a cusp can only be at an intersection of K and N with R_2^4 where a flex line cuts the curve.

The significant thing we note concerning these covariant forms is that they cut out covariant sets of parameter values which are given by binary forms with multiple roots. This multiplicity is such as to be characteristic of the curve with a ramphoid cusp.

Consideration of the group of covariant forms which constitute the complete system of the two binary quartics giving the fundamental involution of the curve shows that one

* Morley invariants of two binary quartics.

member of the system vanishes identically. This form is the quadratic form obtained by operating with the quartic

$$(\beta t)^4 \equiv 12bt^2 + 12ct$$

on the sextic $12(2, H_2)$, where 2 is the second quartic $(\gamma t)^4 \equiv 4dt^4 + 4et^3$ and H_2 is its Hessian. It seems that *the identical vanishing of this covariant form is a necessary and sufficient condition that R_2^4 should have a ramphoid cusp.* We arrive at this conclusion because this covariant does not vanish for any other singularity and its vanishing is independent of the reference scheme giving the curve.

This investigation gives one other theorem that is not known. We noted that the conic K is degenerate when the curve has a ramphoid cusp. Previously* it was pointed out that this was also true when the curve had one or more undulations. So it seems that K is degenerate if there is a line of R_2^4 which cuts out four points of the curve with the same parameter value. Besides the undulations and the ramphoid cusp, the only type of curve where this is true is the limiting form of the oscnode when one loop is drawn up so as to become a point. Any line on this point cuts out three points and the oscnodal line cuts out four. The conic K for this form is the square of the oscnodal line. So we do have the following thorem.

THEOREM 4. *The Stahl conic K degenerates if the quartic curve has a line which cuts out four coincident points with the same parameter value and this line is one of the lines of the conic. The types of quartics with two such lines, the curve with two undulations or an undulation and a ramphoid cusp, have K given by the product of the two lines. When there are three such lines the equation of K vanishes identically.*

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* Neelley, *A note on the rational plane quartic curve with cusps or undulations*, this Bulletin, vol. 34 (1928), pp. 639–645.

ON THE REMOVAL OF FOUR TERMS FROM AN EQUATION BY MEANS OF A TSCHIRNHAUS TRANSFORMATION*

BY RAYMOND GARVER

The removal of the second, third, and fourth terms from an equation $f(x)=0$ by means of a Tschirnhaus transformation is a matter which can be summarized in two statements. First, if $f(x)=0$ is of higher than the fourth degree, the well known Jerrard process, which uses a fourth degree transformation, serves to remove the three terms without requiring the solution of any equation of degree greater than three. Second, if $f(x)=0$ is of the fourth degree, Lagrange first showed that a similar reduction is possible, employing a third-degree transformation.† Lagrange's transformation does not, however, apply to higher degree equations; the fact that the transformed quartic is a binomial equation is essential to its success.

The removal of the second, third, fourth, and fifth terms leads to a larger number of different cases. In general, we can say that, since the equations of condition are of degree 1, 2, 3, 4, we may be led to an equation of the 24th degree. But this maximum can almost always be substantially reduced. Thus Hamilton, in his report on the validity of Jerrard's work, was able to show that, if $f(x)=0$ is of degree greater than ten, the transformation can be effected with the aid of a single quartic equation, and no equation of higher degree, while simply one additional quartic has to be introduced if $f(x)=0$ is of degree ten.‡ Sylvester later obtained

* Presented to the Society, December 27, 1928.

† See his Oeuvres, vol. III, Paris, 1869, pp. 284–295. The article, his well known *Réflexions sur la résolution algébrique des équations*, was first published in 1770–1771.

‡ Sixth Report of the British Association for the Advancement of Science, London, 1837, pp. 295–348. See, in particular, p. 318 ff.

these same results in a somewhat different manner.* Bohlin professed to show that the transformation for $f(x)=0$ of degree ten required only one quartic equation,† but Netto‡ and Heymann§ pointed out that his work was incorrect. Heymann also set up a very simple transformation for the removal of the first four intermediate terms which requires a single quartic, and no equation of higher degree, but it can be applied only if $f(x)=0$ is of at least the twelfth degree. The only other work on the subject seems to be that of Lagrange in connection with the quintic;‖ he showed that the four intermediate terms can be removed without requiring the solution of any equation of higher than the sixth degree. But his method applies only to the quintic, since it makes use of the fact that the transformed equation is binomial.

The purpose of the present paper is to consider the cases which do not seem to be covered in the literature, namely, those in which the equation $f(x)=0$ is of degree 6, 7, 8 or 9. The following theorem gives the principal result:

THEOREM. *The second, third, fourth, and fifth terms of a ninth degree equation can be removed by means of a Tschirnhaus transformation of the eighth degree, the determination of whose coefficients involves the solution of no equation of degree greater than five.*

The method used to obtain this result was suggested by an article of Hilbert.¶ However, the theorem does not at all follow from his work, which leads to an equation of the twenty-seventh degree. In fact the aim of Hilbert's paper is simply to show that the general ninth degree equation is solvable in terms of algebraic functions of not more than four arguments, in the same sense that the general quintic is solvable

* Journal für Mathematik, vol. 100 (1887), pp. 465–486.
† Arkiv för Matematik, Astronomi, och Fysik, vol. 3 (1906–7), No. 5.
‡ Grunert's Archiv, (3), vol. 14 (1909), pp. 125–126.
§ Grunert's Archiv, (3) vol. 19 (1912), pp. 320–330.
‖ Pp. 314–317 of the above reference.
¶ Mathematische Annalen, vol. 97 (1926), pp. 243–250.

in terms of functions of a single argument. The point of view is thus quite different from that of the present paper.

To remove the second, third, fourth, and fifth terms from an equation $f(x)=0$ of the ninth degree by a Tschirnhaus transformation $y=P(x)$, where $P(x)$ is a polynomial of not higher than the eighth degree, we must show that the coefficients of $P(x)$ can be chosen so as to make $\sum y=\sum y^2=\sum y^3=\sum y^4=0$, where $\sum y$ means $\sum_{i=1}^{9} P(x_i)$. If we assume $P(x)$ to be a simple eighth degree polynomial, and attempt to set up the equations of condition directly, the work becomes involved. Hence we proceed as follows.

A theorem of Brioschi* states that, if $f(x)=0$ is of the nth degree, n odd, there exist $(n-1)/2$ polynomials in x, $\phi_1, \phi_2, \cdots, \phi_{(n-1)/2}$, such that (1) each ϕ_i is of lower degree than $f(x)$, (2) the ϕ_i are linearly independent, (3) the ϕ_i possess the properties $\sum\phi_i=0$†, $\sum\phi_i^2=0$, $\sum\phi_i\phi_j=0$, $(i, j=1, 2, \cdots, (n-1)/2, i\neq j)$, (4) the determination of the coefficients of the ϕ_i requires the solution of linear and quadratic equations only. Thus for $n=9$ there are 4 such functions, and if we set up the transformation

$$y = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4 \tag{1}$$

we have at once $\sum y=\sum y^2=0$, independent of the c_i. The two remaining equations of condition, $\sum y^3=0$ and $\sum y^4=0$, are clearly homogeneous equations of degree 3 and 4 respectively in the c_i, with coefficients which can be computed in terms of the coefficients of $f(x)=0$. If these two equations involved only two essential parameters their solution would, in general, require the solution of a 12th degree equation, but the presence of an additional parameter allows it to be carried out without introducing any equation of degree higher than the fifth.

A well known theorem of Sylvester says that the general homogeneous cubic polynomial in four variables may, with

* Rendiconti del Istituto Lombardo, (2), vol. 20 (1887), pp. 364–370. Also his Opere, vol. 3, Milan, 1904, pp. 293–299.

† $\sum\phi_i$ means $\sum_{k=1}^{n}\phi_i(x_k)$, where the x_k are the n roots of $f(x)=0$.

the aid of an equation of the fifth degree, be reduced to the canonical form $M_1^3+M_2^3+M_3^3+M_4^3+M_5^3$, where the M_i^3 are the roots of a quintic whose coefficients are rational in those of the given cubic polynomial, the M_i are linear homogeneous functions of the four variables, and M_5 can be expressed as a linear function of the other M_i, say as $\sum_{i=1}^{4} v_i M_i$. The condition $\sum y^3 = 0$ can then be put in the form

$$(2) \quad M_1^3 + M_2^3 + M_3^3 + M_4^3 + (v_1M_1 + v_2M_2 + v_3M_3 + v_4M_4)^3 = 0,$$

where each M is a linear homogeneous function of c_1, c_2, c_3 and c_4. We wish to show that this equation can be satisfied in such a way as to leave one essential parameter undetermined.

In the exceptional case in which v_1, v_2, v_3, v_4 are all zero, (2) can be satisfied by setting $M_2 = -M_1$, $M_4 = -M_3$. Thus only two linear conditions are imposed on the c_i, and one essential parameter (that is, the ratio of two c's) remains unfixed.

In the general case, at least one v, say v_3, will be different from zero. We may also certainly rearrange subscripts so that the conditions $v_1 = 0$, $v_2^3 = v_3^3$ do not hold simultaneously. Then make the following substitutions

$$(3) \quad \begin{cases} M_1 = s + 1, \\ M_2 = a_2 s + b_2, \\ M_3 = (-v_1 - v_2 a_2)s/v_3 + b_3, \\ M_4 = b_4. \end{cases}$$

Equation (2) then becomes a non-homogeneous cubic in s, which can be satisfied identically in s by proper choice of a_2, b_2, b_3, b_4. First, the coefficient of s^3 is

$$1 + a_2^3 - \left(\frac{v_1 + v_2 a_2}{v_3}\right)^3,$$

and a_2 can be chosen so that this vanishes, under the restrictions on v_1, v_2, v_3, stated at the beginning of this paragraph.

The coefficients of s^2 and s are found to be respectively three times the quantities

$$1 + a_2^2 b_2 + a_3^2 b_3,$$

$$1 + a_2 b_2^2 + a_3 b_3^2,$$

where a_3 stands for $-(v_1+v_2a_2)/v_3$. Now a_2 can be chosen different from zero unless $v_1^3 = v_3^3$, $v_2 = 0$. In this case, $a_3 = -v_1/v_3$, and the coefficients of s^2 and s can both be made to vanish by taking $b_3 = -v_1/v_3$. Secondly, a_3 will be zero if $a_2 = -v_1/v_2$; further, $1+a_2^3$ must equal zero, or $v_1^3 = v_2^3$. In this case both the above coefficients will vanish if $b_2 = -v_1/v_2$. Otherwise a_3 cannot equal zero unless $v_1 = v_2 = 0$; then for $a_2 = b_2 = -1$ the coefficients of s^3, s^2, s will all vanish. In the general case, both a_2 and a_3 will be different from zero, and b_2 and b_3 can obviously be determined so that the coefficients of s^2 and s will vanish.

Finally, the term free of s becomes

$$1 + b_2^3 + b_3^3 + b_4^3 + (v_1 + v_2 b_2 + v_3 b_3 + v_4 b_4)^3.$$

To make this vanish requires simply the solution of a cubic in b_4, unless $1+v_4^3 = 0$. If this condition holds, there remains a quadratic to solve unless $v_1+v_2b_2+v_3b_3 = 0$. However, if this last equation is satisfied, it may be combined with conditions already set up to show that $1+b_2^3+b_3^3$ must also equal zero. Thus the term free of s still vanishes, and the coefficients of (3) have now been determined so that (2) vanishes identically in s.

Now since the M_i are linear homogeneous functions of $c_1, \cdots, c_4$, the c_i may be determined linearly in terms of $M_1, \cdots, M_4$. In general, this determination will be unique, but this will not be so if there is a linear relation between the M_i. By (3), the c_i are given as linear, non-homogeneous functions of s. If now we substitute these values into the expression for $\sum y^4$, the final equation of condition, $\sum y^4 = 0$, becomes a non-homogeneous quartic equation in s, which can

be solved for s.* This completes the proof of the theorem.

If $f(x)=0$ is of degree 7 or 8,† there exist three ϕ-functions having the properties of Brioschi's theorem. We then set up a transformation similar to (1), and the equations $\sum y^3=0$, $\sum y^4=0$ are homogeneous equations in three parameters. The determination of c_1, c_2, c_3 will, in general, lead to an equation of the 12th degree, as was pointed out above.

If $f(x)=0$ is of degree 6 it does not seem possible to lower materially the maximum 24 which was obtained in the second paragraph.

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SOME PROPERTIES OF MULTI-COHERENT CONTINUA‡

BY W. A. WILSON

1. In a recent paper§ C. Kuratowski gives the following definitions:

A continuum C is called *uni-coherent* (or *n-coherent*) if for every decomposition of C into two continua K and L, where $C \neq K \neq L$, $K \cdot L$ has one (or n) components. A continuum C is called *multi-coherent*, if $K \cdot L$ is not connected.

The properties of such continua are later developed in some detail. Among the theorems proved are the following.

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* The very exceptional case in which the coefficients of s^4, s^3, s^2, s all vanish, while the term free of s does not vanish, may be handled by introducing homogeneous parameters in (3), that is, by putting $M=s+t$, $M_2=a_2s+b_2t$, etc. For the present case, t must be zero.

† Although Brioschi's theorem does not mention equations of even degree, $(n-1)/2$ ϕ-functions can be set up for an equation of even degree $n+1$ just as they can for an equation of odd degree n, and in different ways as well.

‡ Presented to the Society, October 27, 1928.

§ C. Kuratowski, *Sur la structure des frontières communes à deux régions*, Fundamenta Mathematicae, vol. 12, pp. 20–42.

(I) *For a continuum C to be multi-coherent it is necessary and sufficient that, for every sub-continuum Q of C, $C-Q$ is connected.*

(II) *If C is a bounded plane set which is the common frontier of two regions, C is a multi-coherent continuum.*

(III) *A bounded plane continuum which is decomposable and bi-coherent is the common frontier of two regions.*

A careless reading of the above definitions may give the erroneous impression that a multi-coherent continuum is n-coherent for some value of n. The continuum given in Ex. 4(a) (loc. cit., p. 23) is multi-coherent, but not n-coherent, since $K \cdot L$ may have either two or three components according to the manner of decomposition of C. Furthermore, as Kuratowski points out, the continuum just cited shows that the converse of Theorem III is not true. The converse of Theorem II also is not valid, and hence the class of multi-coherent plane continua is larger than that of regular frontiers.* Although the problem of finding necessary and sufficient intrinsic conditions for a bounded decomposable continuum to be a regular frontier has been solved,† less general conditions may at times be more convenient to use and the close relationship between multi-coherent continua and regular frontiers seems to the author to make it of interest to determine additional conditions under which the former class of continua have the same intrinsic properties as the latter.

Certain properties follow at once from the equivalent definition of multi-coherent continua given in Theorem I above. We first note that, if the bounded multi-coherent continuum M is the union of two proper sub-continua H and K, both H and K are irreducible about the set $H \cdot K$.

* A set is called a *regular frontier* if it is the frontier of at least two components of its complement.

† See C. Kuratowski, *Sur la séparation d'ensembles situés sur le plan*, Fundamenta Mathematicae, vol. 12, p. 235; and W. A. Wilson, *On bounded regular frontiers in the plane*, this Bulletin, vol. 34 (1928), p. 86.

For otherwise either H or K would contain a sub-continuum disconnecting M. (A continuum M is said to be disconnected by the set C if $M-C$ is non-void and is not connected.) Furthermore, if M is a bounded decomposable multi-coherent continuum, there is always an irreducible decomposition of M into two proper sub-continua H and K.* In this case no proper sub-set of $H \cdot K$ disconnects M and we may say that $H \cdot K$ irreducibly disconnects M.

2. THEOREM. *Let M be a bounded multi-coherent continuum. Let A and B be proper sub-continua of M which are not continua of condensation and let $A \cdot B = 0$. Then $A+B$ disconnects M.*

PROOF. Set $K = M-(A+B)$. If K is connected, $\overline{K}$ is a continuum. Now $A' = A - A \cdot \overline{K}$ and $B' = B - B \cdot \overline{K}$ are not void and $A \cdot B = 0$, by hypothesis. Hence $A'+B'$ is not connected. But $A'+B' = M-\overline{K}$. Hence we have the contradiction that the continuum $\overline{K}$ disconnects M. Therefore K is not connected.

3. THEOREM. *Let M be a bounded multi-coherent continuum. Let A and B be proper sub-continua of M, let $A \cdot B = 0$, and let $A+B$ disconnect M. Then $K = M-(A+B)$ has precisely two components.*

PROOF. Let C be a sub-continuum of M irreducible between A and B.† Then $C - C \cdot (A+B)$ is connected and has limiting points on both A and B. Hence at least one component k of K has limiting points on both A and B and $k+A+B$ is a continuum.

Since $A+B$ disconnects M, $K-k = M-(k+A+B)$ is not void. Since M is disconnected by no sub-continuum, $K-k$ is connected. The theorem is proved.

4. THEOREM. *Let M be a bounded multi-coherent continuum. Let A and B be proper sub-continua of M which are not continua*

* This is shown in the reference at the beginning of this section, p. 25, Theorem (η).

† This does not imply that C contains either A or B, but merely that $C \cdot A \neq 0 \neq C \cdot B$ and that there is no proper sub-continuum of C having this property.

of condensation and let $A \cdot B = 0$. *Let* $K = M - (A + B)$. *Then* $\overline{K}$ *is the sum of two continua which are irreducible between* A *and* B *and have no common points.*

PROOF. By §§2 and 3, K has two components, K' and K'', one of which, say K', has limiting points on both A and B. If K'' had limiting points on A only, we would have the contradiction that A disconnects M. Hence $\overline{K}'$ and $\overline{K}''$ are continua joining A and B and $\overline{K}' \cdot \overline{K}'' = 0$, since otherwise $\overline{K}' + \overline{K}''$ would disconnect M.

Let C and D be sub-continua of $\overline{K}'$ and $\overline{K}''$, respectively, irreducible between A and B. Then $C \supset K'$, for otherwise $C + A + B$ would disconnect M. Thus $C = \overline{K}'$ and in like manner $D = \overline{K}''$, which proves the theorem.

COROLLARY 1. *Let* M *be a bounded multi-coherent continuum. Let* M *contain two proper sub-continua which are not continua of condensation and have no common points. Then* M *is the union of four continua* A, B, C, *and* D, *such that* $A \cdot B = 0 = C \cdot D$, A *and* B *are irreducible between* C *and* D, *and* C *and* D *are irreducible between* A *and* B.

PROOF. Let A' and B' be proper sub-continua of M which are not continua of condensation and have no common points, and let $K = M - (A' + B')$. Then $\overline{K}$ is the sum of two continua C and D which are irreducible between A' and B' and have no common points. Let $L = M - (C + D)$. Then $\overline{L}$ is the sum of two continua A and B which are irreducible between C and D and have no common points. It is evident that $A \subset A'$, $B \subset B'$, and C and D are irreducible between A and B.

COROLLARY 2. *If* M *satisfies the hypotheses of Corollary* 1, *it is the union of two continua* H *and* K *such that* $H \cdot K$ *is the sum of two closed sets between which both* H *and* K *are irreducible.*

To see this set $H = A + C$ and $K = B + D$ in the above corollary.

COROLLARY 3. *If* M *satisfies the hypotheses of Corollary* 1

and is a plane continuum, it cuts the plane and is the frontier of exactly two components of its complement.

This conclusion follows from Corollary 2 and a theorem proved elsewhere.*

5. THEOREM. *Let M be a bounded multi-coherent continuum. Let A and B be proper sub-continua of M, let $A \cdot B = 0$, and let $A+B$ disconnect M. Let $L = M-(A+B)$. Then $\bar{L}$ is the union of two continua, which are irreducible between A and B and whose only common points lie on $A+B$.*

PROOF. Let h and k be the two components of L. Obviously $\bar{L} = \bar{h}+\bar{k}$. Set $H = \bar{h}$ and $K = \bar{k}$. As in the proof of §4 it can be shown that H and K are irreducible between A and B. Since $h = H - H \cdot (A+B)$ and $k = K - K \cdot (A+B)$, and $h \cdot k = 0$, it follows that $H \cdot K \subset A+B$.

COROLLARY 1. *Let M be a bounded multi-coherent continuum. Let A and B be continua of condensation of M, let $A \cdot B = 0$, and let $A+B$ disconnect M. Then M is the union of two continua H and K, $H \cdot K = \alpha+\beta$, where α and β are closed subsets of A and B respectively, and both H and K are irreducible between α and β.*

COROLLARY 2. *In addition to the hypotheses of Corollary* 1 *let M be a plane continuum. Then M cuts the plane and is the frontier of exactly two components of its complement.*

This follows from the reference under §4, p. 736, Lemma IV.

6. *Summary and General Discussion.* The theorems and corollaries of §§4 and 5 show that, if M is a bounded multi-coherent continuum which can be disconnected by the sum of two sub-continua, there are three possible cases: (1) when M can be disconnected by the sum of two continua of condensation; (2) when M can be disconnected by the sum of two sub-continua which are not continua of condensation;

* W. A. Wilson, *On the separation of the plane by irreducible continua*, this Bulletin, vol. 33, p. 739.

(3) when of any two sub-continua disconnecting M, one is a continuum of condensation, but not the other. Cases (1) and (2) are not mutually exclusive; they have been treated in §§4 and 5 and the results there obtained may be summarized as follows.

THEOREM. *Let M be a bounded multi-coherent continuum which can be disconnected by the sum of two sub-continua, both or neither of which are continua of condensation. Then M is the union of two continua whose divisor is the sum of two closed sets between which both continua are irreducible.*

COROLLARY. *Let M satisfy the hypotheses of the above theorem and lie in a plane. Then M is the common frontier of exactly two of its complementary domains.*

To make the third case definite, let A, but not B, be a continuum of condensation of M. If we use the notation of §5, this means that $A \subset H+K$ and $B-B\cdot(H+K)\neq 0$. Also both H and K are indecomposable, for otherwise we would have Case (2). There are two possibilities: $H\cdot K\cdot B=0$ and $H\cdot K\cdot B\neq 0$. If $H\cdot K\cdot B=0$, B contains a sub-continuum C which is irreducible between H and K and may contain $B-B\cdot(H+K)$. If so, $M=C+H+K$, C is indecomposable, and $C+H$ and K are both irreducible between $C\cdot K$ and $H\cdot K$. On the other hand, this is not the only possibility, as the following example shows. Let a, b, c, d, e, and f be six points. Let H be irreducible between each pair of the points a, b, and c; let K be irreducible between each pair of the points a, d, and e; let C be irreducible between each pair of the points b, d, and f; let D be irreducible between each pair of the points c, e, and f; let $B=C+D$; let $H\cdot K=a$, $C\cdot D=f$, $H\cdot B=b+c$, and $K\cdot B=d+e$; and let $M=H+K+B$. Then M is multi-coherent and can be disconnected by the sum of the continua a and B, but M cannot be expressed as the union of two continua having two closed sets in common, between which both are irreducible.

Now suppose that $H\cdot K\cdot B\neq 0$. Set $L=B-B\cdot(H+K)$; then $L=M-(H+K)$ and is connected. Let C be a sub-

continuum of B irreducible about $H \cdot K \cdot B$. Then $C \supset L$, as otherwise C would disconnect M. Thus M is the union of three continua, H, K, and C, where H and K are indecomposable and irreducible between $H \cdot K \cdot A$ and $H \cdot K \cdot B$, and C is irreducible about $H \cdot K \cdot B$ and contained in B, but it can be proved that M is not the union of two continua P and Q such that $P \cdot Q = \alpha + \beta$, where $\alpha \cdot \beta = 0$ and α and β are closed sets between which P and Q are irreducible.

That this case may exist is seen from the following example. Let a, b, c, and d be four points in the XY-plane in three-space; let H be a continuum irreducible between each pair of the points a, b, and c; let K be a continuum irreducible between each pair of the points a, b, and d; let H and K lie on opposite sides of the XY-plane except for these four points; and let C be a continuum lying in the XY-plane and irreducible between each pair of the points b, c, and d, but not containing a. Then $M = H + K + C$ is multi-coherent and can be disconnected by the sum of the two continua a and C; but it is not the union of two continua P and Q having the properties given in the last two lines of the previous paragraph.

7. *Generalization of Previous Theorems.* The properties of multi-coherent continua thus far obtained are readily extended. The following is the analog of §3.

THEOREM. *Let n be an integer greater than 1 and let M be a bounded multi-coherent continuum which cannot be disconnected by any set of $n-1$ or fewer sub-continua. Let $A_1, A_2, \cdots, A_n$ be proper sub-continua of M and let $\sum_1^n A_i$ disconnect M. Then $K = M - \sum_1^n A_i$ has exactly two components.*

PROOF. The theorem has already been proved for $n = 2$; let $n \geqq 3$. Now M cannot be irreducible between each pair of the sets $\{A_i\}$, for then K would be connected.* Let P be a proper sub-continuum of M irreducible between, say, A_1 and A_2; then $P - P \cdot (A_1 + A_2)$ is connected and has

* For M is irreducible between A_1 and $\sum_2^n A_i$ and the assumption that K is not connected leads at once to a contradiction.

limiting points on both A_1 and A_2. Thus for at least one component k of K, $\bar{k}\cdot A_1\neq 0\neq \bar{k}\cdot A_2$.

Suppose first that $\bar{k}\cdot A_i\neq 0$ for r values of i, say for $i\leqq r<n$, and that $\bar{k}\cdot A_i=0$ for $r<i\leqq n$. Then $k+\sum_1^r A_i$ is a continuum not meeting $\sum_{r+1}^n A_i$, and

$$K-k=M-[k+\textstyle\sum_1^r A_i+\sum_{r+1}^n A_i]$$

is not void and is connected by hypothesis, since the set in the brackets is the sum of $n-r+1$ continua. Let

$$k'=K-k+\textstyle\sum_{r+1}^n A_i.$$

Now

$$M=k+k'+\textstyle\sum_1^r A_i;\ \bar{k}\cdot k'=\bar{k}\cdot(K-k)+\bar{k}\sum_{r+1}^n A_i=0;$$

and

$$k\cdot\bar{k}'=k\cdot\overline{(K-k)}+k\cdot\textstyle\sum_{r+1}^n A_i=0.$$

Hence $\sum_1^r A_i$ disconnects M, which is contrary to the hypothesis.

Now let $\bar{k}\cdot A_i\neq 0$ for every i. Then $k+\sum_1^n A_i$ is a continuum and, as before, $K-k=M-(k+\sum_1^n A_i)$ is connected. This proves the theorem.

8. THEOREM. *Let $n\geqq 3$ and let M be a bounded multi-coherent continuum which cannot be disconnected by any set of $n-1$ or fewer sub-continua. Let $A_1, A_2, \cdots, A_n$ be proper sub-continua of M and let $\sum_1^n A_i$ disconnect M. Then M is the union of $\sum_1^n A_i$ and two indecomposable continua H and K such that $H\cdot K\subset\sum_1^n A_i$ and both H and K are irreducible between each pair of the continua $\{A_i\}$.*

PROOF. By §7 there are two components h and k of $M-\sum_1^n A_i$. Set $H=\bar{h}$ and $K=\bar{k}$. Then $H\cdot A_i$ and $K\cdot A_i$ are not void for any value of i.

Let P be a sub-continuum of H irreducible between A_1 and A_2, for example. If $P\cdot A_i=0$ for any i, say $i=3$, we have a contradiction, for then $(A_1+P+A_2)+A_3+\sum_4^n A_i$ disconnects M and contains not more than $n-1$ components, unless $P=H$, in which case M is disconnected by $A_1+A_2+\sum_4^n A_i$, again a contradiction.

Thus $P\cdot A_i\neq 0$ for every i. Now let $P'\subset P$ and be irre-

ducible between A_1 and A_3. By the above reasoning $P' \cdot A_2 \neq 0$. Hence $P' = P$. Hence P is irreducible between each pair of the continua $\{A_i\}$ and so is indecomposable.

Finally $P \supset h$, as otherwise $P + \sum_1^n A_i$ would disconnect M. Hence $P = H$ and the theorem is proved.

COROLLARY. *An n-coherent bounded continuum can be disconnected by n sub-continua, but not by fewer.*

9. THEOREM. *Let $n \geqq 3$ and let M be a bounded multi-coherent continuum which cannot be disconnected by any set of $n-1$ or fewer sub-continua. Let $A_1, A_2, \cdots, A_n$ be proper sub-continua of M, let $\sum_1^n A_i$ disconnect M, and let H and K be the two indecomposable sub-continua of M which are irreducible between each pair of the continua $\{A_i\}$. If every A_i is a continuum of condensation, $M = H + K$ and $H \cdot K = \sum_1^n \alpha_i$, where each α_i is a closed sub-set of A_i. Otherwise there is but one continuum A_i which is not a continuum of condensation and M may not be the union of two continua whose divisor is the sum of two closed sets between which both continua are irreducible.*

PROOF. If two of the continua $\{A_i\}$, say A_1 and A_2, were not continua of condensation of M, we would have a contradiction, since $H + K + \sum_3^n A_i$ would disconnect M. If every A_i is a continuum of condensation, M has the properties stated above by §8. If one of the continua $\{A_i\}$ is not a continuum of condensation, an example similar to that of §6 shows the truth of the last statement of the theorem.

COROLLARY. *Let M satisfy the hypotheses of the above theorem and lie in a plane. If every A_i is a continuum of condensation, M is the common frontier of exactly n components of its complement.*

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CONCERNING COLLECTIONS OF CUTTINGS OF CONNECTED POINT SETS*

BY G. T. WHYBURN

Two point sets are mutually separated if they are mutually exclusive and neither of them contains a limit point of the other. A point set M is said to be not connected or connected according as M is or is not the sum of two non-vacuous mutually separated point sets. A subset X of a connected point set M is said to be a cutting of M, or is said to cut M, provided the set of points $M-X$ is not connected; X is called an irreducible cutting of M provided X cuts M but no proper subset of X cuts M. If A and B are subsets of a connected point set M, the subset X of M is said to separate A and B in M, or to cut M between A and B, provided that $M-X$ is the sum of two mutually separated sets $M_a(X)$ and $M_b(X)$ containing A and B respectively.

In this paper the ordinary notation of point set theory will be employed, for example, $\overline{X}=X+X'$, where X' is the set of all limit points of the set X, $K \subset H$ means that the set K is a subset of the set H, $K \cdot H$ means the set of points common to K and H, $F(R)$ denotes the boundary of the set R, etc. In addition, if X is a cutting of a connected point set M then, unless otherwise stated, the equation $M-X=M_a(X)+M_b(X)$ is to be interpreted as meaning that $M-X$ is the sum of two mutually separated sets $M_a(X)$ and $M_b(X)$ which contain the sets A and B respectively.

1. *Cuttings of Connected Sets in General.* The point sets considered in this section are assumed to lie in a separable metric space which will be denoted by S. It is apparent from the proofs of some of the theorems, however, that they hold in even more general space.

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* Presented to the Society, April 7 and June 2, 1928.

THEOREM 1. *Let G be any uncountable collection of mutually exclusive cuttings of a connected and separable* point set M. Then there exist two points A and B of M which are separated in M by uncountably many elements of G.*

PROOF. Since M is separable, it therefore contains a countable set of points D such that M is a subset of $\overline{D}$. As G is uncountable and D is countable, there exists an uncountable subcollection G_1 of G such that no element of G_1 contains a point of D. Now if X is any element of G_1, $M-X$ is the sum of two mutually separated sets $M_1(X)$ and $M_2(X)$. Let P_1 and P_2 be points of $M_1(X)$ and $M_2(X)$ respectively. Since P_1 is a limit point of D and is not a limit point of $M_2(X)$, then D is not a subset of $M_2(X)$. And since $D \cdot X = 0$, D is not a subset of $M_2(X)+X$. Therefore $M_1(X)$ contains at least one point $A(X)$ of D. Similarly, $M_2(X)$ contains at least one point $B(X)$ of D. Thus every element X of G_1 separates some pair of points $A(X)$ and $B(X)$ of D in M. And since the elements of G_1 are uncountable and the collection of all pairs of points of D is countable, it follows that some two points A and B of D are separated in M by uncountably many elements of G_1.

THEOREM 2. *If G_0 is any collection of mutually exclusive subsets of a connected and separable set M each of which contains a cutting of M, then all save possibly a countable number of the elements of G_0 must be cuttings of M.*

PROOF. Since M is separable, it contains a countable set of points D such that $M \subset \overline{D}$. By hypothesis each element g of G_0 contains a cutting X_g of M. Now let G be the collection of all those elements of G_0 which are not cuttings

* It follows by a theorem of W. Gross (See *Zur Theorie der Mengen, in denen ein Distanzbegriff definiert ist*, Wiener Sitzungsberichte, vol. 123 (1914), pp. 801–819) that every subset of a separable metric space is separable. Hence that M is separable follows from the fact that it is imbedded in the space S. However, the condition that M be separable is explicitly stated in this theorem and in some of the theorems that are to follow because the separability of M is used in the proof and because as thus stated the theorem holds in a more general space than the space S.

of M. For each element g of G, $M-X_g$ is the sum of two mutually separated sets M_1 and M_2; and since $M-g$ is connected and a subset of $M-X_g$, it therefore is a subset either of M_1 or of M_2, say of M_1. Then obviously M_2 is a subset of g. Let P be a point of M_2. Then since P is a limit point of D but not of M_1, D is not a subset of M_1. Hence M_2+X_g contains at least one point of D; and as M_2+X_g is a subset of g, then g contains at least one point of D. Thus every element of G contains at least one point of D. And since the elements of G are mutually exclusive and D is countable, it follows that G is countable.

THEOREM 3. *If X is any cutting of a connected set M between two points A and B of M, and K is any subset of M which contains neither A nor B, then $X+K$ cuts M between A and B.*

PROOF. By hypothesis $M-X=M_a(X)+M_b(X)$. Then $M-(X+K)=[M_a(X)-M_a(X)\cdot K]+[M_b(X)-M_b(X)\cdot K]$, and clearly $[M_a(X)-M_a(X)\cdot K]$ and $[M_b(X)-M_b(X)\cdot K]$ are mutually separated and contain A and B respectively.

THEOREM 4. If *N is any connected subset of a connected and separable set M, then not more than a countable number of the components* of $M-N$ can contain cuttings of M.*

PROOF. By a theorem of Knaster and Kuratowski's,† no component of $M-N$ can cut M. Therefore by Theorem 2, not more than a countable number of the components of $M-N$ can contain any subset which cuts M.

THEOREM 5. *If G_0 is a collection of mutually exclusive connected subsets of a connected and separable set M such that for each element g of G_0, $M-g$ is neither connected nor the sum of two connected sets, then G_0 is countable.*‡

* A component of a point set K is a connected subset of K which is not a proper subset of any connected subset of K. See Hausdorff, *Mengenlehre*.

† *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–253, Theorem 10.

‡ This theorem is a generalization of a theorem due to Kuratowski and Zarankiewicz (see *A theorem on connected point sets*, this Bulletin, vol. 33

PROOF. Suppose, on the contrary, that G_0 is uncountable. Then by Theorem 1, there exist two points A and B of M and an uncountable subcollection G of G_0 each element of which separates A and B in M. Since M is separable, it contains a countable set of points D such that $M \subset \bar{D}$. For each element g of G, $M-g$ is the sum of three mutually separated sets M_a, M_b, and M_g, where M_a and M_b contain A and B respectively. For each element g of G, it follows as in the preceding proofs that M_g+g contains at least one point of D. And since,* for each g, M_a+M_b+g is connected, and contains both A and B, it therefore contains every element of G. Hence for every two distinct elements X and Y of G, M_x+X and M_y+Y are mutually exclusive sets; and as each such set contains a point of D, and D is countable, it follows that G is countable, contrary to supposition. Thus the supposition that Theorem 5 is false leads to a contradiction.

Let A and B be two connected subsets (or points) of a connected point set M, and let G be any collection of mutually exclusive connected subsets of M each of which separates A and B in M. Then the elements of G can be linearly ordered in M from A to B as follows. The element X of G will be said to precede or to follow the element Y of G in M in the order from A to B according as X belongs to $M_a(Y)$ or to $M_b(Y)$, where $M_a(Y)+M_b(Y)$ is one method of expressing $M-Y$ as the sum of two mutually separated sets containing A and B respectively. It is readily deduced from this definition that (a) of two elements X and Y of G, one must always precede the other, (b) X cannot both precede and follow Y, and (c) if X precedes Y, then Y follows X.

(1927), p. 574), who assume the unnecessary condition that the elements of G_0 are closed relative to M. Although the proof here given for Theorem 5 differs markedly from that given by Kuratowski and Zarankiewicz to prove a special case of their theorem, I have found that a proof can be constructed for Theorem 5 based on their methods which is somewhat shorter than the proof they gave in the paper just mentioned and which makes no use of the fact that the set M itself is separable.

* See Knaster and Kuratowski, loc. cit., Theorem 6.

If X and Y are distinct elements of G and X precedes Y in M in the order from A to B, then by the segment $S(X, Y)$ of M from X to Y is meant the set of points $M_b(X) \cdot M_b(Y)$, where $M_a(X)+M_b(X)$ is one way of expressing $M-X$ as the sum of two mutually separated sets containing A and B respectively and similarly for $M_a(Y)$ and $M_b(Y)$; by the interval $I(X, Y)$ of M from X to Y is meant the set of points $[M_b(X)+X]\cdot[M_a(Y)+Y]$. Clearly $I(X, Y)=S(X, Y)+X+Y$. In case all of the sets $M_a(X)$, $M_b(X)$, $M_a(Y)$, and $M_b(Y)$ are connected, as is necessarily the case for all save possibly a countable number of elements of G (see Theorem 5), then $S(X, Y)$ and $I(X, Y)$ are uniquely determined. The following facts are readily proved.

(i). For each pair of elements X and Y of G, where X precedes Y, M is the sum of three connected point sets $M_a(X)+X$, $I(X, Y)$, and $M_b(Y)+Y$, where we have $[M_a(X)+X]\cdot I(X, Y)=X$, $[M_b(Y)+Y]\cdot I(X, Y)=Y$, and $[M_a(X)+X]\cdot[M_b(Y)+Y]=0$.

(ii). No point of $S(X, Y)$ is a limit point of $M-I(X, Y)$.

(iii). If $M_a(X)$, $M_b(X)$, $M_a(Y)$, and $M_b(Y)$ are connected, and $S(X, Y)$ contains an element of G, then $S(X, Y)$ is connected. It may happen that for some element X of G, an element Y of G exists such that $S(X, Y)$ is vacuous. However, Theorem 6 below shows that this can be the case for at most a countable number of elements X of G.

THEOREM 6. *Let G be any uncountable collection of mutually exclusive connected cuttings of a connected set M. Then there exists a subcollection of G^* of G, containing all but a countable number of the elements of G, and such that every two elements of G^* are separated in M by uncountably many elements of G.**

PROOF. Since† M is separable, it contains a countable subset D such that $M \subset \overline{D}$. Let G_1 denote the collection obtained by omitting from the collection G (1) all elements X of G such that $M-X$ is not the sum of two connected point sets, (2) all elements of G which contain at least one

† See Gross, loc. cit.

point of D, and (3) all elements X of G which separate in M some two points of D which are separated in M by only a countable number of elements of G. Then G_1 contains all but a countable number of the elements of G. Let H denote the collection of all possible pairs of points of the set D having the property that each pair A, B in H is separated in M by each element of an uncountable subcollection G_{ab} of the collection G_1. For each pair A, B in H, let E_{ab} denote the collection of all those elements e of G_{ab} such that every neighborhood of e contains points of uncountably many elements of G_{ab}. By a theorem due to Gross (loc. cit.), E_{ab} contains all but a countable number of the elements of G_{ab}. Hence every neighborhood of each element e of E_{ab} contains points of uncountably many elements of E_{ab}.

Now for each pair of elements X and Y of E_{ab}, $S(X, Y)$ is non-vacuous. For suppose $S(X, Y)$ is vacuous, and suppose X precedes Y in M in the order from A to B. Now X contains a point P which is a limit point of $M_b(X)$. And since P is not a limit point of $M_b(Y)$ [for X belongs to $M_a(Y)$], there exists a neighborhood V of P which contains no point of $M_b(Y)$. Now V contains at least one point Q of $M_b(X)$; and since $M_b(X) = M_b(Y) + Y + S(X, Y)$, and $S(X, Y)$ is vacuous, then Q must belong to Y. Now Q is not a limit point of $M_a(X)$, for Q belongs to Y and Y belongs to $M_b(X)$. Hence there exists a neighborhood U of Q which contains no point of $M_a(X)$ and which is a subset of V. Then $U \cdot M$ is a subset of $X + Y$. But $U \cdot M$ must contain at least one point of D; and since X and Y belong to G_1, neither of them can contain a point of D. Thus the supposition that $S(X, Y)$ is vacuous leads to a contradiction.

Now the number of elements X of E_{ab} such that an element Y of E_{ab} exists such that $S(X, Y)$ contains no element of E_{ab} must be countable. For each such segment $S(X, Y)$ must contain at least one point P; and since P is a limit point of D but not of $M - I(X, Y)$, and neither X or Y contains a point of D, then $S(X, Y)$ must contain at least one point of D. And clearly if $S(X_1, Y_1)$ and $S(X_2, Y_2)$ are two

such segments which are different, they must be mutually exclusive. Therefore the number of such segments must be countable, and hence all save a countable number of elements X of E_{ab} must have the property that if Y is any other element whatever of E_{ab}, then $S(X, Y)$ contains at least one element Z of E_{ab}. Let G^*_{ab} denote the collection of all those elements of E_{ab} which have this property. Then if X and Y are any two elements of G^*_{ab}, $S(X, Y)$ contains at least one element Z of E; and since every neighborhood of Z contains points of uncountably many elements of E_{ab}, since no point of Z is a limit point of $M-I(X, Y)$, and since G^*_{ab} contains all but a countable number of the elements of E_{ab}, it follows that $S(X, Y)$ contains uncountably many elements of G^*_{ab}.

Let G^* denote the collection of all those elements g of G such that g belongs to some collection G^*_{ab}. Since H is countable, it follows that G^* contains all save a countable number of the elements of G. Now let X and Y be any two distinct elements of G^*. Then $M-X=M_y+M_1$, where M_y and M_1 are mutually separated and M_y contains Y. It is readily seen that M_1 contains at least one point A of D. Likewise $M-Y=M_x+M_2$, and M_2 contains a point B of D. It is readily seen that both X and Y separate A and B in M. Since X and Y belong to G_1, then A and B are separated in M by uncountably many elements of G. Hence A,B is a pair in H, and X and Y belong to G^*_{ab}. Therefore $S(X, Y)$ contains uncountably many elements $[Z]$ of G^*_{ab}. Since† $M_a(X)+X$ and $M_b(Y)+Y$ are connected and contain A and B respectively but contain no point of Z, for each Z in $[Z]$, then X belongs to $M_a(Z)$ and Y to $M_b(Z)$. Hence Z separates X and Y in M. Thus every two elements X and Y of G^* are separated in M by uncountably many elements of G,* and our theorem is proved.

THEOREM 7. *If G^* is the collection obtained in Theorem 6 and E is the collection of elements e such that $e=\bar{g}\cdot M$, for*

† See Knaster and Kuratowski, loc. cit., Theorem 6.

each element g of G, then the elements of E are mutually exclusive, all save a countable number are cuttings of M and every two of them are separated in M by some third element of E (indeed, by uncountably many elements of E).*

PROOF. Let d and e be two distinct elements of E, where $d=\bar{g}\cdot M$ and $e=\bar{h}\cdot M$, where g and h are elements of G^*. There exists an element i of G^* such that $M-i=M_g+M_h$. Hence h contains no limit point of g; similarly it is shown that i contains no limit point of either g or h. Then since d and e are subsets of M_g+i and M_h+i respectively, and $d\cdot i=e\cdot i=0$, it follows that d and e are subsets of M_g and M_h respectively. Hence d and e are mutually exclusive. A similar argument shows that if f is the element of E such that $f=\bar{i}\cdot M$, then f separates d and e in M. That all save possibly a countable number of the elements of E are cuttings of M follows now from Theorem 2. The truth of Theorem 7 is therefore established.

2. *Cuttings of Continua.* In this section the point sets considered are assumed to lie in a locally compact, metric, and separable space, that is, the same space considered in §1 with the additional restriction that it be locally compact.

THEOREM 8. *No continuum M contains an uncountable collection G of mutually exclusive connected subsets each of which contains a compact proper subset which cuts M.*†

PROOF. Suppose, on the contrary, that some continuum M contains such a collection G. Each element g of G contains a closed‡ and compact set f which cuts M and a point

† For the special case where the continuum M is identical with a whole euclidean space of n dimensions, this theorem is readily deduced from a theorem of Kuratowski's; see C. Kuratowski, *Sur les coupures irréductibles du plan*, Fundamenta Mathematicae, vol. 6 (1924), pp. 130–146, Theorem 8.

‡ For justification of the word "closed" here see R. L. Moore (Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 356–360), who proves, with the aid of a theorem of Knaster and Kuratowski (*Sur les ensembles connexes*, loc. cit., Theorem 37), that every cutting of a bounded plane continuum M between two points A and B of M contains a closed cutting of M between A and B. Although the theorem of Knaster

p not belonging to f. For each element g of G select two such sets f and p, let F denote the collection of sets $[f]$, and let P denote the set of points $[p]$. For each element g of G, let $\delta(f, p)$ be the minimum distance between the sets f and p. There exists a number $\epsilon>0$ and an uncountable subcollection G_1 of G such that for each element g_1 of G_1, $\delta(f_1, p_1) > 4\epsilon$. Let F_1 and P_1 denote the collections $[f_1]$ and $[p_1]$, for all elements g_1 of G_1. There exists a point p which is a point of condensation of P_1. Let R be the set of all points whose distance from p is less than ϵ. Then no element of F_1 contains a point of $\overline{R}$. Let P_2 be the set of points $R \cdot P_1$, and let F_2 and G_2 be the corresponding subcollections of F_1 and G_1 respectively, that is, each point p_2 of P_2 corresponds to an element f_2 in F_2 and an element g_2 in G_2. By a theorem* established in its most general form by R. L. Moore, there exists an element f of F_2 such that every neighborhood of f contains uncountably many elements of F_2. Since f is compact, it follows with the aid of the Borel theorem that there exists a compact open set D which contains f but contains no point whatever of R. Let F_3 be the collection of all those elements of F_2 which lie wholly in D, and let G_3 be the corresponding collection of elements of G_2.

By Theorems 1, 3, and 6 it follows that there exist two points A and B of M and an uncountable subcollection G_4 of G_3 such that if F_4 is the corresponding collection of elements of F, then (1) each element of F_4 and also each element of G_4 separates A and B in M, and (2) every two elements X and Y of G_4 are separated in M by uncountably

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and Kuratowski used by Moore does not necessarily remain true in the space considered in this paper, nevertheless the methods of proof used by Knaster and Kuratowski in proving their theorem suffice to prove Moore's theorem in such a space, and indeed to prove the following more general theorem: *If K is a cutting of a continuum M (bounded or not) between any two subsets A and B of M, then K contains a closed cutting of M between A and B.*

* See R. L. Moore, *Concerning triodic continua in the plane*, to appear in Fundamenta Mathematicae; for the case of a euclidean space see a theorem of C. Zarankiewicz in his paper *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), Theorem 2.

many elements of G^4. Let T denote the point set consisting of the sum of all the point sets of the collection G_4, and for each element g of G_4, let e denote the component of $M-(T-g)$ which contains g, and let E denote the collection of sets $[e]$ thus obtained. Then each element e of E must be closed; for if some element h of G_4, different from the element of G_4 which belongs to e, contained a limit point of g, then $e+h$ would be connected and would contain both e and h, contrary to the fact that an element j of G_4 exists which contains no point of $e+h$ but which separates g and h in M. Hence each element of E is a continuum, and a similar argument shows that the elements of E are mutually exclusive and that every two elements of E are separated in M by some third element of E. Since E is uncountable, it follows readily from Theorems 3 and 5 together with the theorem of Moore-Zarankiewicz quoted above that there exists an element e of E such that (1) e separates A and B in M and both $M_a(e)$ and $M_b(e)$ are connected, (2) either every point of e is a limit point of $M_a(e)$ or every point of e is a limit point of $M_b(e)$, suppose the former, and (3) there exists a point p of e and a sequence of points $p_1, p_2, p_0, \cdots$, such that for each i, p_i belongs to an element e_i of E which belongs to $M_b(e)$, and such that p is the sequential limit point of the sequence of points $p_1, p_2, p_3, \cdots$. For each i, let q_i denote the component of $e_i \cdot \bar{D}$ which contains p_i, and let K denote the sequential limiting set of the sequence $q_1, q_2, q_3, \cdots$. By a theorem due to Janiszewski,* for each i, q_i is a compact continuum containing at least one point of $F(D)$. Hence by a theorem proved in its most general form by R. G. Lubben,† it follows that K is a continuum. Clearly K contains at least one point X of $F(D)$. Now K contains no point of $T-e$. For suppose K does contain a point Y of $T-e$; then Y belongs to some element j of E; and

* *Sur les continus irréductibles entre deux points*, Journal de l'École Polytechnique, (2), vol. 16 (1912).

† See R. G. Lubben, *Concerning limiting sets in abstract spaces*, Transactions of this Society, vol. 30 (1928), pp. 668–685.

there exists an element d of E such that $M-d=M_e(d)+M_j(e)$. Either $M_e(d)$ or $M_j(e)$ must contain infinitely many elements of the sequence $q_1, q_2, q_3, \cdots$. But $M_e(d)$ cannot contain infinitely many elements of this sequence; for if it did, then Y, a point of $M_j(d)$, would be a limit point of $M_e(d)$. Likewise $M_j(d)$ cannot contain infinitely many elements of this sequence; for if so, then p, a point of $M_e(d)$, would be a limit point of $M_j(d)$. Thus the supposition that K contains a point of $T-e$ leads to a contradiction. Then since K is a connected subset of M which contains p but contains no point of $T-e$, and e is the component of $M-(T-e)$ which contains p, K must be a subset of e. Hence e contains the point X. Then X is a limit point of $M_a(e)$, for every point of e is a limit point of $M_a(e)$; and X is also a limit point of $M_b(e)$. Hence $M_a(e)+X+M_b(e)$ is connected and contains both A and B. But this is contrary to the fact that e contains an element f of F which lies wholly in D (and hence does not contain X) and which separates A and B in M. Thus the supposition that Theorem 8 is false leads to a contradiction.

COROLLARY. *If G is any collection of mutually exclusive connected subsets of a continuum M each of which contains a compact cutting of M, then all save possibly a countable number of the elements of G are themselves compact continua which are irreducible cuttings of M.*

THEOREM 9. *If G is any collection of mutually exclusive connected subsets of a continuum M each of which contains a compact cutting of M, and T denotes the point set obtained by adding together all the point sets of the collection G, then all save possibly a countable number of the elements of G are components of $M-(T-g)$; indeed, all save a countable number of the elements g of G have the property that every connected subset of M which contains g but is not identical with g must contain points of uncountably many elements of G.*

PROOF. By Theorems 6 and 8, G contains a subcollection G_1 which contains all but a countable number of the elements

of G and such that the elements of G_1 are compact continua which are irreducible cuttings of M and have the property that every two of them are separated in M by uncountably many elements of G_1. Now let E denote the collection of all those elements g of G_1 such that g is a proper subset of some connected subset X of M which has points in common with at most a countable number of elements of G. Let $[X]$ denote the collection of sets X. Then since every two elements of G_1 are separated in M by uncountably many elements of G_1, it follows that no element of $[X]$ contains points of more than one element of G_1 and that no two elements of $[X]$ have a point in common. And since each element of $[X]$ contains a compact proper subset which cuts M, it follows by Theorem 8 that $[X]$ must be countable. Therefore E is countable.

THEOREM 10. *If G_0 is any uncountable collection of mutually exclusive connected subsets of a continuum M each of which contains a compact cutting of M, then G_0 contains a subcollection G which contains all save possibly a countable number of the elements of G_0 and has the following properties*: (1) *every element g of G is a compact continuum which is an irreducible cutting of M and is a component of the set of points $M-(T-g)$, where T denotes the point set obtained by adding together all the point sets of the collection G,* (2) *every two elements of G are separated in M by uncountably many elements of G, and* (3) *G is upper semi-continuous.**

PROOF. That G_0 contains a collection G having properties (1) and (2) follows immediately from Theorems 6, 8, and 9. I shall now show that any collection G of mutually exclusive

* A collection G of point sets is said to be upper semi-continuous provided that if g is any element of G and $g_1, g_2, g_3, \cdots$ is any infinite sequence of elements of G containing points $P_1, P_2, \cdots$, respectively, such that the sequence of points $P_1, P_2, \cdots$ has a sequential limit point which belongs to g, then g contains the entire sequential limiting set of the sequence of sets $g_1, g_2, g_3, \cdots$. See R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27 (1925), pp. 416–428.

cuttings of M having properties (1) and (2) must be upper semi-continuous. Let $g_1, g_2, g_3, \cdots$ be any sequence of elements of G containing points $P_1, P_2, P_3, \cdots$, respectively, such that the sequence $P_1, P_2, P_3, \cdots$ has a sequential limit point P belonging to some element g of G. Let L denote the sequential limiting set of the sequence $g_1, g_2, \cdots$. Then g must contain L. For if not, then since g is compact, it follows readily that g is a proper subset of the component K of $g+L$ which contains g. And, since g is a component of $M-(T-g)$, K must contain a point Q of some other element h of G different from g. And by (2) there exists an element f of G which separates g and h in M, that is, $M-f=M_g(f)+M_h(f)$. Now either $M_g(f)$ or $M_h(f)$ must contain infinitely many sets of the sequence $g_1, g_2, \cdots$. But it is easy to see that neither of these sets can contain infinitely many sets of this sequence, because each of these sets contains a point of L. Hence g contains L, and therefore G is an upper semi-continuous collection.

COROLLARY. *If G is any collection of mutually exclusive compact subcontinua of a continuum M such that* (1) *each element g of G is saturated with respect to the property of being a subcontinuum of $M-(T-g)$, where T is the point set obtained by adding together all the point sets of the collection G, and* (2) *every two elements of G are separated in M by some third element of G, then G is upper semi-continuous.*

Suppose M is a compact continuum and G is a collection of mutually exclusive sets having properties (1) and (2) in the statement of Theorem 10. Then by property (1) it follows that every component of $M-T$ is closed and hence is a compact continuum. And if G_0 denotes the collection whose elements are the elements of G together with all the components of $M-T$, then the sum of all the elements of G_0 is identical with M; and using property (2) of G it is readily shown that M is an acyclic continuous curve (that is, a connected im kleinen continuum which contains no simple closed curve) with respect to the elements of G_0, and thus G_0 is upper semi-continuous. The same is true if instead of as-

suming M compact we assume merely that each component of $M-T$ is compact. Thus we have the following theorem.

THEOREM 11. *If M is any compact continuum and G is any uncountable collection of mutually exclusive connected subsets of M each of which contains a cutting of M, then there exists an upper semi-continuous collection G_0 of mutually exclusive compact subcontinua of M such that* (1) *all save possibly a countable number of elements of G are elements also of G_0,* (2) *the sum of all the elements of G_0 is identical with M, and* (3) *M is an acyclic continuous curve with respect to the elements of G_0.*

3. *The Order of Various Cuttings and im Kleinen Cuttings of Continua.** With the aid of the results established above it follows that all of the theorems, except Theorem 10, in the author's paper *Concerning the cut points of continua*† hold true in any separable metric space which is locally compact. In particular, the theorem (loc. cit., Theorem 7) that *all save possibly a countable number of the cut points of any continuum M are points of Menger order two of M* holds true in such a space. Using the results proved above and a method essentially the same as that used to prove Theorem 7 in the paper of the author's just mentioned, it is not difficult to prove the following more general theorem.

THEOREM 12. *If M is any continuum and G is any collection of mutually exclusive compact subcontinua of M each of which cuts M, then all, save possibly a countable number, elements X of G have the property that for each $\epsilon>0$, an open set R exists containing X and every point of which is at a distance $<\epsilon$ from some point of X and such that $F(R)\cdot M=Y+Z$, where Y and Z are continua of the collection G.*

Thus all save possibly a countable number of elements of G are continua of Menger order (or Urysohn index) two of

* The space considered in this section is the same as the one considered in §2.

† Transactions of this Society, vol. 30 (1928), No. 3, pp. 597–609.

M in a certain generalized sense. Menger* and Urysohn* call a point P of a continuum M a point of "Verzweigungs" order (or index) n of M provided that for each $\epsilon>0$, P can be ϵ-separated† by n subcontinua of M but not by $n-1$ such continua, that is, for each $\epsilon>0$ an open set R exists which contains P and is of diameter $<\epsilon$ and such that $F(R)\cdot M$ contains at most n components and such that n is the least integer such that this property is preserved. If we extend this notion to include "continua P" as well as "points P," and say that a subcontinuum P of a continuum M is ϵ-separated by n subcontinua of M provided that for each $\epsilon>0$ an open set R exists which contains P and such that (1) every point of R is at a distance $<\epsilon$ from some point of P and (2) $F(R)\cdot M$ has at most n components, then Theorem 12 states that all save possibly a countable number of the elements of G are continua of "Verzweigungs" order two of M, or indeed, that all save possibly a countable number of the elements of G can, for each $\epsilon>0$, be ϵ-separated by two continua *which are elements of* G.

If C is a subcontinuum of a continuum M such that for each $\epsilon>0$, an open set R exists which contains C and such that (1) every point of R is at a distance $<\epsilon$ from some point of C and (2) C is a cutting of the component of $M\cdot\overline{R}$ which contains C, then C will be called an *im kleinen cut continuum* of M. For the special case where C is a single point, C will be called an *im kleinen cut point*, or a *cut point im kleinen*, of M. The notion of a cut point im kleinen of a continuum is embodied in Urysohn's notion of an "unvermeidbar" (unavoidable) point of a continuum‡ and in R. L. Moore's

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* See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 276–306, and P. Urysohn, *Über im kleinen zusammenhängende Kontinua*, Mathematische Annalen, vol. 98 (1927), pp. 296–308, and earlier papers by the same authors referred to therein.

† See P. Urysohn, *Sur la ramification des lignes Cantoriennes*, Comptes Rendus, vol. 175 (1922), p. 481.

‡ See P. Urysohn, *Über im kleinen zusammenhängende Kontinua*, loc. cit.; and R. L. Moore, *Concerning triods in the plane and the junction points*

notion of a "junction" point of a continuum. For the case of a continuous curve (that is, a connected im kleinen continuum) it has been observed by W. L. Ayres and the author that the notions of an unavoidable point in the sense of Urysohn and of an im kleinen cut point as above defined are equivalent.

THEOREM 13. *If H denotes the set of all the points of a continuum M at which M is connected im kleinen and K denotes the set of all the im kleinen cut points of M, then all save possibly a countable number of the points of $K \cdot H$ are points of Menger order two of M.*

PROOF. Suppose, on the contrary, that $H \cdot K$ contains an uncountable subset N no point of which is a point of order two of M. For each point P of N there exists a number $E_p > 0$ such that if R_p denotes the component of $M \cdot S(P, E_p)$ containing P, where $S(P, E_p)$ denotes the set of all points whose distance from P is $\leqq E_p$, then P is a cut point of R_p. There exists a number $e > 0$ and an uncountable subset N_1 of N such that for each point P_1 of N_1, $E_{p_1} > 4e$. There exists a point Q of N_1 which is a point of condensation of N_1. Let W denote the component of $M \cdot S(Q, e)$ which contains Q. Since M is connected im kleinen at Q and Q is a point of condensation of N_1, there exists an uncountable subset N_2 of N_1 every point of which belongs to W and is at a distance $< e/2$ from Q. Then for each point X of N_2, R_x must contain W, because $E_x > 4e$. Hence, by a lemma of R. L. Moore's,* each point of N_2 is a cut point of W. Then by the theorem of the author's stated in the first paragraph of this section it follows that there exists at least one point Y of N_2 which is a point of Menger order two of W. And since Y is not a limit point of $M - W$ (for M is connected im kleinen at Y), then Y is a point of order two of M, contrary to hypothesis. Thus the supposition that Theorem 13 is false leads to a contradiction.

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of plane continua, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 85–88.

* *Concerning triods* · · · , loc. cit., Lemma 2.

COROLLARY 1. *All save possibly a countable number of the cut points im kleinen of any continuous curve M are points of Menger order two of M.*

COROLLARY 2. *The junction points* of any continuous curve are countable.*

THEOREM 14. *If every subcontinuum of a continuous curve M contains uncountably many im kleinen cut points of M, then M is a Menger regular curve.*†

Theorem 14 follows at once from Theorem 13 and Menger's theorem (loc. cit., Theorem 8) that the set of all non-regular points of any continuum either is vacuous or else contains a continuum. For the case where M lies in the plane, Theorem 13 is related to Theorem 11 in the author's paper *Concerning certain accessible points of plane continua.*‡

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* A junction point of a continuum M is a cut point im kleinen of M which is the emanation point of some triod which lies in M, where by a triod is meant a continuum which is the sum of three continua AO, BO and CO, irreducible between the points A and O, B and O, and C and O respectively, and such that O is the only point common to any two of them, and by the emanation point of a triod is meant the point corresponding to the point O in the definition just given. (See R. L. Moore, *Concerning triods* · · ·, loc. cit.) Moore shows that the junction points of any *plane* continuum are countable, whether M is a continuous curve or not. That Moore's theorem is not true in 3-space is shown by the following example. Let K be a non-dense perfect set on an interval I of the X-axis. For each point X of K, with X-coordinate x, let T_x denote the triod which is the sum of the straight line interval joining the points $(x, 0, 0)$ and $(x, 2, 0)$ and the one joining $(x, 1, 0)$ and $(x, 1, 1)$. Let $M = I + \sum_{x \subset K} T_x$. Then for each point X of K, the point $(x, 1, 0)$ is a junction point of the continuum M. Corollary 2 shows that the condition in Moore's theorem that the continuum M lie in the plane may be replaced by the condition that M be a continuous curve. Incidentally, Corollary 2 gives a more general result than the results of Wazewski-Menger and Alexandroff referred to in the introduction of the above mentioned paper of Moore's, even in n dimensions or in any locally compact, metric, and separable space.

† A Menger regular curve is a continuum M all of whose points are regular in the sense of Menger, that is, each point of M can, for each $\epsilon > 0$, be ϵ-separated by a finite number of points of M; see K. Menger, *Grundzüge einer Theorie der Kurven*, loc. cit.

‡ Monatshefte für Mathematik und Physik, vol. 35 (1928), pp. 289–304.

COROLLARY. *If every point of a continuous curve M is a cut point im kleinen of M, (or is an unavoidable point of M in the sense of Urysohn), then M is a Menger regular curve.**

A continuous curve having the property stated in this corollary, that is, every one of its points is an im kleinen cut point, has a number of interesting simple properties, such as, for example, every subcontinuum of any one of its maximal cyclic curves C contains an arc segment which is an open subset of C.

Using the theorems in the preceding sections together with methods similar to those used in the proof of Theorem 13, it follows that *if G is any collection of mutually exclusive compact im kleinen cut continua of a continuous curve M, then all save possibly a countable number of the continua of G are continua of "Verzweigungs" order two of M.* Likewise if we follow R. L. Moore (loc. cit.) and call a subcontinuum N of a continuum M a *junction continuum* of M provided N is an im kleinen cut continuum of M and is the emanation continuum of some *analog of a triod*† belonging to M, then *no continuous curve contains an uncountable collection of mutually exclusive compact junction continua of itself.*

For the case of the plane, it follows from the results of this paper by methods of proof essentially the same as used by R. L. Moore in proving Theorem 5 of his paper *Concerning triods in the plane and the junction points of plane continua* (loc. cit.) that *no plane continuum contains an uncountable collection of mutually exclusive compact junction continua of itself.*

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* For the case where M lies in the plane, this corollary is closely related to Theorem 2 of my paper *Concerning accessibility in the plane and regular accessibility in n dimensions*, this Bulletin, vol. 34 (1928), pp. 504-510.

† See R. L. Moore, *Concerning triodic continua in the plane*, to appear in Fundamenta Mathematicae. By an *analog of a triod* is meant a continuum which differs from a triod only in that it emanates from a continuum instead of from a point.

APPROXIMATE SOLUTIONS OF CERTAIN GENERAL TYPES OF BOUNDARY PROBLEMS FROM THE STANDPOINT OF INTEGRAL EQUATIONS*

BY ROSE WHELAN

1. *Introduction.* The principle of Rayleigh†, which is used by physicists to obtain practical solutions of certain boundary problems, assumes that the system under consideration may be replaced by an approximating algebraic system so chosen that the difference between the solutions of the two systems is negligible. It is evident that the practical value of this principle depends not only on the possibility of choosing an approximating system which can be easily solved, but also on the possibility of estimating the order of difference between the solutions.

In a paper published in 1923, M. Plancherel‡ presented a justification of the use of the difference system as an approximating system for a second order self adjoint linear differential equation with simple boundary conditions. He does not discuss the order of approximation of solutions. Such a discussion has been given by N. Bogoliouboff and N. Kryloff.§ R. Courant,‖ in a paper which appeared in 1926, showed that certain integro-differential boundary problems could also be approximated by the method of

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* Presented to the Society, September 7, 1928. This paper is an abbreviated version of a thesis submitted to Brown University in candidacy for the degree of Doctor of Philosophy.

† Lord Rayleigh, *Theory of Sound*, vol. 1, pp. 89–96.

‡ M. Plancherel, Bulletin des Sciences Mathématiques, (2), vol. 47 (1923), pp. 153-160, 170–177.

§ N. Bogoliouboff and N. Kryloff, Annals of Mathematics, (2), vol. 29 (1928), pp. 255–275.

‖ R. Courant, Acta Mathematica, vol. 49 (1926), pp. 1–67.

difference equations. In this paper we use an essentially different method.* The class of problems to which it can be applied is not restricted to the class of problems that originated in the calculus of variations, with which Courant has been concerned. In some respects, however, the method used by Courant is more general since it can be applied to non-linear as well as to partial differential boundary problems.

2. *Preliminary Theorems: Approximate Solutions of the Fredholm Integral Equation.*† We obtain for reference in this and later sections an estimate of the differences between the characteristic numbers of the Fredholm integral equation

$$(A_1) \qquad u(x) + \lambda \int_a^b k(x, s)u(s)ds = f(x),$$

and those of

$$(A_2) \qquad u(x,\epsilon) + \lambda \int_a^b k(x,s,\epsilon)u(s,\epsilon)ds = f(x,\epsilon),$$

where, in the region $a \leqq \overset{x}{s} \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, the functions $k(x, s, \epsilon)$ and $k(x, s)$ are bounded and integrable (in the Lebesgue sense) and

$$k(x,s,\epsilon) - k(x,s) = O(\epsilon)\ ; \quad f(x,\epsilon) - f(x) = O(\epsilon).\ddagger$$

If ϵ_0 is sufficiently small, then

$$k(x,s,\epsilon) - k(x,s) = \epsilon h(x,s,\epsilon)$$

where $h(x, s, \epsilon)$ is uniformly bounded in the region $a \leqq \overset{x}{s} \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, and $f(x, \epsilon) - f(x) = \epsilon h(x, \epsilon)$, where $h(x, \epsilon)$ is uniformly

* This method was suggested by Professor J. D. Tamarkin to whom the author is indebted for advice and criticism during the preparation of this paper.

† E. Schmidt, Mathematische Annalen, vol. 63, pp. 467–472, and vol. 64, pp. 161–174; H. Bateman, Proceedings of the Royal Society, A, vol. 100, pp. 441–449; F. Tricomi, R. Accademia dei Lincei, vol. 33, sem. 1, pp. 483–486, and vol. 33, sem. 2, pp. 26–30; R. B. Adams, Thesis in Candidacy for the degree of Doctor of Philosophy submitted to Radcliffe College, 1921; H. Block, Lunds Universitets Årsskrift, N. F. Afd. 2, 7, no. 1, (1911); G. C. Evans, American Mathematical Monthly, vol. 34, pp. 148–150.

‡ By definition $f(\epsilon) = O(\epsilon^\alpha)$ if $f(\epsilon)/\epsilon^\alpha$ is bounded.

bounded in the region $a \leqq x \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$. The integral equation

$$(1) \qquad u(x,\rho,\epsilon) + \lambda \int_a^b k(x,s,\rho,\epsilon)u(s,\rho,\epsilon)ds = f(x,\rho,\epsilon)$$

in which

$$k(x,s,\rho,\epsilon) \equiv k(x,s) + \rho h(x,s,\epsilon),$$
$$f(x,\rho,\epsilon) \equiv f(x) + \rho h(x,\epsilon),$$

reduces to (A_1) when $\rho = 0$ and to (A_2) when $\rho = \epsilon$. For any finite value of ρ, the Fredholm determinant and the Fredholm minors of $k(x, s, \rho, \epsilon)$ exist and are analytic for all finite values of λ. Also since $k(x, s, \rho, \epsilon)$ is analytic in ρ and possesses a resolvent for the particular value $\rho = 0$, the Fredholm determinant $D(\lambda, \rho, \epsilon)$ and the Fredholm minors

$$D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda, \rho, \epsilon\Big), \qquad (\mu = 1,2,3, \cdots),$$

are analytic for all finite values of ρ, and are not identically zero.*

Let $\lambda = \lambda_0$ be a characteristic number of multiplicity p and index q† associated with the kernel $k(x, s, 0, \epsilon) \equiv k(x, s)$. Then‡

$$(2) \qquad D(\lambda,0,\epsilon) \equiv D(\lambda) = (\lambda - \lambda_0)^p d(\lambda), \qquad d(\lambda_0) \neq 0;$$

and

$$(3) \qquad D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda, 0, \epsilon\Big) \equiv D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda\Big)$$
$$= (\lambda - \lambda_0)^{q-\mu} d_\mu\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda\Big), \quad (\mu = 1,2, \cdots, q),$$

where, in general,

$$d_\mu\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda_0\Big) \neq 0.$$

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* See, e.g. J. D. Tamarkin, Annals of Mathematics, (2), vol. 28 (1927), pp. 127–152.

† A characteristic number is said to be of index q if there are exactly q characteristic functions associated with it.

‡ These formulas follow as a direct consequence of the assumption that λ_0 is a characteristic number of multiplicity p and index q.

Also, for all values of μ, including $\mu=0$,*

$$\partial D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big)/\partial\rho$$

$$= \sum_{i,j=1}^{\mu} (-1)^{i+j} h(x_i,s_j,\epsilon) D\begin{pmatrix} x_1 \cdots x_{i-1}x_{i+1}\cdots x_\mu \\ s_1 \cdots s_{j-1}\, s_{j+1}\cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big)$$

$$+ \lambda D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big) \int_a^b h(\xi,\xi,\epsilon)\,d\xi$$

$$(4)\quad - \lambda \sum_{i=1}^{\mu} \int_a^b h(x_i,\xi,\epsilon) D\begin{pmatrix} x_1 \cdots x_{i-1}\xi x_{i+1}\cdots x_\mu \\ s_1 \cdots s_{i-1}s_i s_{i+1}\cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big)\,d\xi$$

$$- \lambda \sum_{i=1}^{\mu} \int_a^b D\begin{pmatrix} x_1 \cdots x_{i-1}x_i x_{i+1}\cdots x_\mu \\ s_1 \cdots s_{i-1}\,\xi\, s_{i+1}\cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big) h(\xi,s_i,\epsilon)\,d\xi$$

$$- \lambda^2 \int_a^b h(\xi,\eta,\epsilon) D\begin{pmatrix} x_1 \cdots x_{\mu\eta} \\ s_1 \cdots s_{\mu\xi} \end{pmatrix},\lambda,\rho,\epsilon\Big)\,d\xi d\eta.$$

Assume that there is no characteristic number of $k(x, s)$ other than $\lambda=\lambda_0$ within or on the boundary of the region $|\lambda-\lambda_0| \leqq \delta_1$. The analytic function $D(\lambda, \rho, \epsilon)$ is represented in the region $|\lambda-\lambda_0| \leqq \delta_1$; $0 \leqq \rho \leqq \delta_2 < 1$ by the convergent power series

$$(5)\quad D(\lambda,\rho,\epsilon) = \sum_{\alpha,\beta=0}^{\infty} A_{\alpha,\beta}(\epsilon)(\lambda-\lambda_0)^\alpha \rho^\beta, \qquad |A_{\alpha,\beta}(\epsilon)| \leqq A,$$

where A is a finite constant.

It follows from (2), (3), and (4) that

$$A_{\alpha,0} = 0,\ \alpha < p, \qquad A_{\alpha,\beta} = 0,\ \alpha+\beta < q, \qquad A_{p,0} \neq 0,$$

and, in general,

$$A_{\alpha,\beta} \neq 0, \quad \alpha+\beta = q.$$

We shall assume for the present that $A_{q-1,1} \neq 0$ and, by collecting terms of the same degree, express (5) in the form

* G. C. Evans, loc. cit.

$$D(\lambda,\rho,\epsilon) - D(\lambda) =$$
$$[A_{q-1,1}(\lambda-\lambda_0)^{q-1}\rho + A_{q-2,2}(\lambda-\lambda_0)^{q-2}\rho^2 + \cdots + A_{0,q}\rho^q]$$
$$+ [A_{q,1}(\lambda-\lambda_0)^q\rho + A_{q-1,2}(\lambda-\lambda_0)^{q-1}\rho^2 + \cdots + A_{0,q+1}\rho^{q+1}]$$
$$+ \cdots .$$

Consequently, for $\lambda-\lambda_0$ and ϵ sufficiently small,*

$$|D(\lambda,\epsilon) - D(\lambda)| < 2A\epsilon[|\lambda-\lambda_0| + \epsilon]^{q-1}.$$

Hence

$$D(\lambda,\epsilon) = D(\lambda)[1 + \psi(\lambda,\epsilon)]$$

where

$$|\psi(\lambda,\epsilon)| < 2A\epsilon[|\lambda-\lambda_0| + \epsilon]^{q-1}|D(\lambda)|_1^{-1}.$$

Consider a circle (γ) about λ_0 of radius $r \leqq \delta_1$. Since $d(\lambda)$ in (2) can not vanish on (γ), then on (γ) $|d(\lambda)| \geqq d_0^{-1} > 0$ where d_0 is a constant different from zero. If now we choose r so that

$$2Ad_0\epsilon[r+\epsilon]^{q-1}r^{-p} = C < 1 \tag{6}$$

then on (γ), $|\psi(\lambda,\epsilon)| < 1$ and the equation $D(\lambda,\epsilon) = 0$ will have the same number of zeros within (γ) as $D(\lambda) = 0$. It is readily seen that (6) implies $r = O(\epsilon^{1/(p-q+1)})$.

THEOREM 1. *If $\lambda = \lambda_0$ is a characteristic number of (A_1) of multiplicity p and index q, then for a fixed ϵ sufficiently small (A_2) will possess exactly p characteristic numbers $\lambda = \lambda_{0,i}(\epsilon)$ $(i = 1, 2, \cdots, p)$ such that $\lambda_0 - \lambda_{0,i}(\epsilon) = O(\epsilon^{1/(p-q+1)})$ at least.*

By an analogous method it is possible to discuss the special cases that arise when $A_{q-1,1} = 0$. We shall find that if any of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta = q$, are different from zero, then $r = O(\epsilon^{1/(p-q+1)})$; if all of the coefficients $A_{\alpha,\beta}, \alpha+\beta = q$, vanish, but one of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta = q+1$, is different from zero, then $r = O(\epsilon^{1/(p-q)})$; if $A_{\alpha,\beta}$, $\alpha+\beta < q+2$, vanish, but one of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta = q+2$ is different from zero, then $r = O(\epsilon^{1/(p+q-1)})$; and so forth.

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* $D(\lambda,\epsilon,\epsilon) \equiv D(\lambda,\epsilon)$.

THEOREM 2. *If* $\{\phi_j(x)\}$ $(j=1, 2, \cdots, q)$ *is a normalized set of characteristic functions of* (A_1) *associated with the characteristic number* $\lambda=\lambda_0$ *and* $\phi(x, \epsilon)$ *is a normalized characteristic function of* (A_2) *associated with the characteristic number* $\lambda=\lambda_{0,i}(\epsilon)$, *then there exists an approximately normalized* linear combination* $\sum_{j=1}^{q} c_j(\epsilon)\phi_j(x)$ *such that*

$$\phi(x,\epsilon) - \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) = O(\epsilon^{1/(p-q+1)})$$

at least.

Consider the equation

$$\phi(x,\epsilon)+\lambda_{0,i}(\epsilon)\int_a^b k(x,s,\epsilon)\phi(s,\epsilon)ds = 0, \tag{7}$$

where

$$\int_a^b |\phi(x,\epsilon)|^2 dx = 1.$$

This may be written as

$$\phi(x,\epsilon) + \lambda_0 \int_a^b k(x,s)\phi(s,\epsilon)ds = \Omega(x,\epsilon), \tag{8}$$

where

$$\Omega(x,\epsilon) = \lambda_0 \int_a^b [k(x,s) - k(x,s,\epsilon)]\phi(s,\epsilon)ds + [\lambda_0 - \lambda_{0,i}(\epsilon)] \int_a^b k(x,s,\epsilon)\phi(s,\epsilon)ds.$$

Since

$$\lambda_{0,i}(\epsilon) - \lambda_0 = O(\epsilon^{1/(p-q+1)}), \quad k(x,s,\epsilon) - k(x,s) = O(\epsilon),$$

and since $k(x, s, \epsilon)$ is bounded and $\phi(x, \epsilon)$ is normalized, we have $\Omega(x,\epsilon) = O(\epsilon^{1/(p-q+1)})$ at least.

But (8) is a non-homogeneous equation in which λ_0 is a characteristic number and $\phi(x, \epsilon)$ is a solution. Accordingly, if $R(x, s)$ is the pseudo-resolvent of $k(x, s)$, then

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* A function $\omega(x, \epsilon)$ is said to be approximately normalized if $\int_a^b |\omega(x, \epsilon)|^2 dx = 1+\eta(\epsilon)$ where $\eta(\epsilon)\to 0$ as $\epsilon\to 0$.

$$\phi(x,\epsilon) = \Omega(x,\epsilon) - \lambda_0 \int_a^b R(x,s)\Omega(s,\epsilon)ds + \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x).$$

Since $R(x, s)$ is bounded,

$$(9) \quad \phi(x,\epsilon) = \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) + \eta(x,\epsilon), \quad \eta(x,\epsilon) = O(\epsilon^{1/(p-q+1)}),$$

which yields the result

$$\int_a^b \left| \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) \right|^2 dx = 1 + \bar{\eta}(x,\epsilon), \quad \bar{\eta}(x,\epsilon) = O(\epsilon^{1/(p-q+1)}).$$

THEOREM 3. *If $u(x)$ represents a solution of (A_1) corresponding to a fixed value of the parameter which is not a characteristic number, then for ϵ sufficiently small and the same value of the parameter there is a solution $u(x, \epsilon)$ of (A_2) such that*

$$u(x,\epsilon) - u(x) = O(\epsilon).$$

If λ is not a characteristic value of (A_1), then from Theorem 1 it follows that for ϵ sufficiently small λ is not a characteristic value of (A_2). We may write (A_2) as

$$u(x,\epsilon) + \lambda \int_a^b k(x,s)u(s,\epsilon)ds = g(x,\epsilon),$$

$$(x,\epsilon) = f(x,\epsilon) - \lambda \int_a^b [k(x,s,\epsilon) - k(x,s)]u(s,\epsilon)ds.$$

Let $\mathfrak{k}(x, s, \lambda)$ represent the resolvent of $k(x, s)$. Then

$$u(x,\epsilon) + \lambda \int_a^b \mathfrak{k}(x,s,\lambda)g(s,\epsilon)ds = g(x,\epsilon),$$

or

$$u(x,\epsilon) + \lambda \int_a^b \mathfrak{k}(x,s,\lambda,\epsilon)u(s,\epsilon)ds = g(x,\lambda,\epsilon)$$

where

$$\mathfrak{k}(x,s,\lambda,\epsilon) = k(x,s,\epsilon) - k(x,s) - \lambda \int_a^b \mathfrak{k}(x,\xi,\lambda)[k(\xi,s,\epsilon) - k(\xi,s)]d\xi,$$

$$g(x,\lambda,\epsilon) = f(x,\epsilon) - \lambda \int_a^b \mathfrak{k}(x,s,\lambda)f(s,\epsilon)ds.$$

Since $\mathfrak{k}(x, s, \lambda, \epsilon) = O(\epsilon)$, the Neumann series representing the resolvent of $\mathfrak{k}(x, s, \lambda, \epsilon)$ will converge and for ϵ sufficiently small will also be $O(\epsilon)$. Consequently, $u(x, \epsilon)$ is bounded.

If we set $v(x,\epsilon) = u(x, \epsilon) - u(x)$, then from (A_1) and (A_2) it follows that

$$v(x,\epsilon) + \lambda \int_a^b k(x,s)v(s,\epsilon)ds = \eta(x,\lambda,\epsilon),$$

where

$$\eta(x,\lambda,\epsilon) = f(x,\epsilon) - f(x) - \lambda \int_a^b [k(x,s,\epsilon) - k(x,s)]u(s,\epsilon)ds = O(\epsilon).$$

Hence

$$v(x,\epsilon) = \eta(x,\lambda,\epsilon) - \lambda \int_a^b \mathfrak{k}(x,s,\lambda)\eta(s,\lambda,\epsilon)ds = O(\epsilon),$$

which proves Theorem 3.

3. *Reduction of the Differential System and the Associated Difference System to Fredholm Integral Equations.* We consider the differential system represented in matrix notation*

$$L(Y\cdot) + \lambda Q(x)Y\cdot = F(x)\cdot,$$

$$(B_1) \qquad U(Y\cdot) \equiv W_0Y(0)\cdot + W_1Y(1)\cdot = 0, \ (0 \leqq x \leqq 1),$$

$$L(Y\cdot) \equiv dY\cdot/dx - A(x)Y\cdot,$$

where $A(x)$, $Q(x)$, and $F(x)\cdot$ are matrices whose elements satisfy a Lipschitz condition; W_0 and W_1 are matrices of constants; and λ is a scalar parameter.

Let the interval (0, 1) be divided into m equal parts by the points $x_0 \equiv 0, x_1, x_2, \cdots, x_m \equiv 1$. Then associated with the system (B_1) is a difference system

* Throughout the remainder of this paper capital letters—with the single exception of O—will be used to represent matrices with n rows and n columns. A dot following a capital indicates that the columns of the matrix are identical.

$$L(Y_l\cdot) + \lambda Q(x_l)Y_l\cdot = F(x_l)\cdot,$$

$$(B_2) \qquad U(Y_l\cdot) \equiv W_0Y_0\cdot + W_1Y_m\cdot = 0, \quad (l = 1,2,\cdots,m),$$

$$L(Y_l\cdot) \equiv \Delta Y_{l-1}\cdot/\Delta x - A(x_l)Y_l\cdot\ ;\ \ \Delta Y_l\cdot = Y_{l+1}\cdot - Y_l\cdot,$$

$$\Delta x = x_{l+1} - x_l = 1/m.$$

We may assume without loss of generality that $\lambda = 0$ is not a characteristic number of either (B_1) or (B_2). Then for an arbitrary $F(x)\cdot$ there exists a unique solution of the system

$$(10) \qquad \begin{aligned} L(Y\cdot) &= F(x)\cdot, \\ U(Y\cdot) &= 0, \end{aligned} \qquad (0 \leqq x \leqq 1),$$

and this solution can be expressed in the form

$$Y(x)\cdot = \int_0^1 G(x,s)F(s)\cdot ds,$$

where $G(x, s)$ is called the Green's matrix of the system and is given explicitly by the formula*

$$G(x,s) = \overline{G}(x,s) - Y_h(x)\left[U(Y_h(x)\right]^{-1}U\left[\overline{G}(x,s)\right],$$

where

$$\overline{G}(x,s) = \begin{cases} \frac{1}{2}Y_h(x)\left[Y_h(s)\right]^{-1}, & s < x \\ -\frac{1}{2}Y_h(x)\left[Y_h(s)\right]^{-1}, & s > x \end{cases}\ ;\ \ L(Y_h(x)) = 0.$$

Hence

$$(11) \qquad Y(x)\cdot + \lambda\int_0^1 K(x,s)Y(s)\cdot ds = \overline{F}(x)\cdot,$$

where

$$K(x,s) = G(x,s)Q(s), \qquad \overline{F}(x)\cdot = \int_0^1 G(x,s)F(s)\cdot ds,$$

is equivalent to (B_1). In the ordinary notation, the system (11) represents a system of n linear integral equations

---

* Birkhoff and Langer, Proceedings American Academy of Arts and Sciences, vol. 58 (1923), pp. 51–128.

$$
\begin{aligned}
&y_i(x) + \lambda \int_0^1 \sum_{j=1}^{n} k_{i,j}(x,s)\, y_j(s)\, ds = \bar{f}_i(x), \\
&k_{i,j}(x,s) = \sum_{\alpha=1}^{n} g_{i,\alpha}(x,s)\, q_{\alpha,j}(s), \quad (i,j = 1,2, \cdots, n).
\end{aligned} \tag{12}
$$

If we define

$$
\begin{aligned}
k(x,s) &= k_{i,j}(x - i + 1,\ s - j + 1), \quad 0 < \frac{x - i + 1}{s - j + 1} < 1, \\
f(x) &= \bar{f}_i(x - i + 1), \qquad\qquad 0 < x - i + 1 < 1,
\end{aligned}
$$

then

$$
y(x) + \lambda \int_0^n k(x,s)\, y(s)\, ds = f(x), \tag{13}
$$

where $y(x) = y_i(x-i+1)$, $0 < x-i+1 < 1$, is equivalent to (12).*

It is also possible by means of a Green's matrix to reduce the system (B_2) to a single Fredholm integral equation. For this purpose we shall need the following lemmas, the proofs of which do not present any difficulty:

i. *If* $Y_h^{(m)}(x_l)$, $(l = 0, 1, 2, \cdots, m)$, *is a solution of the system*

$$
L(Y_l) = 0, \qquad (l = 1,2, \cdots, m), \tag{14}
$$

and if† $\left| Y_h^{(m)}(1) \right| \neq 0$, *there exists a number* m_0 *such that for all* $m > m_0$, $\left| Y_h^{(m)}(x_l) \right| \neq 0$, $(l = 0, 1, 2, \cdots, m)$.

ii. *If* $\left| Y_h^{(m)}(1) \right| \neq 0$, *then for a sufficiently large* m *the most general solution of* (14) *is* $Y_h^{(m)}(x_l)C$, $(l = 0, 1, 2, \cdots, m)$, *where* C *is a matrix of constants.*

iii. *If* $\overline{Y}^{(m)}(x_l)$, $(l = 0, 1, 2, \cdots, m)$, *is a particular solution of the system*

$$
L(Y_l) = F(x_l), \qquad (l = 1,2, \cdots, m), \tag{15}
$$

---

* I. Fredholm, Acta Matematica, vol. 27 (1903), pp. 378–379.

† The notation $|A|$ is used to indicate the determinant of the matrix A.

and if $|Y_k^{(m)}(1)| \neq 0$, *then for m sufficiently large*

$$Y^{(m)}(x_l) = \overline{Y}^{(m)}(x_l) + Y_k^m(x_l)C, \qquad (l = 0,1,2, \cdots, m),$$

is the most general solution of (15).

iv. *A particular solution of* (15) *is*

$$\overline{Y}^{(m)}(x_l) = \sum_{k=1}^{m} \overline{G}^{(m)}(x_l, x_{k-1})F(x_k)\Delta x, \qquad (l = 0,1,2, \cdots, m),$$

where

$$\overline{G}^{(m)}(x_l, x_{k-1}) = \begin{cases} \frac{1}{2}Y_k^{(m)}(x_l)\left[Y_k^{(m)}(x_{k-1})\right]^{-1}, & k \leqq l, \\ -\frac{1}{2}Y_k^{(m)}(x_l)\left[Y_k^{(m)}(x_{k-1})\right]^{-1}, & k > l. \end{cases}$$

By an argument closely analogous to that used by Birkhoff and Langer* for the differential system, it is readily proved that there exists a unique solution of the system

$$(16) \qquad L(Y_l \cdot) = F(x_l)\cdot, \quad U(Y_l \cdot) = 0, \quad (l = 1,2, \cdots, m),$$

and that this solution can be expressed in the form

$$Y^{(m)}(x_l)\cdot = \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})F(x_k)\cdot\Delta x, \quad (l = 0,1,2, \cdots, m),$$

where $G^{(m)}(x_l, x_{k-1})$ is called the Green's matrix of the system and is given explicitly by the formula

$$G^{(m)}(x_l, x_{k-1}) = \overline{G}^{(m)}(x_l, x_{k-1}) - Y_k^{(m)}(x_l)\left\{U\left[Y_k^{(m)}(x_l)\right]\right\}^{-1}U\left[\overline{G}^{(m)}(x_l, x_{k-1})\right], \quad (l = 0,1,2, \cdots, m;\ k = 1,2, \cdots, m).$$

Consequently the system

$$(17) \quad Y^{(m)}(x_l)\cdot + \lambda \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})Q(x_k)Y^{(m)}(x_k)\cdot\Delta x = \overline{F}^m(x_l)\cdot,$$

where

$$\overline{F}^{(m)}(x_l)\cdot = \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})F(x_k)\cdot\Delta x,$$

is equivalent to (B_2).

* Birkhoff and Langer, loc. cit.

On setting $G^{(m)}(x_l, x_{k-1}) = G^{(m)}(x, s)$ in the intervals

$$I_x: \quad x_{l-1} < x < x_l, \qquad (l = 1,2, \cdots, m),$$
$$I_s: \quad x_{k-1} < s < x_k, \qquad (k = 1,2, \cdots, m),$$

and

$$\overline{F}^{(m)}(x_l)\cdot = \overline{F}^{(m)}(x)\cdot, \quad Q(x_l) = Q^{(m)}(x), \quad Y^{(m)}(x_l)\cdot = Y^{m}(x)\cdot,$$

in the interval I_x, we reduce (17) to an equivalent system of integral equations

$$(18) \qquad Y^{(m)}(x)\cdot + \lambda \int_0^1 K^{(m)}(x,s)Y^{(m)}(s)\cdot ds = \overline{F}^{(m)}(x)\cdot,$$
$$K^{(m)}(x,s) = G^{(m)}(x,s)Q^{(m)}(s).$$

Then (18) is equivalent to

$$(19) \qquad y^{(m)}(x) + \lambda \int_0^n k^{(m)}(x,s)y^{(m)}(s)ds = f^{(m)}(x),$$

where

$$k^{(m)}(x,s) = k_{i,j}(x - i + 1, s - j + 1), \quad 0 < \frac{x - i + 1}{s - j + 1} < 1,$$
$$f^{(m)}(x) = \bar{f}_i^{(m)}(x - i + 1),$$
$$y^{(m)}(x) = y_i^{(m)}(x - i + 1), \qquad 0 < x - i + 1 < 1.$$

As a final step it is necessary to estimate the difference between the kernels $k(x, s)$ and $k^{(m)}(x, s)$. For this purpose we shall need the additional lemmas:

v. *If $U_1(x)$ and $U_2(x)$ are continuous matrices which, except at a finite number of points, satisfy the relations*

$$dU_1/dx = A(x)U_1 + \Omega_1, \quad U_1(1) = C_0,$$
$$dU_2/dx = A(x)U_2 + \Omega_2, \quad U_2(1) = C_0,$$

where $A(x) \ll a_0$; $\Omega_1 \ll \epsilon_1$ and $\Omega_2 \ll \epsilon_2$, ϵ_1 and ϵ_2 being positive scalars; then*

* $A << B$ indicates that each element of the matrix A is less in absolute value than the corresponding element of the matrix B. Similarly, $A << b$ indicates that each element of the matrix A is less in absolute value than the scalar b.

$$U_1(x) - U_2(x) \ll (\epsilon_1 + \epsilon_2)(e^{a_0 n x} - 1)/a_0 n.*$$

vi. *The elements of the matrices*

$$Y_k(x), \quad [Y_k(x)]^{-1}, \quad U[Y_k(x)], \quad \{U[Y_k(x)\}^{-1}$$

are bounded; the same is true of the elements of the matrices

$$Y_k^{(m)}(x_l), \quad [Y_k^{(m)}(x_l)]^{-1}, \quad U[Y_k^{(m)}(x_l)], \quad \{U[Y_k^{(m)}(x_l)]\}^{-1},$$

$$U[\overline{G}^{(m)}(x_l, x_{k-1})], \quad (l = 0,1,2, \cdots, m\ ;\ k = 1,2, \cdots, m),$$

for m sufficiently large.†

vii. $Y_k^{(m)}(x_l) - Y_k(x) = O(1/m)$ *in* I_x.‡

Let the continuous matrix $Y_k^{(m)}(x)$ be defined in I_x by the relation

$$dY_k^{(m)}(x)/dx = \Delta Y_k^{(m)}(x_{l-1})/\Delta x = A(x_l)Y_l,$$

where $Y_l = Y_k^{(m)}(x_l)$. Then

$$dY_k^{(m)}(x)/dx = A(x)Y_k^{(m)}(x) + \Omega^{(m)}(x),$$

where

$$\Omega^{(m)}(x) = A(x_l)[Y_k^{(m)}(x_l) - Y_k^{(m)}(x)] + [A(x_l) - A(x)]Y_k^{(m)}(x).$$

It is evident that

$$\Omega^{(m)}(x) = O(1/m).$$

The lemma now follows as a consequence of (v).

viii. $\overline{G}^{(m)}(x_l, x_{k-1}) - \overline{G}(x,s) = O(1/m)$ *in* I_x, I_s.

ix. $U[Y_k^{(m)}(x_l)] - U[Y_k(x)] = O(1/m)$.

x. $\{U[Y_k^{(m)}(x_l)]\}^{-1} - \{U[Y_k(x)]\}^{-1} = O(1/m)$.

---

* The method of proof is similar to that used in the case of a single differential equation. (Ch. J. de la Vallée Poussin, *Cours d'Analyse Infinitéesimale*, vol. 2, ed. 4, p. 135.)

† See Birkhoff and Langer, loc. cit.

‡ $A(\epsilon) = O(\epsilon)$ if each element of $A(\epsilon)$ is $O(\epsilon)$.

Combining lemmas (v) to (x) and using the fact that $Q(x)$ and $F(x)\cdot$ satisfy a Lipschitz condition, we obtain the final lemmas:

xi. $G^{(m)}(x_l, x_{k-1}) - G(x,s) = O(1/m)$ *in* I_x, I_s.

xii. $$K^{(m)}(x,s) - K(x,s) = O(1/m)\ ;$$

$$k^{(m)}(x,s) - k(x,s) = O(1/m)\ ;$$

$$f^{(m)}(x) - f(x) = O(1/m).$$

Equations (13) and (19) can now be treated by methods developed in §2.

4. *Reduction of the Integro-Differential System and the Associated Difference System to Fredholm Integral Equations.* Let us consider the integro-differential system

$$(C_1)\qquad \begin{aligned} L(Y\cdot) + \lambda Q(x)Y\cdot &= F(x)\cdot + \int_0^1 H(x,s)Y(s)\cdot ds, \\ U(Y\cdot) &= 0, \end{aligned}$$

where $L(Y\cdot)$, $U(Y\cdot)$, $Q(x)$, $F(x)\cdot$, and λ have been defined above, and $H(x, s)$ satisfies a Lipschitz condition in (x, s).

Associated with this system is a difference system

$$(C_2)\qquad \begin{aligned} L(Y_l\cdot) + \lambda Q(x_l)Y_l\cdot &= F(x_l)\cdot + \sum_{k=0}^{m} H(x_l, x_k)Y_k\cdot \Delta x, \\ U(Y_l\cdot) &= 0, \qquad (l = 1, 2, \cdots, m). \end{aligned}$$

We can assume without loss of generality that $\lambda = 0$ is not a characteristic number* of either (C_1) or the differential system

* The discussion of characteristic numbers and characteristic functions of a single integro-differential equation satisfying boundary conditions, has been given by J. D. Tamarkin (Transactions of this Society, vol. 29 (1927), pp. 775–800). His results have been extended to a system of integro-differential equations by F. C. Jonah in a thesis to be submitted to Brown University in candidacy for the degree of Doctor of Philosophy.

(B_1). Since $\lambda = 0$ is not a characteristic number of (C_1), the system

$$(20)\qquad \begin{aligned} L(Y\cdot) &= F(x)\cdot + \int_0^1 H(x,s)Y(s)\cdot ds, \\ U(Y\cdot) &= 0, \end{aligned}$$

possesses a unique solution for an arbitrary $F(x)\cdot$ and this can be expressed in the form $Y(x)\cdot = \int_0^1 \Gamma(x, s)F(s)\cdot ds$. The matrix $\Gamma(x, s)$ is uniquely determined at its points of continuity and is called the Green's matrix of (20). It is readily seen that $\Gamma(x, s)$ is given explicitly by the formula

$$\Gamma(x,s) = G(x,s) + \int_0^1 \mathfrak{H}(x,\xi)G(\xi,s)d\xi,$$

if $\mathfrak{H}(x, s)$ is the resolvent of the matrix $\overline{H}(x, s)$, and

$$Y(x)\cdot = \int_0^1 \overline{H}(x,s)Y(s)\cdot ds + \overline{F}(x)\cdot,$$

$$(21)\qquad \overline{H}(x,s) = \int_0^1 G(x,\xi)H(\xi,s)d\xi,$$

$$\overline{F}(x)\cdot = \int_0^1 G(x,s)F(s)\cdot ds.$$

Hence

$$(22)\quad Y(x)\cdot + \lambda\int_0^1 \Gamma(x,s)Q(s)Y(s)\cdot ds = \int_0^1 \Gamma(x,s)F(s)\cdot ds$$

is equivalent to (C_1).

Similarly, an explicit expression for the Green's matrix of the system

$$(23)\qquad \begin{aligned} L(Y_l\cdot) &= F(x_l)\cdot + \sum_{k=0}^{m} H(x_l,x_k)F(x_k)\cdot\Delta x, \\ U(Y_l\cdot) &= 0, \qquad\qquad (l = 1,2,\cdots,m), \end{aligned}$$

can be obtained. Since $\lambda = 0$ is not a characteristic number of

(B_1), then from Theorem 1 it follows that for m sufficiently large $\lambda = 0$ is not a characteristic number of (B_2). Hence

$$Y^{(m)}(x_l)\cdot = \sum_{k=0}^{m} \overline{H}^{(m)}(x_l, x_k) Y^{(m)}(x_k)\cdot \Delta x + \overline{F}(x_l)\cdot , \qquad (l = 0,1,2, \cdots , m),$$

$$\overline{H}^{(m)}(x_l, x_k) = \sum_{j=1}^{m} G^{(m)}(x_l, x_{j-1}) H(x_j, x_k)\Delta x ,$$

$$\overline{F}(x_l)\cdot = \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1}) F(x_k)\cdot \Delta x ,$$

or

$$Y^{(m)}(x)\cdot = \int_0^1 \overline{H}^{(m)}(x,s) Y^{(m)}(s) ds + \overline{F}^{(m)}(x)\cdot , \tag{24}$$

where

$$H^{(m)}(x,s) = H(x_l, x_k) \quad \text{in} \quad I_x, I_s,$$

$$\overline{H}^{(m)}(x,s) = \int_0^1 G^{(m)}(x,\xi) H^{(m)}(\xi, s) d\xi = \overline{H}^{(m)}(x_l, x_k) \quad in \quad I_x, I_s,$$

$$\overline{F}^{(m)}(x) = \int_0^1 G^{(m)}(x,s) F^{(m)}(s)\cdot ds = \overline{F}(x_l)\cdot \quad in \quad I_x ,$$

is equivalent to (23).

In virtue of lemma (xi) and the assumption that $H(x, s)$ satisfies a Lipschitz condition, we have the lemma

xiii. $$\overline{H}^{(m)}(x,s) - H(x,s) = O(1/m),$$

$$\overline{F}^{(m)}(x)\cdot - \overline{F}(x)\cdot = O(1/m).$$

In consequence of Theorem 1, since (21) has a unique solution, then for m sufficiently large (24) will also possess a unique solution. So, for m sufficiently large, there exists a resolvent matrix $\mathfrak{H}^{(m)}(x, s)$ associated with $\overline{H}^{(m)}(x, s)$, and

$$Y^{(m)}(x)\cdot = \int_0^1 \Gamma^{(m)}(x,s) F^{(m)}(s)\cdot ds,$$

where

$$\Gamma^{(m)}(x,s) = G^{(m)}(x,s) + \int_0^1 \mathfrak{H}^{(m)}(x,\xi)G^{(m)}(\xi,s)d\xi.$$

Hence

$$(25) \qquad Y^{(m)}(x)\cdot + \lambda \int_0^1 \Gamma^{(m)}(x,s)Q^{(m)}(s)Y^{(m)}(s)\cdot ds = \int_0^1 \Gamma^{(m)}(x,s)F^{(m)}(s)\cdot ds,$$

is equivalent to (C_2), for $l<m$.

It is necessary to estimate the order of difference between the Green's matrices $\Gamma^{(m)}(x, s)$ and $\Gamma(x, s)$. For this purpose we need the lemma

xiv. $$\mathfrak{H}^{(m)}(x,s) - \mathfrak{H}(x,s) = O(1/m).$$

The resolvent matrices $\mathfrak{H}^{(m)}(x, s)$, $\mathfrak{H}(x, s)$ and the kernels $\overline{H}^{(m)}(x, s)$, $\overline{H}(x, s)$ satisfy the relations

$$\mathfrak{H}(x,s) - \overline{H}(x,s) = \int_0^1 \overline{H}(x,\xi)\mathfrak{H}(\xi,s)d\xi = \int_0^1 \mathfrak{H}(x,\xi)\overline{H}(\xi,s)d\xi,$$

$$\mathfrak{H}^{(m)}(x,s) - \overline{H}^{(m)}(x,s) = \int_0^1 \overline{H}^{(m)}(x,\xi)\mathfrak{H}^{(m)}(\xi,s)d\xi = \int_0^1 \mathfrak{H}^{(m)}(x,\xi)\overline{H}^{(m)}(\xi,s)d\xi.$$

Let

$$\overline{H}^{(m)}(x,s) - \overline{H}(x,s) = H_1^{(m)}(x,s) = O(1/m),$$
$$\mathfrak{H}^{(m)}(x,s) - \mathfrak{H}(x,s) = \mathfrak{U}^{(m)}(x,s).$$

Then

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 \overline{H}^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_1^{(m)}(x,s),$$

$$(26) \qquad \mathfrak{F}_1^{(m)}(x,s) = H_1^{(m)}(x,s) + \int_0^1 H_1^{(m)}(x,\xi)\mathfrak{H}(\xi,s)d\xi = O(1/m).$$

The equation (26) is equivalent to

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 \overline{H}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_2^{(m)}(x,s)\,,$$

$$\mathfrak{F}_2^{(m)}(x,s) = \mathfrak{F}_1^{(m)}(x,s) + \int_0^1 H_1^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi.$$

But $\overline{H}(x, s)$ possesses a resolvent $\mathfrak{H}(x, s)$. Hence

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 H_2^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_3^{(m)}(x,s),$$

where

$$H_2^{(m)}(x,s) = H_1^{(m)}(x,s) + \int_0^1 \mathfrak{H}(x,\xi)H_1^{(m)}(\xi,s)d\xi = O(1/m),$$

$$\mathfrak{F}_3^{(r)}(x,s) = \mathfrak{F}_1^{(m)}(x,s) + \int_0^1 \mathfrak{H}(x,\xi)\mathfrak{F}_1^{(m)}(\xi,s)d\xi = O(1/m).$$

But for m sufficiently large, the elements of $\mathfrak{H}_2^{(m)}(x, \xi)$, the resolvent matrix of $H_2^{(m)}(x, \xi)$, are bounded, and

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 \mathfrak{H}_2^{(m)}(x,\xi)\mathfrak{F}_3^{(m)}(\xi,s)d\xi + \mathfrak{F}_3^{(m)}(x,s),$$

whence

$$\mathfrak{U}^{(m)}(x,s) = O(1/m).$$

In virtue of this lemma,

$$\Gamma^{(m)}(x,s) - \Gamma(x,s) = O(1/m).$$

We have shown that the systems (C_1) and (C_2) can be reduced respectively to the integral equations (22) and (25), respectively which possess the same character as the integral equations (11) and (18). So, (22) and (25) can be reduced to Fredholm integral equations to which the theorems of §2 apply.

THE UNIVERSITY OF ROCHESTER

LINNAEUS WAYLAND DOWLING

IN MEMORIAM

The following minute has been adopted by the Faculty of the University of Wisconsin, and is printed here in memory of Professor Dowling, who had been a member of this Society since 1897.

Linnaeus Wayland Dowling was born at Medina, Michigan, on Decem ber 7, 1867. He graduated from Adrian College in 1890 and obtained the degree of Doctor of Philosophy at Clark University in 1895. He gave his entire collegiate service of thirty-three years to the University of Wisconsin, becoming successively Instructor, Assistant Professor, Associate Professor, and Professor of Mathematics. His special field was geometry, and he was the author of textbooks on Analytic Geometry, Projective Geometry, and Mathematics of Insurance.

Professor Dowling's instruction was characterized by an exceptional clarity of exposition which, with his magnetism, geniality of manner, and interest in his students, made him a universally popular and beloved teacher. He was a man of unusual breadth of interests: a talented musician, an eager and well-informed botanist, and a student of the humanities. His knowledge of the language and literature of Italy, where he studied under the geometer Segre, was exceptional; and he was an ardent reader and lover of poetry. These talents explain, in part, the bonds of sympathy that united him with so many colleagues and students. But above all his attraction was due to his personal qualities, his cheerful disposition and his kindly character. His influence in these private associations was no less great than in his admirable public service as a teacher, and he will long be remembered and missed by his friends.

Be it resolved that we as a faculty hereby express our sense of loss in the death of our colleague, Professor Dowling, and that this resolution be spread upon our minutes and a copy of it sent to Mrs. Dowling and to his son Dr. Philip Dowling.

[Resolution of the Faculty of the University of Wisconsin]

GRASSMANN'S PROJECTIVE GEOMETRY; VOLUME II

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung. Volume II: *Ternäres.* Part 1. By Hermann Grassmann. Leipzig und Berlin, B. G. Teubner, 1913. xii+410 pp.

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung. Volume II: *Ternäres.* Part 2. By Hermann Grassmann. Leipzig und Berlin. B. G. Teubner, 1927. xvi+522 pp.

The first volume of this work was published in 1909 and reviewed in this Bulletin (1913) by L. W. Dowling. The first part of the second volume appeared in 1913, but, owing to the world war, the volume was not completed before December 1921, when the finished second part was deposited as manuscript in the author's desk. Hermann Grassmann, Jr. died the following month, Jan. 21, 1922. One of his students, G. Wolff in Hannover, took charge of the publication of this second part, but owing to financial difficulties the work did not appear before 1927. Among those giving financial aid we note our own E. Carus.

The first part, which has four chapters (Hauptteile), introduces us to the ternary field of projective geometry. A point x in the plane is given by the equation $x=\xi_1 e_1+\xi_2 e_2+\xi_3 e_3$, where e_1, e_2, e_3 are the vertices of the fundamental triangle; ξ_1, ξ_2, ξ_3, the "Ableitszahlen," may be considered as the Grassmann coordinates of the point x. The dual of the point is the "stab," or directed line-segment which is bound to the line on which it lies, in the sense that it can only be carried along the line in its own direction or in the opposite direction. Any stab U in the plane is represented by the equation $U=u_1E_1+u_2E_2+u_3E_3$, where u_1, u_2, u_3 are the Grassmann line-coordinates and E_1, E_2, E_3 are "stäbe" on the lines joining the fundamental points, or "grundstäbe," defined by the equations $E_1=[e_2e_3]$, $E_2=[e_3e_1]$, $E_3=[e_1e_2]$. The unit point e, mass m, is now defined as a point with coordinates (1, 1, 1), so that $e=e_1+e_2+e_3$; the masses m_1, m_2, m_3 may be so chosen that the exterior three-point product $[e_1e_2e_3]=1$ (blatteinheit). Dually we have a unit "stab" of length l such that $E=E_1+E_2+E_3$ and $[E_1E_2E_3]=[e_1e_2e_3]=1$. It is then shown that the ratios $\xi_1:\xi_2:\xi_3$ and $u_1:u_2:u_3$ may be geometrically interpreted as double ratios

$$\frac{\xi_2}{\xi_3}=\frac{p_2}{p_2'}:\frac{p_3}{p_3'},\qquad \frac{\xi_3}{\xi_1}=\frac{p_3}{p_3'}:\frac{p_1}{p_1'},\qquad \frac{\xi_1}{\xi_2}=\frac{p_1}{p_1'}:\frac{p_2}{p_2'},$$

where p_1, p_2, p_3 are the distances of the point x from the sides of the fundamental triangle, and p_1', p_2', p_3' those of the unit point e from the same sides respectively. Dually we obtain for the ratios $u_1:u_2:u_3$ the values

$$\frac{u_2}{u_3}=\frac{q_2}{q_2'}:\frac{q_3}{q_3'},\qquad \frac{u_3}{u_1}=\frac{q_3}{q_3'}:\frac{q_1}{q_1'},\qquad \frac{u_1}{u_2}=\frac{q_1}{q_1'}:\frac{q_2}{q_2'},$$

q_1, q_2, q_3 being the lengths of the perpendiculars from the vertices e_1, e_2, e_3 on the stab U, and q_1', q_2', q_3' those of the perpendiculars from the same

three vertices on the unit stab E. By letting E_3 recede to infinity the author obtains the Hessian coordinate system familiar to students of Clebsch-Lindemann. The chapter concludes with a section on the harmonic relations in the quadrilateral-quadrangular configuration.

The Grassmann algorithm when applied to projective geometry does not however produce a pure coordinate geometry. The coordinates ξ_i and u_i are used in proving certain important formulas involving exterior and combinatory products, and the geometric and projective properties are then derived from the interpretation of these formulas; in other words, direct analysis is employed as far as possible. That coordinates cannot be entirely dispensed with seems to substantiate Study's dictum, at least in this case, that "without coordinates there is no geometry."

The second chapter deals with the fundamental properties of collineations, their double-elements and certain combinatory products of two and three collineations. A collineation k is defined by the symbolic or "extensive" fraction

$$k = \frac{a_1, a_2, a_3}{e_1, e_2, e_3},$$

which means that $e_1k = a_1$, $e_2k = a_2$, $e_3k = a_3$, and that $(\xi_1e_1+\xi_2e_2+\xi_3e_3)k = \xi_1a_1+\xi_2a_2+\xi_3a_3$, or $xk = y$; to the vertices e_1, e_2, e_3 of the fundamental triangle correspond the vertices a_1, a_2, a_3 of another triangle. Any point-row $y+hz$ is transformed into a point-row $yk+hkz = (\xi_1+h\zeta_1)a_1+(\xi_2+h\zeta_2)a_2+(\xi_3+h\zeta_3)a_3$, and the double-ratio of four points x, y, z, u on a line is invariant.

Closely related to the collineation k is the so-called adjoint collineation

$$K = \frac{A_1, A_2, A_3}{E_1, E_2, E_3}$$

which is a line-to-line transformation (Stab-Stab Abbildung) carrying a stab U into another stab UK, that is $U = \sum u_iE_i$ is transformed into $V = \sum u_iA_i$, where $E_1 = [e_2e_3]$, $E_2 = [e_3e_1]$, $E_3 = [e_1e_2]$, $A_1 = [a_2a_3]$, $A_2 = [a_3a_1]$, $A_3 = [a_1a_2]$. From the equation $[x, U] = 0$, which means that x is on U, it follows that $[xk, UK] = 0$, that is, the transform xk of the point x by k is on the transform UK of U by K.

If k and l are two collineations, the combinatory product $[kl]$ is defined as follows:

$$[kl] = \frac{[yz \cdot kl]}{[yz]} = \frac{[yk \cdot zl] - [zk \cdot yl]}{2[yz]}, \tag{1}$$

the right side of which is shown not to depend on the two points y and z; hence the symbol $[kl]$ is justified. If then in (1) we put $y = e_1$, $z = e_2$; $y = e_2$, $z = e_3$; $y = e_3$, $z = e_1$ in succession, it is easily seen that $[kl]$ may be put in the form of an extensive fraction

$$[kl] = \frac{[e_2e_3 \cdot kl], [e_3e_1 \cdot kl], [e_1e_2 \cdot kl]}{[e_2e_3], [e_3e_1], [e_1e_2]}.$$

If $k = l$, we have $[k^2] = K$, an important formula.

The combinatory product of three collineations is defined thus:

$$(2)\quad [klm] = \frac{[xyz\cdot klm]}{[xyz]} = \frac{\begin{array}{r}[xk\cdot yl\cdot zm]+[yk\cdot zl\cdot xm]+[zk\cdot xl\cdot ym]\\ -[xk\cdot zl\cdot ym]-[yk\cdot xm\cdot zl]-[zk\cdot ym\cdot xl]\end{array}}{6[xyz]},$$

and it is shown that the fraction on the right side does not depend on the points x, y, z, only on the numerators a_i, l_i, c_i of the extended fractions k, l and m. If $k=l=m$, we have the combinatory cube

$$[k^3] = \frac{[a_1a_2a_3]}{[e_1e_2e_3]} = \frac{[e_1k\cdot e_2k\cdot e_3k]}{[e_1e_2e_3]}$$

which is another fundamental formula. The author then proceeds to classify the various singular collineations, namely those for which $[k^3]=0$:

1. $[a_1a_2a_3]=0$, at least one of the products $[a_ia_k]$ being different from zero.
2. $[a_1a_2a_3]=0$, all the products $[a_ia_k]$ vanishing.
3. $[a_1a_2a_3]=0$, all three a's vanishing.

Starting with the adjoint K and the adjoint to K, denoted by $\bar{k}$,

$$\bar{k} = \frac{[A_2A_3], [A_3A_1], [A_1A_2]}{[E_2E_3], [E_3E_1], [E_1E_2]}$$

and introducing the combinatory product $[Kl]$ of two stab-stab collineations, he proves that $[K^2]=\bar{k}=ak$, where $a=[a_1a_2a_3]=[k^3]$. The combinatory product $[KLM]$ is then defined in a manner analogous to that for $[klm]$ and an expression for $[K^3]$ is obtained, viz.: $[K^3]=[A_1A_2A_3]=[a_1a_2a_3]=a$. A classification of singular stab-stab collineations then follows, dual to the above for point-collineations.

It appears here, as elsewhere throughout the work, that the author discusses the dual case in detail instead of merely stating the result, applying the principle of duality, as is usually done, in projective geometry. This tends to increase materially the number of calculations and becomes wearisome in the long run, but the elementary student who attacks the subject for the first time *via* Grassmann will perhaps be satisfied, as he is saved the trouble of proving the dual case. It is all "cut and dry." In the second part of the volume he sometimes, but not often, omits the calculation.

The double elements of the collineation are obtained starting with the condition $d_tk=r_td_t$ where d_t is a point and r_t a pure number. The question is then: What point, or points d_t are left in situ in a collineation k, the point d_t changing at most its mass which is multiplied by a number r_t? The answer to this question leads to a cubic equation having r_1, r_2, r_3 for roots. This is *the characteristic equation*. We are thus led to a collineation

$$k = \frac{r_1d_1, r_2d_2, r_3d_3}{d_1, d_2, d_3},$$

d_1, d_2, d_3 being the fundamental triangle. The dual case, which is carried out in detail, gives rise to the stab-stab collineation

$$K = \frac{r_2r_3[d_2d_3], r_3r_1[d_3d_1], r_1r_2[d_1d_2]}{[d_2d_3], [d_3d_1], [d_1d_2]}.$$

The various subcases are then considered, the result being a slightly modified form of the classification found in Veblen and Young's treatise, vol. I, pp. 271–276. It may be noted in this connection that Grassmann needs 25 pages to exhaust the subject of double elements and types of collineations, while Veblen and Young need only 5 pages. This is not due to a more complicated algorithm, for the calculations in Grassmann are simple, and the geometry can be read from the symbolic equations without difficulty, but rather to a more detailed consideration of the geometry involved. In this respect the method of Grassmann is carried out very carefully; he never forgets that he is a geometer.

The last section (29) of this chapter deals with the geometric meaning of the vanishing of the combinatory product $[klm]$ which was defined above, (eq. 2). If $l=m$ we get the product $[kll]=0$ or

$$[xkyz] + [ykzx] + [zkxy] = 0. \tag{3}$$

If (3) is satisfied the collineation has an *inscribed triangular position*. This theorem is due to Pasch. A few more theorems of a similar kind then follow. The remainder of the chapter deals with point-triples. The locus of points x such that the two pencils $xabc$ and $xa'b'c'$ are in involution is a cubic called the cubic of involution; the special case when this cubic degenerates into three lines, i.e. when a, b, c and a', b', c' are collinear point triples is then considered, and a number of theorems on point-triples are deduced.

In the sixth chapter the author considers first correlations in the plane and then the more special polarities with their pole-curves of the second order and class. He sets up the extensive fractions for correlations and and polarities together with a number of combinatory products of two and three correlations (polarities). Starting with these formulas, the remaining part of the chapter is devoted to the consideration of poles and polars, the polar triangle, curves (that is pole-curves) of the second order and class. etc. The discussion of singular or "entartete" systems of polarity occupies about 30 pages, and the various forms of conics, considered as point-loci and stab-loci and depending on the various positions of their polar triangles, another 57 pages. The last section deals with conics in Cartesian point-coordinates (oblique axes) and Hessian line-coordinates, one side of the polar triangle being at infinity. The last, but not the least interesting, chapter deals with the theory of pencils of conics and their duals, ranges of conics. We shall not give an account of the various cases considered. The work is done with meticulous care, and the dual case (ranges) is treated in detail. The author finds no place for a discussion of group-properties of collineations, correlations and polarities, although in the first volume he did consider the product (folgeprodukt) kl of two projectivities in the binary field. The notion of an invariant, as well as invariant configurations connected with the various subgroups of projective transformations, is also absent. Nevertheless, the first part of this volume is a decided success from the standpoint of the Grassmann point-calculus; he no doubt had two main objects in view: To write a projective geometry based on the Grassmann point-calculus, and to make it a magisterial work that a

student can easily master. Any mathematician who has found the elder Grassmann's presentation of his point-calculus, such as is found in Grassmann's *Ausdehnungslehre* of 1862 rather "abschreckend" will be pleasingly surprised at the ease with which the calculus can be mastered when presented in a masterly way.

The second part begins with a section on the projective geometry on a curve of the second order and class. Involutions on a conic are carefully considered and a following short section deals with the equianharmonic projectivities.

In the next chapter the author turns to the consideration of range-pencils and pencil-ranges of conics. Special cases are considered such as homo-asymptotic hyperbolas and ellipses. All collineations are then found that leave pencil-ranges invariant, and incidentally also the collineations which transform a conic into itself. The general theory of reciprocities (correlations) is treated in the remainder of the chapter. Let

$$r = \frac{A_1, A_2, A_3}{e_1, e_2, e_3}, \quad R = r^2 = \frac{a_1, a_2, a_3}{E_1, E_2, E_3}$$

be a reciprocity and its adjoint. If $[A_1A_2A_3] \neq 0$, and $[a_1a_2a_3] \neq 0$, we have also the inverse

$$\frac{1}{r} = \frac{e_1, e_2, e_3}{A_1, A_2, A_3}, \qquad \frac{1}{R} = \frac{E_1, E_2, E_3}{a_1, a_2, a_3};$$

for the special case of polarity we have $r = a/R$, $a = [a_1a_2a_3]$. The curves $[x \cdot xr] = 0$, and $[U \cdot UR] = 0$ are called the *nuclei* of the reciprocity. The first expresses the condition that a point shall lie on the corresponding line, and the second equation that the line U shall pass through the corresponding point. These curves are also called respectively *the pole-curve* and the *polar curve*. The reciprocity conjugate to r is

$$r' = \frac{A_1', A_2', A_3'}{e_1, e_2, e_3} = \frac{a}{R},$$

where $A_i' = a_{1i}E_1 + a_{2i}E_2 + a_{3i}E_3$. A null-system of the second order is one for which $a_{ii} = 0$, $a_{ik} = -a_{ki}$, and for a null-system of the second class we have $A_{ii} = 0$, $A_{ik} = -A_{ki}$, the A_{ik} being the minors of the determinant $|a_{11}a_{22}a_{33}|$. A reciprocity of this kind may be put in the form

$$n = \frac{[ae_1], [ae_2], [ae_3]}{e_1, e_2, e_3},$$

in which a stands for the null-point of the reciprocity. Dually we get

$$N = \frac{[AE_1], [AE_2], [AE_3]}{E_1, E_2, E_3},$$

where A is the null-axis of the reciprocity. Of these two singular correlations, the one carries a point x into a line nx joining x to the null-point a so that the pole-curve becomes a double point a, and the second carries a line U into the point of intersection of the line with the null-axis, i.e. A, as a double line, is the polar curve.

The author then discusses conjugate and adjoint reciprocities and the relative positions of their nuclei. Although for a reciprocity, which is not a polarity, we must always have $r \neq r'$, the equality $cr = cr'$ may be satisfied for a special point. This point is shown to have the coordinates $c_1 : c_2 : c_3 = a_{23} - a_{32} : a_{31} - a_{13} : a_{12} - a_{21}$ and is called the "kernpunkt" of the reciprocity, and dually, for the coordinates of the "kerngerade" $C_1 : C_2 : C_3 = A_{23} - A_{31} : A_{32} - A_{13} : A_{12} - A_{21}$, and he proves that the "kernpunkt" of a reciprocity is the pole of the "kerngerade" with respect to both nuclei of the reciprocity.

In the tenth and eleventh chapters the author takes up the subject of apolarity. He derives the condition for apolarity of two reciprocities in the form

$$(4) \qquad [rS] = \frac{1}{3[xyz]}\{[xr \cdot XS] + [yr \cdot YS] + [zr \cdot ZS]\} = 0,$$

where

$$X = [yz], \quad y = [zx], \quad Z = [xy], \quad S = [st].$$

If in this equation we put $e_i r = A_i$ and $E_i S = b_i$; $x = e_1$, $y = e_2$, $z = e_3$, $X = E_1$, $Y = E_2$, $Z = E_3$, the condition $[rS] = 0$ takes the well known form

$$a_{11}B_{11} + \cdots + (a_{23}B_{23} + a_{32}B_{32}) + \cdots = 0,$$

and when r and S are two polar systems p and Q, $a_{11}B_{11} + \cdots + 2a_{23}B_{23} + \cdots = 0$.

In Chapter XI the author introduces two new symbolic expressions, the *Lückenform* and the *Potenzform*. We shall not try to give English equivalents for these terms; *lacunary forms* would perhaps do for the first term, but *powerform* does not sound well, and we shall therefore use the German terms throughout. Suppose we have a polarity of the second order

$$p = \frac{A_1, A_2, A_3}{e_1, e_2, e_3},$$

where $A_i = \sum a_{ik}E_k$ and $a_{ik} = a_{ki}$; then if $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$, we have the polar $xp = \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3$. Multiplying x exteriorly by E_1, E_2, E_3 we get $[xE_1] = \xi_1$, $[xE_2] = \xi_2$, $[xE_3] = \xi_3$ and the polar xp becomes

$$xp = [xE_1]A_1 + [xE_2]A_2 + [xE_3]A_3 = \sum [xE_i]A_i,$$

and the quadratic form $[x \cdot xp]$ is represented by the expression

$$[x \cdot xp] = [xE_1][xA_1] + [xE_2][xA_2] + [xE_3][xA_3] = \sum_1^3 [xE_i][xA_i].$$

Since x occurs twice in each term on the right side, we may, if we wish, put x^2 outside and fill in the spaces left vacant by a letter l (Lücke:lacuna) so that the quadratic form is written,

$$(5) \qquad [x \cdot xp] = \{[lE_1][lA_1] + [lE_2][lA_2] + [lE_3][lA_3]\}x^2 = \sum_1^3 [lE_i][lA_i]x^2.$$

The factor of x^2 is called a "Lückenform" of the second order; the small letter l in the expression means that the space is to be filled in by a point

such as x, y, etc.; l is therefore so to speak "the ghost of a departed point." We denote the lückenform by the letter A_2 and write

$$A_2 = \sum_1^3 [lE_i][lA_i]$$

and, introducing for the A's their values $\sum a_{ik}E_k$, we have

$$A_2 = a_{11}[lE_1]^2 + a_{22}[lE_2]^2 + a_{33}[lE_3]^2 + 2a_{23}[lE_2][lE_3] + 2a_{31}[lE_1][lE_3] + 2a_{12}[lE_1][lE_2]$$

and $[x \cdot xp] = A_2x^2$, so that $A_2x^2 = 0$ becomes the equation of the polecurve. In the same way we have for the bilinear form $[y \cdot xp] = \sum [yE_i][xA_i]$; but, since for any polarity we have the fundamental equation $[y \cdot xp] = [x \cdot yp]$, we have

$$[x \cdot yp] = [y \cdot xp] = \sum_1^3 \frac{[xE_i][yA_i] + [yE_i][xA_i]}{2},$$

so that we may write

$$[x \cdot yp] = [y \cdot xp] = A_2xy = \sum [lE_i][lA_i]\{yx\} = \frac{[xE_i][yA_i] + [yE_i][xA_i]}{2} = A_2yx;$$

xy is an algebraic product since $A_2xy = A_2yx$.

It is important to note that each term of (5) is a product of two factors which may be considered as lacunary forms of the first order and the product is not changed by interchanging the factors; the product $[xE_1][xA_1]$, for example, vanishes whenever the point is on the line-pair containing the stäbe E_1 and A_1, and in the case of the quadratic terms the line-pair becomes a double line. We may therefore say that the products in the separate terms of the lückenform (5) represent line-pairs, considered as reducible curves of the second order. In order to simplify the expression we leave out the l's and the brackets; the remaining stab-factors $E_1, A_1, \cdots$ we combine into products E_1A_1, $E_2A_2, \cdots$ of the algebraic kind, since the original products were algebraic just as the product xy. The new expression for the sum of such factors we shall call a *potenzform of the second order* and denote it by the symbol $A^{(2)}$ so that

$$A^{(2)} = \sum_1^3 E_iA_i = a_{11}E_1^2 + a_{22}E_2^2 + a_{33}E_3^2 + 2a_{23}E_3E_2 + 2a_{31}E_3E_1 + 2a_{21}E_2E_1,$$

and this potenzform shall henceforth mean precisely what is expressed by the equation $A_2x^2 = 0$, that is, a curve of the second order. The quantities $E_1, {}^2E_2, {}^2E_3, {}^2E_2E_3, E_3E_1, E_1E_2$ are *the six units of the second order*. Conversely, the lückenform may easily be constructed, the potenzform being given. Moreover, if we want the curve itself from the potenzform, all we have to do is to substitute for the fundamental stäbe E_1, E_2, E_3 the coordinates ξ_1, ξ_2, ξ_3. Thus the potenzform $a_{23}E_2E_3 + a_{31}E_3E_1 + a_{12}E_1E_2$ represents the curve of the second order $a_{23}\xi_2\xi_3 + a_{31}\xi_3\xi_1 + a_{12}\xi_1\xi_2 = 0$ which is circumscribed about the fundamental triangle; the lückenform is

$$a_{23}[lE_2][lE_3] + a_{31}[lE_3][lE_1] + a_{12}[lE_1][lE_2].$$

In the dual case, starting with a polarity of the second class, we obtain a ternary lückenform of the second class

$$A_2 = [e_1L][a_1L] + [e_2L][a_2L] + [e_3L][a_3L] = \sum [e_iL_i][a_iL]$$

or, introducing the values $a_i = \sum A_{ik}e_k$, $A_{ik} = A_{ki}$,

$$A_2 = A_{11}[e_1L]^2 + A_{22}[e_2L]^2 + A_{33}[e_3L]^2 + 2A_{32}[e_3L][e_2L] + 2A_{31}[e_3L][e_1L] + 2A_{12}[e_2L][e_1L]$$

and the polar curve of the polarity P has the equation $a_2U^2 = 0$. Two lines U and V are conjugate when the bilinear form $a_2UV = 0$. The potenzform corresponding to the lückenform of the second class is now

$$A^{(2)} = A_{11}e_1^2 + A_{22}e_2^2 + A_{33}e_3^2 + 2A_{32}e_3e_2 + 2A_{31}e_3e_1l + 2A_{12}e_1e_2,$$

which represents the curve $a_2U^2 = 0$. The most general potenzform of the second class is derived from the *six units of the second class*, viz.: e_1^2, e_2^2, e_3^2, e_2e_3, e_3e_1, e_1e_2, and the polar curve may be obtained from it by substituting e_i for u_i in the equation of a conic of the second class.

The combinatory product of a potenzform of the second order and a potenzform of the second class is now defined as follows: $[A^{(2)}b^{(2)}] = A_2b^{(2)}$ and $[b^{(2)}A^{(2)}] = b_2A^{(2)}$, that is, we put the combinatory product equal to the product obtained by substituting for the first potenzform the corresponding lückenform and consider the extensive factor of the second potenzform as "filling in" factor intended for the lacunae of the lückenform. The following theorem is then proved: *The combinatory product* $[A^{(2)}b^{(2)}]$ *vanishes when, and only when, the two potenzformen are apolar.* There is however a "fly in the ointment" right here. This elaborate preparation of an algorithm to express the apolarity of two polar systems is not applicable to non-involutary or general reciprocities. We have to go back to the former condition $[rS] = 0$, equation (4).

In the remainder of the chapter special cases of apolarity are given and two theorems of von Staudt on polar triangles and quadrilaterals are proved.

In our journey through 950 pages of this treatise of 1250 pages we have met with the cubic only once, the cubic of involution mentioned above. It is therefore a pleasant relief when the author in the following Chapters XII and XIII allows us to renew our old acquaintances, the curves of the third order and class, although in a stranger garb. A lückenform of the third order is written

$$A_3 = a_{111}[lE_1]^3 + \cdots + 3a_{112}[lE_1]^2[lE_2] + \cdots + a_{123}[lE_2][lE_2][lE_3] + \cdots,$$

and a lückenform of the third class

$$a_3 = A_{111}[e_1L]^3 + \cdots + 3A_{112}[e_1L]^2[e_2L] + \cdots + 3A_{123}[e_1L][e_2L][e_3L] + \cdots,$$

the corresponding curves of the third order and the third class being

$$A_3x^3 = \{\sum a_{iks}[lE_i][lE_k][lE_s]\}x^3 = 0,$$
$$a_3U^3 = \{\sum A_{iks}[e_iL][e_kL][e_sL]\}U^3 = 0.$$

Several familiar theorems on polars of a cubic are then proved. The Hessian of a cubic is expressed by the vanishing of a combinatory product of three lacunary forms $H_3z^3=[A_3ze_1\cdot A_3ze_2\cdot A_3ze_3]$, where $H_3=[A_3le_1\cdot A_3le_2\cdot A_3le_3]$ is a ternary lückenform of the third order. The Hessian of a curve of the third class is represented as a combinatory product of three lückenformen of the third class in the form $h_3W^3=[a_3WE_1\cdot a_3WE_2\cdot a_3WE_3]=0$. The Grassmann point-calculus seems to be well adapted to a careful treatment of the inflectional tangents to a cubic, and the author takes special care in his discussion of the real and imaginary configurations connected with these tangents. The same care is also taken with the dual configuration of the 9 cuspidal tangents to the cubic of the third class.

Chapter XIV deals with nets of conics and their duals, webs of conics. The chapter ends with a section on domains of polarity, and a final section on the Cayleyan curve of a net (web). In the last two chapters metrical properties of conics and their relation to the circular points at infinity are considered. Oblique and rectangular axes are used in turn, and a host of special theorems are produced. The projective properties of some special nets and webs of conics are given due consideration in the last chapter.

As a magisterial work the Projective Geometry of H. Grassmann, Jr. takes a high rank. Although somewhat diffuse at times, it is a marvel of painstaking care and exactitude. It is singularly free from missprints. On page 145, (vol. 2, Part 2) sixth line from bottom, the words "der Ableitszahlen" should be inserted after the word "Funktionen." On p. 284, line 19 from above, the word "und 789" should be inserted after the number "791." On p. 346, last line, "Potenzreihen" should read "Potenzformen."

It is to be regretted that the author had to depart before he could accomplish what was doubtless his intention, an extension of the work to the quaternary field. The notes published by the author in Volume I of Grassmann's Gesammelte Werke, pages 438–464, seem to be a tentative beginning in this direction.

"Hans bok kom ikke, han selv gik derhen,
Hvor tankernes lov ikke skrives med pen."
—B. Björnson

JOHN EIESLAND

THE NEW RIEMANN-WEBER—VOLUME II

Die Differential- und Integralgleichungen der Mechanik und Physik. Zweiter, physikalischer Teil, edited by P. Frank with the assistance of H. Faxén, R. Fürth, Th. v. Kármán, Fr. Noether, C. W. Oseen, A. Sommerfeld, E. Trefftz. Braunschweig, Vieweg, 1927. 863 pp.+88 figures. Paper, 53 Rentenmark; bound, 58 Rentenmark.

The first part, or mathematical section of this work, was reviewed by the present writer in the May-June 1927 number of this Bulletin. It outlines in a very broad manner the mathematical tool-box of the theoretical physicist of the present day, which covers nearly the whole realm of analysis and large portions of the other departments of mathematics.

For the second volume, the goal adopted by the editors was to accomplish for the last Riemann-Weber volume of 1910, that which Weber did in 1900, for Riemann's original *Partielle Differentialgleichungen der mathematischen Physik.* We may quote freely from the preface to the effect that in spite of fad and fancy in physical hypotheses, the mathematical formulations and methods are relatively constant. Examples are cited in the application of the planetary perturbation theory of Laplace and Lagrange to atomic states of energy; in the identity between the equations for the transfer of heat due to Fourier and that of the phenomena of diffusion in liquids; in the universality of the wave equations of D'Alembert, Euler and Gauss in sound, light, and atomistics; in the appearance of classical potential theory of Laplace, Gauss, Cauchy and Riemann in the modern theories of lift and drag on an airplane wing. Contrarywise, the fluid theories of heat and electricity are in the discard, atoms, molecules and electrons may even follow, Newton's hypothesis of Universal Gravitation may go out of fashion, but the fundamental analytic framework in terms of which the corresponding phenomena are described is relatively permanent. More recently, the influence of the mathematical frame of reference on hypothesis has been considerable.

The second volume of the present work joins on to the first in very much the same manner as superstructure to foundation. Despite the fact that volume two is the work of seven writers, in each case different from each of the eight collaborators on the first, a unified picture of the entire field wherein differential and integral equations function in the field of mathematical physics has been secured. There is no discussion of merit or lack of merit of physical hypothesis, very little comparison between mathematical result and experimental observation, but for the sake of unity the entire purpose is concentrated on the development of those particular analytic formulations in the realm of physical phenomena, particularly to recent developments such as quanta, Brownian movements, radiation, etc., together with technical applications of special importance such as aeronautics and radio. Some of this material on these topics appears in print for the first time.

The following is a summary by chapter headings:

Part I (by P. Frank). Analytic mechanics. 1. The differential equations of general mechanical systems. 2. Stability and small oscillations. 3. The equations of motion of rigid bodies. 4. Methods in perturbation theory. 5. Problems of celestial and atomic mechanics.

Part II (by R. Fürth). Conduction of heat and diffusion. 6. Free conduction of heat and diffusion. 7. Induced conduction of heat and diffusion.

Part III (by Fr. Noether). The stationary and quasi-stationary electromagnetic fields. 8. Electrostatics. 9. Steady electric currents. 10. Magnetostatics. 11. Quasi-stationary currents and waves.

Part IV (by A. Sommerfeld). Electromagnetic oscillations. 12. General theorems and methods. 13. Theory of refraction. 14. Resistance and skin effect. 15. Waves on wires. 16. Wireless telegraphy.

Part V (by E. Trefftz). Mechanics of continua. 17. Mathematical foundations of the theory of elasticity. 18. Problems in elastic equilibrium. 19. Dynamical problems of elastic theory. 20. Ideal fluids (by Th. v. Kármán). 21. Motion of viscous fluids (by Faxén and Oseen).

Part I is especially complete on the general equations of mechanics and their solution. The treatment is entirely classical and prepares the ground for the problems of classical atomic mechanics. Relativistic mechanics is not included. The general equations of motion and stability of an airplane are treated. Part II on heat and diffusion follows in the main the classical treatment, except that the problems of diffusion are treated in parallel with those of heat flow. Part III on the static and quasi-static electromagnetic fields covers both the classic and the modern portions of the subject. The same may be said of electromagnetic oscillations. Here, the emphasis is on the theory of radiation and electric wave phenomena with applications to radio-activity, light and wireless. Finally, Part IV treats the mechanics of continuous media such as elasticity, motion of fluids, both perfect and viscous, and related problems. The last two chapters of this section contain new material of importance in aerodynamics.

It is a fairly simple matter to point out omissions in the writing of as comprehensive a work as the present one. There is the immediate reply to this of the authors of a desire to avoid expansion into encyclopedic proportions. However, it does seem that it would have been in order to include some applications of difference equations or algebraic systems related to differential and integral equations. One might also venture to suggest the inclusion of some chapters on the newer geometry and the theory of geodesics with applications to problems in gravitation, elasticity, etc.

Though definitely outside of the program of the present two volumes, if one were to add in an appendix, chapters on statistics and probability in relation to states of energy, and group and number theory in relation to the concept of symmetries in crystal structure, one would have a truly representative and comprehensive picture of the interlacing of mathematical reference frames and physical hypotheses in the description of the natural phenomena of the universe.

H. J. Ettlinger

SHORTER NOTICES

Theorie der Kristallstruktur. By Arthur Schoenflies. Berlin, Gëbrüder Borntraeger, 1923.

This book is essentially a new edition of Professor Schoenflies' *Krystallsysteme und Krystallstruktur* which appeared in 1891. Like its predecessor its primary concern is the deduction of the 230 crystallographically significant space groups. Though remaining purely geometric in the details of its reasoning, this deduction has been so rewritten as to make it somewhat shorter and more concise. The insertion of numerous figures is a great help towards the picturing of individual groups. As before these groups are described through the statement of both their sub-groups and the coordinates of equivalent positions within them. This description has, however, been amplified and improved through a listing of the symmetry properties associated with points lying in elements of symmetry.

Since the discovery of X-ray diffraction the theory of space groups has become of immediate and every day use in experimental physics. The realization of this changed importance has inevitably influenced Professor Schoenflies' treatment and has led him to lay greater stress upon items of applied crystallographic interest. A chapter has accordingly been inserted which outlines the practical usefulness of space groups in studies of the positions of atoms in crystals. Although this account will not meet the needs of the practicing crystal analyst, Professor Schoenflies' books remain the only suitable source of information for those interested in the derivation of space groups.

R. W. G. Wyckoff

Mathematics in Liberal Education. By F. Cajori. Boston, Christopher, 1928.

Do you believe in the "transfer of training"? Do you think that the study of mathematics yields a certain discipline which can be applied to situations outside of mathematics—which can be "transferred"? If you do, psychologists today would mostly agree with you.

But when they proceed by standard tests and other devices to measure the amount of this transfer, their findings are meager or nil. Professor Cajori doubts the psychologists' methods and suggests that in the absence of adequate objective measures it is not unscientific to consider subjective measures such as the opinions of philosophers and men of science.

His little book presents the recorded opinions of leading thinkers of all ages on the disciplinary value of mathematics. These quotations are arranged in chronological order, and include all opinions *contra* as well as *pro*. Which being counted, it appears that the *pro*'s have it.

This is a book for the believer, for the regeneration of faith in the faithful. And for him who battles daily with those who value mathematics only as a tool, or who sit in the seat of the scornful, it increaseth strength.

Ralph Beatley

Gewöhnliche Differentialgleichungen. By Dr. G. Hoheisel. Sammlung Göschen. Berlin and Leipzig, Walter de Gruyter, 1926. 159 pp.

This little book contains a surprising amount of material. Beginning with the very elements of the subject, it is remarkably comprehensive.

As a text-book for the American or English student, it will not be found entirely satisfactory. The treatment is not very systematic, nor is the usual elementary ground completely covered. And there are no exercises for the student. But for the enquiring student, and as collateral reading this little book will undoubtedly be found very attractive.

The author presents relatively simply some fairly difficult topics. While he does not go deeply into them, he introduces quite skillfully such subjects as behavior of integral curves in the neighborhood of singular points, integration by means of infinite series and definite integrals, and boundary value problems.

A. COHEN

Theorie der Differentialgleichungen. Vorlesungen aus dem Gesamtgebiet der gewöhnlichen und der partiellen Differentialgleichungen. Zweite neubearbeitete Auflage.* By Ludvig Bieberbach. Berlin, Julius Springer, 1926. ix+358 pp.

As in the original edition, the author has set himself the difficult task of carrying the reader from the very elements of the subject to the comprehension of some of the recent developments. In his preface he recognizes the impossibility of covering the whole field. So he endeavors to make a selection of topics. That he succeeds in making a most attractive one there can be no doubt. But skillful as the author is in masterful presentation, it seems almost an impossible undertaking to carry out the aims of the book in so few pages.

The second edition follows the general plan of the earlier one. There are many minor changes and additional cuts. Several chapters are practically rewritten. Although larger by some fifty pages, here too some of the attempts to give a hasty survey of advanced theories seem less readable (because of their brevity) than the original memoirs.

But the book a pioneer of its kind in the subject of differential equations, is a valuable addition to the literature, and serves excellently as a guide towards some very interesting and important fields of development.

A. COHEN

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* For a review of the first edition see this Bulletin, vol. 31 (1925), p. 279.

NOTES

The concluding number of volume 30 of the Transactions of this Society (October, 1928) contains the following papers: *Some non-linear problems in approximation*, by D. Jackson; *Second-order differential systems with integral and k-point boundary conditions*, by W. M. Whyburn; *Analytic functions of hypercomplex variables*, by P. W. Ketchum; *Concerning limiting sets in abstract spaces*, by R. G. Lubben; *Riesz summability for double series*, by F. M. Mears; *The inverse problem of the calculus of variations in higher space*, by D. R. Davis; *Types of motion of the gyroscope*, by A. H. Copeland; *The invariant integral and the inverse problem in the calculus of variations*, by T. H. Rawles; *The projective differential geometry of systems of linear homogeneous differential equations of the first order*, by E. P. Lane; *The behavior of a boundary value problem as the interval becomes infinite*, by W. E. Milne; *The second derivative of a polygenic function*, by E. Kasner; *Diophantine equations in division algebras*, by R. G. Archibald; *On the degree of approximation to an analytic function by means of rational functions*, by J. L. Walsh; *Existence and oscillation theorems for non-linear differential systems of the second order*, by W. M. Whyburn. The Index of volumes 21–30 (1920–28) is bound with this number.

The concluding number of volume 50 of the American Journal of Mathematics (October, 1928) contains: *Affinely connected function space manifolds*, by A. D. Michal; *A theorem concerning the affine connection*, by T. Y. Thomas; *Concerning subsets of a continuous curve which can be connected through the complement of the continuous curve*, by W. L. Ayres; *Admissible numbers in the theory of probability*, by A. H. Copeland; *The differential invariants of inversive geometry*, by B. C. Patterson; *Expansions of the Neumann type in terms of products of Bessel functions*, by G. Stevenson; *Canonical forms for ordinary homogeneous linear differential equations of the second order with periodic coefficients*, by E. Swift; *Periodic orbits of three finite masses about the equilateral triangle points*, by H. E. Buchanan; *The problem of plane involutions of order $t>2$*, by F. R. Sharpe. A portrait of Goursat is bound with this number.

Professor G. D. Birkhoff of Harvard University delivered a lecture on *The mathematical basis of art* at the Ohio State University on December 3, 1928.

Professor E. R. Hedrick, Editor-in-Chief of this Bulletin, visited and lectured at the following institutions during November and December, 1928: University of Missouri, November 26; Ohio State University, November 27–28; University of Cincinnati, November 30–December 3; University of Kentucky, December 4; Oberlin College, December 5–6; University of Michigan, December 7; University of Wisconsin, December 10; Northwestern University, December 11; University of Indiana, De-

cember 12; University of Pittsburgh, December 13–17; University of West Virginia, December 18; University of Pennsylvania, December 19.

A new journal entitled Boletín del Seminario Mathemático Argentino has been founded, under the editorship of Professor J. Rey Pastor. The first number appeared in September, 1928.

The Société Mathématique Suisse has founded a new mathematical journal, the Comentarii Mathematici Helvetici; Professor Andreas Speiser, of the University of Zurich, is editor-in-chief. The journal is published by Orell-Füssli, of Zurich.

On the occasion of the meeting of the British Association for the Advancement of Science at Glasgow, the University of Glasgow conferred honorary doctorates on Sir William Bragg, president of the Association, and Professor P. Zeeman, of the University of Amsterdam, among others. The Association will hold its summer meeting of 1929 in South Africa, at Cape Town, Johannesburg, and Pretoria.

At the thirteenth annual meeting of the Optical Society of America, held in Washington, October 31 to November 3, 1928, under the joint auspices of that society and the Bureau of Standards, was celebrated the semicentennial of the publication of Professor A. A. Michelson's first communication on the velocity of light. On this occasion, Professor Michelson and his collaborators, Messrs. Pease and Pearson, delivered an address on *Results of repetition of the Michelson-Morley experiment.*

The professors and students of the Faculty of Sciences of the University of Turin have placed a commemorative medallion in honor of the late Professor Corrado Segre in one of the rooms of the Mathematical Institute of the University.

The Sacchetti Prize designed to acknowledge eminent scientific work of the teaching staff of the University of Bologna has been awarded to Professor S. Pincherle, who has retired.

Sir Ernest Rutherford has been elected a foreign honorary member and Professor Erwin Schrödinger a foreign corresponding member of the Vienna Academy of Sciences.

Professor Raymond Pearl, of Johns Hopkins University, has been elected president of the International Union for the Study of Demographic Problems.

The following have been appointed to National Research Fellowships in mathematics for 1928–29: A. A. Albert, W. L. Ayres, F. R. Bamforth, Theodore Bennett, Alonzo Church, T. F. Cope, C. M. Cramlet, Jesse Douglas, D. A. Flanders, J. J. Gergen, M. S. Knebelman, Lincoln LaPaz, Morris Marden, T. W. Moore, Hillel Poritsky, N. E. Rutt, I. M. Sheffer, Marie J. Weiss. Messrs. Ayres, Church, and Douglas hold fellowships supported by the International Education Board.

Professor L. Fantappié, of the University of Cagliari, has been transferred to the chair of algebraic analysis in the University of Palermo.

Associate Professor J. Lense, of the Munich Technical School, has been promoted to a full professorship of mathematics.

Professor B. Levi, of the University of Parma, has been transferred to the chair of theory of functions in the University of Bologna.

Dr. G. Mammana has been appointed associate professor of calculus in the University of Cagliari.

Associate Professor R. Ortvay, of the University of Szeged, has been appointed associate professor of theoretical physics at the University of Budapest.

Professor Johann Radon, of the University of Erlangen, has been appointed professor of mathematics at the University of Breslau.

Dr. C. Weber has been appointed professor of mechanics at the Dresden Technical School.

Professor Gregor Wenzel, of the University of Leipzig, has been appointed professor of theoretical physics at the University of Zurich.

Mr. P. T. Copp has been appointed head of the department of mathematics at Valparaiso University.

Dr. Mary G. Haseman has been appointed professor of mathematics at Hartwick College, Oneonta, N. Y.

Associate Professor L. S. Johnston, of Pennsylvania State College, has been appointed head of the department of engineering mathematics at the University of Detroit.

Assistant Professor J. R. Kline has been promoted to a full professorship at the University of Pennsylvania.

Dr. O. J. Peterson has been appointed head of the department of mathematics at the State Teachers' College at Emporia, Kansas.

Mr. M. R. Richardson has been appointed assistant professor of mathematics at the Georgia School of Technology.

Dr. W. C. Risselman has been appointed assistant professor of mathematics at the University of Pittsburgh.

Assistant Professor J. B. Scarborough, of the United States Naval Academy, has been promoted to an associate professorship of mathematics.

During the second half of the present academic year, beginning about February 1, Professor H. W. Tyler, of the Massachusetts Institute of Technology, will be on leave of absence in Washington, D.C., in order to establish there a permanent office of the American Association of University Professors, of which he has acted as Secretary for many years.

Dr. John Williamson has been appointed associate in mathematics at Johns Hopkins University.

The following appointments to instructorships in mathematics are announced:

George Washington University, Mr. E. W. Woolard of the United States Weather Bureau;

University of Michigan, Mr. P. M. Swingle;

Northwestern University, Mr. R. H. Bardell;

Pennsylvania State College, Messrs. L. T. Dunlap, W. O. Gordon, J. A. Hamilton, P. V. Kunkel, and R. E. Peterson;

St. John's College, Mr. G. C. Vedova.

Professor Richard Birkeland, of the University of Oslo, died April 10, 1928, at the age of forty-nine.

Professor Victor Eberhard, of the University of Halle, died April 28, 1928, at the age of sixty-seven.

Mr. Jun Hirai, of the Shizuoka High School, has died.

Professor O. Henrici, retired, of the London Technical College, has died, at the age of eighty-eight.

Professor E. Wiechert, of the University of Göttingen, has died at the age of fifty-seven.

Dr. Wilhelm Wien, professor of physics at the University of Munich, died August 30, 1928, at the age of sixty-five.

Professor R. Sterneck, of the University of Graz, has died, at the age of fifty-seven.

Professor H. B. Fine, of Princeton University, was fatally injured by an automobile on the evening of Friday, December 21, and died about 1 A.M. on Saturday, December 22, 1928, at the age of seventy. He had been a member of the American Mathematical Society since 1891, and was president of the Society during the years 1911–1912.

Miss Marion Reilly, of Bryn Mawr College, died in January, 1928. Miss Reilly had been a member of the American Mathematical Society since 1913.

Professor C. D. Rice, of the University of Texas, died May 28, 1928, at the age of sixty-five. Professor Rice had been a member of the American Mathematical Society since 1920.

Professor Alexander Ziwet, of the University of Michigan, died November 18, 1928. He had been a member of the American Mathematical Society since 1891, and had served for many years as a member of the board of editors of this Bulletin.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ARISTOTLE. The works of Aristotle, translated into English under the editorship of W. D. Ross. Volume 1. London, Oxford University Press, 1928. 4+652 pp.

VAN DEN BERG (P. J.). Die scherpe hoek. Velp, 1928. 22+25+22 pp.

BERNAYS (P.). See HILBERT (D.).

BIANCHI (L.). Congruenze di sfere di Ribaucour e superficie di Peterson. Bologna, Zanichelli, 1928. 103 pp.

BIERNACKI (M.). Sur les équations algébriques contenant des paramètres arbitraires. (Thèse, Paris.) Cracovie, Imprimerie de l'Université, 1928. 145 pp.

BLANC (E.). See PICKARD (E.).

BORTOLOTTI (E.). La scuola matematica di Bologna. Bologna, Zanichelli, 1928. 75 pp.

BOULIGAND (G.). Leçons de géométrie analytique. Paris, Vuibert, 1928. 498 pp.

CAJORI (F.). A history of mathematical notations. Volume I: Notations in elementary mathematics. Chicago, Open Court, 1928. 16+451 pp. $6.00.

—— Mathematics in liberal education. Boston, Christopher, 1928. 169 pp.

CARTAN (E.). Leçons sur la géométrie des espaces de Riemann. Paris, Gauthier-Villars, 1928. 6+273 pp.

FARWICK (W.). See LEINEWEBER (N.).

HARDY (G. H.). A course of pure mathematics. Cambridge, University Press, 1928. 12+455 pp.

HARMEGNIES (R.). See JULIA (G.).

HILBERT (D.). Die Grundlagen der Mathematik. Mit Zusätzen von Hermann Weyl und Paul Bernays. (Hamburger Mathematischen Einzelschriften, Heft 5.) Leipzig, Teubner, 1928. 28 pp.

JULIA (G.). Exercices d'analyse rédigés par R. Harmegnies et R. Julia. Tome I. Paris, Gauthier-Villars, 1928. 8+454 pp.

JULIA (R.). See JULIA (G.).

JUNGE (G.). Einführung in Wesen und Wert der Mathematik. Karlsruhe, G. Braun, 1928. 92 pp.

KNOPP (K.). Theory and application of infinite series. Translated from the second German edition by R. C. Young. London, Blackie, 1928. 12+571 pp.

LEBESGUE (H.). Leçons sur l'intégration et la recherche des fonctions primitives. 2e édition. Paris, Gauthier-Villars, 1928. 16+342 pp.

LEINEWEBER (N.) und FARWICK (W.). Wertetafel. 2te Auflage. Münster, Aschendorffsche Verlagsbuchhandlung, 1927. 8+112 pp.

LIEDLOFF (W.). Beiträge zur Psychologie der mathematischen Schulbegabung. Langensalza, Verlag Beltz, 1928. 94 pp.

Lipps (H.). Untersuchungen zur Phänomenologie der Erkenntnis. Teil 2: Aussage und Urteile. Bonn, Friedrich Cohen, 1928. 105 pp.

Menger (K.). Dimensionstheorie. Leipzig, Teubner, 1928. 4+320 pp.

von Mises (R.). Wahrscheinlichkeit, Statistik und Wahrheit. Berlin, Springer, 1928. 98 pp.

Padalino (F.). Five sheets of diagrams illustrating the Delian problem, rectification of arches, and trisection of angles. New York, 1928.

Phillips (H. B.). Calculus. London, Chapman and Hall, 1927. 353 pp.

Picard (E.). Leçons sur les équations fonctionnelles avec des applications à divers problèmes d'analyse et de physique mathématique. Rédigées par E. Blanc. Paris, Gauthier-Villars, 1928. 187 pp.

—— Selecta. Paris, Gauthier-Villars, 1928. 8+304 pp.

Prasad (G.). Six lectures on recent researches in the theory of Fourier series. Calcutta, University of Calcutta, 1928. 14+139 pp.

Ross (W. D.). See Aristotle.

Rowe (J. E.). Introductory mathematics. London, Pitman, 1928. 293 pp.

Sierpiński (W.). Leçons sur les nombres transfinis. (Borel Series.) Paris, Gauthier-Villars, 1928. 6+240 pp.

Symon (A.). The new geometry. Part 2. Glasgow, Gibson, 1928.

Turnbull (H. W.). The theory of determinants, matrices, and invariants. London, Blackie, 1928. 16+338 pp.

Werkmeister (P.). Einführung in die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Stuttgart, Konrad Wittwer, 1928. 6+198 pp.

Weyl (R.). See Hilbert (D.).

Young (R. C.). See Knopp (K.).

PART II. APPLIED MATHEMATICS

Adami (F.). Die Elektrizität. 4te Auflage. Teil 2. Leipzig, Reclam Verlag, 1928. 180 pp.

Auerbach (F.) und Hort (W.). Handbuch der physikalischen und technischen Mechanik. Band II, 1te Lieferung; Band V, 1te und 2te Lieferung; Band VI, 1te Lieferung; Band VII, 1te Lieferung. Leipzig, Barth, 1927–28. 8+404+9+274+6+718+8+460+4+238 pp.

Basu (K.) and Karmakar (S.). An intermediate course of statics. Allahabad, Indian Press, 1928. 18+308 pp.

Baur (F.). Korrelationsrechnung. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1928. 6+57 pp.

Bavink (B.). Die Hauptfragen der heutigen Naturphilosophie. I. Berlin, Salle, 1928. 121 pp.

Berthoud (A.). Photochimie. Paris, Doin, 1928. 8+323 pp.

Beyer (K.). See Mohr (O.).

Bird (G. W.). Examples in the strength and elasticity of materials. London, Arnold, 1927.

Bothe (W.). See Kohlrausch (F.).

Bowley (A. L.). Edgeworth's contributions to mathematical statistics. London, Royal Statistical Society, 1928. 7+139 pp.

Brill (A.). Vorlesungen über allgemeine Mechanik. München, Oldenbourg, 1928. 8+356 pp.

Brillouin (L.). See de Broglie (L.).

Brodhun (E.). See Kohlrausch (F.).

de Broglie (L.) and Brillouin (L.). Selected papers on wave mechanics. London, Blackie, 1928. 151 pp.

Candy (H. C. H.). A manual of physics, theoretical and practical, for medical students. London, Cassell, 1928. 496 pp.

Congrès International pour l'Essai des Matériaux. Proceedings of the International Congress for Testing Materials, Amsterdam, 1927. 2 volumes. The Hague, Nijhoff, 1928. 20+581+12+737 pp.

Coppel (T.), Fournier (G.), et Yovanitch (—.). Quelques suggestions concernant la matière et le rayonnement. Paris, Blanchard, 1928. 46 pp.

Coventry (W. B.). The mechanics of rowing. London, Spon, 1928. 8+70 pp.

Creighton (H. J.). Principles and applications of electro-chemistry. Volume 1: Principles. New York, Wiley, 1928. 16+488 pp.

Czwalina (A.). See Kleomedes.

Danneel (H.). Elektrochemie und ihre physikalisch-chemischen Grundlagen. Band 4: Elektrolyse. (Sammlung Göschen.) Berlin, de Gruyter, 1928. 144 pp.

Flechsenhaar (A.). Einführung in die Finanzmathematik. 2te Auflage, bearbeitet in Verbindung mit F. Fleege-Althoff. Leipzig, Teubner, 1927. 6+109 pp.

Fleege-Althoff (F.). See Flechsenhaar (A.).

Forsyth (C. H.). Introduction to the mathematical theory of finance. New York, Wiley, 1928. 5+205 pp.

Fortrat (R.). Introduction à l'étude de la physique théorique. Fascicule 3: Thermodynamique. Paris, Hermann, 1927. 189 pp.

Fotheringham (J. K.). See Langdon (S.).

Fournier (G.). See Coppel (T.).

Gibson (C. R.). Modern conceptions of electricity. London, Seeley, Service and Company, 1928. 284 pp.

Giebe (E.). See Kohlrausch (F.).

Graetz (L.). Die Elektrizität und ihre Anwendungen. 23te wesentlich neugearbeitete Auflage. Stuttgart, Engelhorn, 1928. 16+818 pp.

Grotrian (W.). Graphische Darstellung der Spektren von Atomen und Ionen mit ein, zwei und drei Valenzelektronen. 1ter und 2ter Teil. (Struktur der Materie in Einzeldarstellungen, Band 7.) Berlin, Springer, 1928. 13+245+10+168 pp.

Grüneisen (E.). See Kohlrausch (F.).

von Hahn (F. V.). Dispersoidanalyse. Die Methoden der Teilchengrössenbestimmung und ihre theoretischen Grundlagen. Dresden, Steinkopf, 1928. 24+553 pp.

Harwood (P. J.). A theory of the solar system. Part 2. Brighton, the Author, 1928. 2+64 pp.

Holborn (L.). See Kohlrausch (F.).

Hort (W.). See Auerbach (F.).

JEANS (J. H.). Eos, or the wider aspects of cosmogony. London, Kegan Paul, and New York, Dutton, 1928.

JELLINEK (K.). Lehrbuch der physikalischen Chemie. 2te, vollständig umgearbeitete Auflage. Band 1, und Band 2, Lieferung 4. Stuttgart, Enke, 1928. 54+966+272 pp.

JOHNSON (R. C.). Spectra. London, Methuen, 1928. 7+104 pp.

KARMAKAR (S.). See BASU (K.).

KENNELLY (A. K.). Vestiges of pre-metric weights and measures persisting in metric-system Europe. New York, Macmillan, 1928. 13+189 pp.

KLEOMEDES. Die Kreisbewegung der Gestirne. Übersetzt und erläutert von A. Czwalina. (Ostwalds Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1927. 93 pp.

KOHLRAUSCH (F.). Lehrbuch der praktischen Physik. 15te, stark vermehrte Auflage. Neu bearbeitet von W. Bothe, E. Brodhun, E. Giebe, E. Grüneisen, L. Holborn, K. Scheel, und O. Schönrock. Leipzig, Teubner, 1927. 30+832 pp.

KRÜGER (F.). Materie und Energie im Welt-Geschehen. Greifswald, Verlag Bamberg, 1928. 29 pp.

LANGDON (S.) and FOTHERINGHAM (J. K.). The Venus tablets of Ammizaduga. A solution of Babylonian chronology by means of the Venus observations of the first dynasty. With tables for the computation by C. Schoch. London, Oxford University Press, 1928. 6+109+16 pp.

MAHLER (G.). Physikalische Formelsammlung. 5te, völlig umgearbeitete Auflage, besorgt von K. Mahler. Berlin, de Gruyter, 1927. 162 pp.

MAHLER (K.). See MAHLER (G.).

MARTIN (G.). A treatise on chemical engineering applied to the flow of industrial gases, steam, water and liquid chemicals, including the pneumatic transport of powders and granulated materials. London, Crosby and Lockwood, 1928. 23+424+19 pp.

MOELLER (H.). Statistik. Berlin, Spaeth und Linde, 1928. 14+149 pp.

MOHR (O.). Abhandlungen aus dem Gebiete der technischen Mechanik. 3te erweiterte Auflage zur Jahrhundertfeier der Technischen Hochschule Dresden. Herausgegeben von K. Beyer und H. Spangenberg. Berlin, Wilhelm Ernst, 1928. 15+622 pp.

MOTTA (M.). La costituzione e l'evluzioni della materia secondo gli ultimi risultati della fisico-chimica. Roma, Loescher, 1928.

NADAI (A.). Der bildsame Zustand der Werkstoffe. Berlin, Springer, 1927. 171 pp.

OLLENDORFF (F.). Erdströme. Berlin, Springer, 1928. 8+260 pp.

ORTHNER (R.). Über physikalische und mathematische Abhängigkeit. Umrisse eines neuen Verfahrens zur Aufstellung von Gleichungen zwischen geometrischen und physikalischen Grössen. Leipzig und Wien, Deuticke, 1928. 28 pp.

SCHEEL (K.). See KOHLRAUSCH (F.).

SCHOCH (C.). See LANGDON (S.).

SCHÖNROCK (O.). See KOHLRAUSCH (F.).

SPANGENBERG (H.). See MOHR (O.).

YOVANITCH (—.). See COPPEL (T.).

THE THANKSGIVING MEETING OF THE SOUTHWESTERN SECTION

The twenty-first regular meeting of the Southwestern Section was held at the University of Kansas on Saturday, December 1, 1928. The total attendance was forty-seven, including the following twenty-three members:

N. L. Anderson, Ashton, Wealthy Babcock, Florence Black, Brenke, Gaba, Garrett, Harshbarger, Hyde, Louis Ingold, Kempner, Luby, McShane, U. G. Mitchell, Mossman, O. J. Peterson, Richert, Roever, G. W. Smith, Stouffer, J. S. Turner, J. J. Wheeler, Wyant.

Professor Mitchell occupied the chair, being relieved during the morning session by Professor Roever. The morning session was devoted to the reading of the papers listed below. During the afternoon session Professor A. J. Kempner gave a special lecture by invitation of the program committee on *The development of the analytical theory of numbers in the present century.*

The titles and abstracts of the papers read are given below. The papers of Altshiller-Court, Reid, and Whyburn were read by title.

1. Professor Louis Ingold: *Generalization of the tensor laws of transformation.*

In the theory of tensors two types of transformation of sets of quantities are employed, namely, the covariant, and the contravariant laws of transformation. These involve the first derivatives of the new variables with respect to the old, or of the old with respect to the new. In connection with certain invariant expressions there occur other types of transformation involving derivatives of one set of variables with respect to the other, of order higher than the first. These are the generalizations considered in this paper.

2. Miss Nola L. Anderson: *First normal spaces in Riemannian geometry.*

If a space is defined by a general vector function of the coordinates u^i, say $f(u^1, u^2, \cdots, u^n)$, the first derivatives $f_i = \partial f/\partial u^i$ of f at each point determine the tangent space. Every vector orthogonal to the tangent space is called a normal vector. A normal vector expressible in terms of the tangent vectors f_i and their derivatives is called a first normal. It is the pur-

pose of this paper to investigate the relations connecting various first normal vectors invariant under change of coordinates. Some of these normals are independent of any loci passing through the point under consideration while others are associated with curves, surfaces or other loci passing through the point. The details are given only for two- and three-dimensional spaces.

3. Professor U. G. Mitchell: *Note on Mersenne numbers.*

At the February 1925 meeting of the Kansas Section of the Mathematical Association of America the author presented a paper summarizing the results previously obtained and the chief methods employed in factoring Mersenne and Fermat numbers. A simple method of attack based upon geometrical considerations was suggested and some results obtained by the method were also presented. In the present note is shown as a further result obtained by the method the fact that $2^{1321}-1$ has the factor 7927, proof of which is established by two independent series of congruences.

4. Professor W. C. Brenke: *On polynomial solutions of a class of linear differential equations of the second order.*

In this paper certain polynomial solutions of the differential equation $py''+qy'+r_n y=0$ are considered, when p and q are polynomials in x and r_n a function of n alone. In particular, simple formulas for the normalizing constants and generating functions are obtained, the latter by an application of the Laplace expansion as used by Darboux for the Jacobi polynomials.

5. Professor Nathan Altshiller-Court: *Some tetrahedral complexes.*

A tetrahedral complex is defined as the totality of straight lines which cut the four faces of a tetrahedron in four points having a constant anharmonic ratio. The author proves several theorems on such complexes.

6. Professor W. H. Roever: *A geometric representation of a line integral.*

By regarding the differential expression $Pdx+Qdy$ as defining surface elements in space which cut from a vertical cylindrical surface of directrix c in the xy plane lineal elements which have for unions a one-parameter family of curves on this cylindrical surface, it is possible to give a very simple geometrical representation to the value of the line integral $\int_c Pdx+Qdy$ and to see geometrically under what conditions this line integral is zero for a closed path.

7. Professor Wealthy Babcock: *On the geometry associated with certain determinants with linear elements.*

If the linear elements of the determinantal form in three variables are set equal to zero, each represents a straight line. The geometry of these lines may be studied by placing restrictions on the invariants and covariants of the linear forms which are the elements of the determinant. Certain

facts concerning the geometry of the lines which are obtained from the elements of the determinantal form of the second order and their relation to the conic, and of the lines which are obtained from the determinantal form of the third order and their relation to the plane cubic curve were brought out in this paper. From the determinantal form of the equations of plane curves, some theorems concerning the generation of plane curves of order n by projective pencils of curves of order less than n, may be readily proved.

8. Professor J. S. Turner: *The factoring of large numbers by means of positive determinants of Seelhoff type.*

This is a continuation of former papers (this Bulletin, vol. 37, pp. 163, 399). A list (probably complete) of 14 positive determinants D is given, for which: (*a*) D contains no square factor, (*b*) $T^2-D=1$, (*c*) there are two cycles of reduced forms in each genus. The method of applying these determinants to the factorization of large numbers is explained. The prime or composite character of a number of 10 digits can thus be determined in two or three hours.

9. Professor J. S. Turner: *The expression of numbers from* 12,000 *to* 30,000 *in the form* $w^3+x^3+2y^3+2z^3$.

The computation was undertaken at the instance of Professor L. E. Dickson as part of a project to verify von Sterneck's table (Wiener Sitzungsberichte, vol. 112, II*a*, (1903), pp. 1627–66) showing the number of cubes required to represent all numbers $\leqq 40{,}000$. The method adopted is described in the American Mathematical Monthly (vol. 35 (1928), p. 113). The verification, now completed, that every number following 8041 up to 40,000 is the sum of six cubes is of the greatest importance, since this result is employed in the proof of Waring's theorem that every number is the sum of nine cubes. In addition to $p=(111111111)$ the following theorems (L. E. Dickson, American Mathematical Monthly, April, 1927, p. 177) are rendered independent of von Sterneck's table: $p=(11111112)$ (that is, Theorem III), $p=(1111114)$, where p is any positive integer $\leqq 40{,}000$. These theorems are likewise true of the forms derived from them by partitioning 2 and 4.

10. Professor E. B. Stouffer: *A canonical expansion associated with developable surfaces.*

The author obtains a canonical expansion for the equation of a developable surface by means of the fourth order differential equation associated with the edge of regression and also determines the geometrical location of the vertices of the associated tetrahedron of reference. All the work is far more simple than when the developable surface is treated as a special case of a curved surface, the method which has been used previously.

11. Mr. W. T. Reid: *Countable differential systems containing a parameter.*

A system of integral equations of the form $y(x, \mu)=\int_0^x A(t, \mu)y(t, \mu)dt$

$+a(\mu)$ is considered, where $A(x, \mu) \equiv (A_{ij}(x, \mu))$ is a square matrix with a countable number of rows and columns, $y(x, \mu) = (y_i(x, \mu))$ and $a(\mu) \equiv (a_i(\mu))$, $(i, j = 1, 2, \cdots)$. We will call a solution of this system a set of functions $y_i(x, \mu)$, $i = 1, 2, \cdots$, each of which is absolutely continuous in x for fixed value of μ and such that $y(x, \mu)$ is a vector in Hilbert space. If (1) for each μ on L, $L_1 < \mu < L_2$, $A_{ij}(x, \mu)$ is a measurable function on X, $0 \leqq x \leqq 1$; (2) $a(\mu)$ for μ on L is a single valued vector in Hilbert space; (3) there exists a non-negative summable function $\phi(x)$ on X such that for each vector ω in Hilbert space $|A(x, \mu)\omega| \leqq \phi(x)|\omega|$; then there exists a unique solution of this system on XL. Sufficient conditions are given for the solution $y(x, \mu)$ to be continuous in (x, μ) on XL, and also to possess a partial derivative with respect to the parameter μ. The above system is also considered for each element of the matrix $A(x, \mu)$ analytic in the parameter μ.

12. Mr. W. T. Reid: *Matrix solutions of countable differential systems and their adjoints.*

If $A(x)$ is a square matrix in a countable number of dimensions limited by a summable function $\phi(x)$ on $X: a \leqq x \leqq b$, that is, for every pair of points ξ and η, of Hilbert space and for every n, $|\sum_{i,j=1}^{n} A_{ij}(x)\xi_i\eta_j| \leqq \phi(x)|\xi||\eta|$, then there exists a matrix $Y(x)$ of absolutely continuous functions satisfying equation (1): $Y'(x) = A(x) \cdot Y(x)$ "almost everywhere" on X. It is proved that if $Y(x)$ is limited at some point of X, then $Y(x)$ is limited uniformly on X. Similarly, there exists an absolutely continuous matrix $Z(x)$ satisfying the adjoint equation (2): $Z'(x) = -Z(x) \cdot A(x)$ "almost everywhere" on X, and if $Z(x)$ is limited at a point of X, then it is limited uniformly on X. If $Y(x)$ is limited and at a point of X possesses a unique limited reciprocal, then $Y(x)$ possesses on X a unique, limited, absolutely continuous reciprocal which is a solution of (2). We will call a matrix solution of (1) a limited, absolutely continuous matrix which satisfies (1) "almost everywhere" on X and has a unique reciprocal. If $Y(x)$ is a matrix solution of (1) the most general matrix solution is $Y(x) \cdot C$ where C is a constant limited matrix possessing a unique reciprocal. Furthermore, if $Z(x)$ is a matrix and $Z(x) \cdot Y(x) = C$ on X, where $Y(x)$ is a matrix solution on (1) and C is a constant matrix possessing a unique reciprocal, then $Z(x)$ is a matrix solution of (2).

13. Professor G. T. Whyburn: *On regular points of continua and regular curves of at most order n.*

This paper appears in full in the present issue of this Bulletin.

E. B. STOUFFER,
Secretary of the Section

THE THANKSGIVING MEETING IN CINCINNATI

The two hundred sixty-fourth, thirtieth regular Western, meeting of the Society was held at the University of Cincinnati on Friday and Saturday, November 30–December 1, 1928. Over one hundred persons attended the meeting, among whom were the following fifty members of the Society:

B. E. Allen, W. E. Anderson, C. L. Arnold, Bareis, Barnett, Birkhoff, Blumberg, Brand, Byrne, Coble, L. W. Cohen, H. T. Davis, P. D. Edwards, Feinler, Fry, C. A. Garabedian, Hancock, E. R. Hedrick, Archibald Henderson, Ingraham, M. M. Johnson, Kersten, Kindle, Kuhn, Latimer, LeStourgeon, MacDuffee, Metz, Michal, C. N. Moore, Mullings, T. S. Peterson, Pollard, Rainich, Rasor, C. E. Rhodes, Roth, Shohat, Sibert, W. G. Simon, Spenceley, Tripp, J. H. Weaver, Wiley, K. P. Williams, C. O. Williamson, Winston, Yanney, Yowell, Zehring.

Friday afternoon was devoted to addresses given upon invitation of the Program Committee. Professor E. R. Hedrick spoke on *Recent developments regarding non-analytic functions*. Professor Archibald Henderson spoke on *Relativity: Survey and Outlook*. These papers were followed by a lecture on *The mathematical basis of art* by Professor G. D. Birkhoff. The paper of Dr. T. C. Fry on *The use of continued fractions in the design of electrical networks* which appeared on the program on Saturday morning was also upon invitation of the Program Committee.

On Friday evening fifty-five members and their guests attended dinner in the Hotel Gibson. Professor Hancock acted as toastmaster. The toastmaster called upon Professor Birkhoff to speak of his recent trip around the world, on Dr. Fry to talk of mathematics in industry, while he himself spoke on the future of mathematical meetings in the Ohio Valley region, and called on Professors Hedrick, C. N. Moore and Blumberg to discuss the same topic. The dinner was a complimentary one for the out of town members of the Society.

On Saturday morning a resolution was passed thanking the authorities of the University of Cincinnati, the Chamber of Commerce of the City of Cincinnati, the Local Press, the Hotel Gibson and the Local Committee, of which Professor C. N. Moore was Chairman, for the excellent arrangements that had been made for the program and for the comfort and pleasure of the members of the Society. A resolution was also passed thanking Professor Hedrick, Professor Henderson, Professor Birkhoff, and Dr. Fry for their interesting and stimulating addresses.

The papers whose abstracts appear below were read as follows: papers 1–7 on Friday morning, and papers 8–17 on Saturday morning. Papers 14, 15 and 17 were read by title, the rest in person. Professor E. R. Hedrick presided Friday morning, Professor Coble presided during the addresses of Professors Hedrick and Henderson; Professor C. N. Moore presided during Professor Birkhoff's lecture; and Professors Kuhn and Birkhoff presided Saturday morning. Mr. Miller was introduced by Professor R. L. Wilder, and Mr. Churchill by Professor Rainich.

1. Professor J. A. Shohat: *On the polynomial and trigonometric approximation of measurable bounded functions, on a finite interval.*

This paper shows that the notion of the polynomial (or trigonometric) "best approximation" (in Tchebycheff sense) can be extended, *in two ways*, to the case of measurable bounded functions defined on a finite interval, making use: (a) of the upper bound, or (b) the "measurable upper bound" (C. Haskins) of such functions. The second kind of "polynomials of best approximation" is more important, for it gives an indefinitely close approximation, as in the case of continuous functions. It is also closely related to $\min \int_a^b p(x)\,|f(x)-G_n(x)\,|^m dx$, as $m\to\infty$ ($p(x)\geqq 0$, $f(x)$ are defined on (a, b), $G_n(x)$ is an arbitrary polynomial of degree $\leqq n$). Considering the same minimum for m, $n\to\infty$, we get a sequence of polynomials converging, under very general conditions, to $f(x)$ uniformly over the whole interval (a, b), with an approximation of the same order (for $n\to\infty$), as the best approximation. (See the work of G. Pólya, D. Jackson, and the author.)

2. Dr. L. W. Cohen: *On sequences of functions, of sets, and of Lebesgue integrals.*

For $f(x)\geqq 0$, we define $h(f; A)$; $H(f; A)$ as the set of all points $P(x, y)$ such that $x\subset A$, $0\leqq y<f(x)$; $0\leqq y\leqq f(x)$ respectively. Theorem I: If $f_n\geqq 0$

on A and $\lim_n f_n = f$, then (1) $h(f; a) \subset \lim\inf_n h(f_n; A) \subset \lim\inf_n H(f_n; A) \subset H(f; A)$; (2) $h(f; A) \subset \lim\sup_n h(f_n; A) \subset \lim\sup_n H(f_n; A) \subset H(f; A)$, and conversely. Let W_p be the cube $-p \leqq x_i \leqq p (i = 1, 2, \cdots, n; p = 1, 2, \cdots)$ and A_k be a set in n-space. Theorem II: If meas $A_k < +\infty$ and $\lim_p \text{meas}\,(A_k - A_k W_p) = 0$, uniformly in k, then $\lim\sup_k$ meas $A_k \leqq$ meas $\lim\sup_k A_k$. Theorem III: If $\lim_k \text{meas}\, A_k = \text{meas}\, A$ and meas $\lim\inf_k A_k = \text{meas}\, A$ then $\lim_p \text{meas}\,(A_k - A_k W_p) = 0$, uniformly in k. From I and II it follows that the usual sufficient condition $|f_n| \leqq S$, where S is summable, can be replaced by the more general condition $\lim \tau = +\infty \int_A \max(0; |f_n| - \tau) = 0$, uniformly in n, in theorems on the term-by-term integration of sequences of functions. From I and III it follows that the above condition is necessary in the case of term-by-term integration of monotonic sequences or of sequences of non-negative functions.

3. Professor H. T. Davis: *A general theory of operators with explicit application to differential equations of infinite order with polynomial coefficients of degree p.*

The present theory of operators is founded upon the Pincherle-Bourlet generatrix equation $[X \cdot F] = 1$ where the bracket symbolizes the product of the differential operators $X(x, z)$ and $F(x, z)$ in which $z = d/dx$. The case where $F(x, z)$ expands into a power series in z with constant coefficients leads to the methods of the Heaviside calculus and to formal solutions of difference equations with constant coefficients. When $F(x, z)$ expands into a series in negative powers of z the operational solution of the Volterra integral equation of the closed cycle is obtained. When the coefficients of $F(x, z)$ are polynomials of degree not greater than p the solution of the generatrix equation yields three formal series: (1) in powers of z; (2) in negative powers of x; (3) in inverse factorials in x. The regions of convergence of the solutions obtained from these formal operators are discussed.

4. Professor C. C. MacDuffee: *The discriminant matrices of a linear associative algebra.*

A sufficient condition in order that the first matrices $R_1, R_2, \cdots, R_n$ of the basal numbers of an algebra be linearly independent is found to be the non-singularity of a certain matrix called the *first discriminant matrix*, since its determinant proves to be the discriminant of the algebra as defined by Speiser. From the second matrices $S_1, S_2, \cdots, S_n$ of the basal numbers we obtain similarly a *second discriminant matrix*. These matrices are symmetric, and under a linear transformation of units are transformed like the matrix of a quadratic form. Other properties are noted.

5. Professor W. E. Roth: *On the solution of the matric equation $P(A, X) = 0$ for X commutative with A.*

The present paper takes up the matric equation $P(A, X) = 0$, where $P(\lambda, \mu)$ is a polynomial in λ and μ, and where A is a given square matrix. Solutions of the equation subject to the restriction that they be commutative with A are found. This restriction is much less severe than that im-

posed in a paper previously announced by the writer, where solutions of the same equation expressible as polynomials in the given matrix were sought. See abstract in this Bulletin, vol. 34, p. 151.

6. Mr. E. W. Miller: *The most general closed point set in n-dimensional space through which it is possible to pass an arc.*

It was shown by R. L. Moore and J. R. Kline in 1919 (Annals of Mathematics, vol. 20, pp. 218–223) that in order that a closed and bounded *plane* point set M should be a subset of an arc, it is necessary and sufficient that every component of M should be either a point or an arc, and that if t is an arc of M, no point of t, with the exception of its end points, is a limit point of $M-t$. It is shown in the present paper that if M is a closed and bounded point set in n-dimensional euclidean space, these same conditions are necessary and sufficient in order that M should be a subset of an arc.

7. Professor Henry Blumberg: *Extension and applications of the theorem on the measurable bounds of an arbitrary function.*

This theorem, communicated at the New York 1928 April meeting, is as follows: If $f(x)$—to confine the statement to the case of one variable—is an arbitrary real function, there exist two measurable functions $m_1(x)$ and $m_2(x)$ such that $m_1(x) \leqq f(x) \leqq m_2(x)$ almost everywhere, and the "metric approach," in the sense of exterior measure, via the curve $y=f(x)$ is non-infinitesimal at every point of $y=m_1(x)$ and $y=m_2(x)$. This theorem seems to be of marked utility in yielding extensions of certain properties of measurable functions to unconditioned functions. Thus the author shows that the Denjoy-Young theorems on derivatives of measurable functions carry over rather directly to general functions; likewise, for example—in a certain sense—the Arzela-Fréchet necessary and sufficient condition that an infinity of measurable functions be such as to allow the extraction of a sequence, convergent on the average. There is application, too, in the theory of integration. The theorem is extended to give information also concerning the manner of the clustering of the points of $y=f(x)$ between its measurable bounds.

8. Professor Marie M. Johnson: *Tensors of the calculus of variations.*

The tensors, considered in this paper, are connected with the non-parametric problem of the calculus of variations for a space of $n+1$ dimensions. The n expressions in the Euler differential equations, multiplied by dx, and an additional function form a covariant tensor of rank 1. An application is made of this tensor to "canonical" transformations as defined by Carathéodory. It is found that the expression which, when set equal to zero, defines transversality is an invariant. The quadratic form of the Legendre condition after it is multiplied by $1/dx$ becomes an invariant. When the Weierstrass E-function is transformed, a factor is introduced. There are $n+1$ functions which are connected with the Jacobi differential equations that form a covariant tensor. The laws of transformation of the

two determinants which are used to find the conjugate points of the Jacobi condition are discussed. In connection with fields of extremals it is shown that the expression in the Hamilton-Jacobi partial differential equation is an invariant and that the necessary conditions for an integral which is independent of the path have tensor character.

9. Professor Louis Brand: *The calculus of variations from a vector standpoint.*

This paper will appear in full in an early issue of this Bulletin.

10. Dr. T. C. Fry: *The use of continued fractions in the design of electrical networks.*

The problem in electrical circuit theory which is most frequently touched upon in textbooks is "Given a particular electrical system and a known driving force to find how the system responds." There is, however, a converse problem which is equally important, viz., "Given a known driving force and the response which it is desired to produce, to find a system which will so respond." One method of attacking this problem is through the medium of continued fractions, particularly those of the Stieltjes type. The present paper explains in a general way the part that continued fractions play in the solution of such problems and discusses some extensions which must be made in the Stieltjes theory in order to meet certain mathematical restrictions that are inherent in the technical problem.

11. Professor G. D. Birkhoff: *A theorem on the singular points of ordinary differential equations.*

In the differential equations $dx_i/dt = X_i$, $(i = 1, 2)$, in which the real analytic functions X_i vanish at the singular point (0, 0), the characteristic numbers m_1, m_2 may be either (1) real and of the same sign, (2) real and of opposite signs, or (3) conjugate imaginaries. Certain associated series converge in general cases (1) and (3), but may diverge in case (2). Nevertheless this paper establishes in this case for m_1/m_2 irrational, that by a one-to-one local transformation of the x_1, x_2 plane, in which the functions involved are continuous together with all their partial derivatives, analytic except along the two invariant analytic curves through (0, 0) and possessed of certain other properties, the differential equations reduce to $dx_i/dt = m_i x_i$, $(i = 1, 2)$. This is essentially the same normal form previously obtained in the other two general cases by means of a one-to-one analytic transformation based upon the convergent series available in these cases.

12. Professor G. Y. Rainich: *Plane waves in curved space-time.*

The group of Lorentz transformations is found to which the electromagnetic field of a plane wave belongs, and a metric is sought which admits an isomorphic group. Using curvilinear coordinates introduced before, the general symmetric tensor of rank two admitting the group is found: it is taken as the fundamental tensor of a metric and subjected to the condition

that the contracted Riemann tensor of that metric has the form of an electromagnetic stress energy tensor. The consequence of this restriction is that the space has zero curvature. This result is interpreted as indicating that in general relativity plane waves do not exist—a conclusion arrived at by Baldwin and Jeffery in a different way. It follows that the only type of fields in curved space-time which possess symmetry properties of radiation are fields with plane singularities. It is indicated that a field with a great number of such singularities may be considered as having the symmetry of a plane wave if local discontinuities are disregarded much as a part of space with many point singularities corresponding to material particles may be regarded as a continuous body by neglecting local discontinuities.

13. Mr. R. V. Churchill: *On the geometry of the Riemann tensor.*

At any point of a four-dimensional Riemann space a tensor of the second type is determined by a skeleton (that is, a set of four mutually perpendicular directions intrinsically related to the tensor) and three numbers. Moreover the skeleton is uniquely determined by this tensor. A tensor of the first type is determined by a skeleton and five numbers, but the skeleton is not uniquely determined by this tensor. From any one skeleton and its set of five numbers five other skeletons with their corresponding sets of five numbers can be found. Each of these six skeletons with its set of numbers serves in the same way in the determination of their tensor. It has been shown that the Riemann tensor at any point is the sum of a tensor of the first type and a tensor of the second type. Hence the Riemann tensor at a point of a four-dimensional Riemann space is determined by two skeletons and eight numbers. In this paper most of these results are obtained by using six-vectors for the arguments of the Riemann tensor after a restatement and extension of the theory of six-vectors is made.

14. Dr. I. M. Sheffer: *On the theory of sum-orthogonal polynomials.*

Let there be given two sets of real numbers: $\{p_k\}$, $\{x_k\}$, $k=0, 1, \cdots, \infty$, with $p_k>0$ and $x_i\neq x_j$ for $i\neq j$. We denote by $S[u(x)]$ (when it exists) the weighted sum for $u(x)$ relative to $\{p_k\}$ over the range $\{x_k\}$: $S[u(x)] = \sum_{k=0}^{\infty} P_k u(x_k)$. Two functions $u(x)$, $v(x)$ are defined to be sum-orthogonal if $S[uv]=0$. This paper is concerned with the existence and properties of a (unique) system of polynomials $\{T_n(x)\}$ which are normal and sum-orthogonal.

15. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem and continuous curves in n-space.*

The author proves that a continuous curve (bounded) in n-space without a cut point which satisfies the Janiszewski-Mullikin theorem must be the simple sphere of analysis situs, establishing in a sense a "converse theorem." Using generalizations of certain plane theorems he is able to discard the condition of no cut point and to discuss in full the most general solution.

16. Professor K. P. Williams: *A generalization of the Cauchy-Riemann equations.*

The sum, product, and quotient of two analytic functions are analytic. This shows that there is a formal connection between the rules for differentiating the product and quotient of two functions and the Cauchy-Riemann equations, and suggests the following problem considered in this paper. Let $u(x, y)$, $v(x, y)$ and $u^*(x, y)$, $v^*(x, y)$ be two solutions of $au_x+bu_y+cv_x+dv_y=0$ where a, b, c, d are constants. Let it be required to annex other conditions such that U and V will also satisfy the given and annexed conditions, where $U+iV$ is the sum, product, or quotient of $u+iv$ and u^*+iv^*. It is found that one is led to the system of equations $Au_x+u_y+v_x=0$, $u_x-Av_x-v_y=0$. These equations can be regarded as a natural generalization of the Cauchy-Riemann equations. It is found that $u+iv$ can be considered as an analytic function of $(x+iv)-Ay$.

17. Professor A. D. Campbell: *Note on the Plücker equations for plane algebraic curves in the Galois fields.*

This paper appears in full in the November-December 1928 issue of this Bulletin.

M. H. Ingraham,
Associate Secretary

THE ANNUAL MEETING IN NEW YORK

The thirty-fifth Annual Meeting of the American Mathematical Society was held in New York City, from Wednesday to Monday, December 26–31, 1928, in conjunction with the meetings of the American Association for the Advancement of Science and the Mathematical Association of America. All the sessions of the Society were held at Columbia University: the Gibbs Lecture in the Casa Italiana, the Symposium on Quantum Mechanics in Room 401 of the Physics Laboratories, all other sessions in Barnard College. The meeting opened on Wednesday evening with a session of the Board of Trustees at the Hotel Lincoln (the headquarters of the mathematical organizations). Thursday was devoted to sectional sessions of the Society. The third award of the Bôcher Memorial Prize was announced at a general session on Friday forenoon; this session was followed by the annual business meeting, at which trustees, officers and members of the Council were elected. The first part of Friday afternoon was devoted to a joint session of the Society, the Mathematical Association and Section A of the American Association for the Advancement of Science; this session was followed by the sixth Josiah Willard Gibbs Lecture under the joint auspices of the Society and the A.A.A.S. On Saturday morning the Society held a joint session with Section K of the A.A.A.S. (Social and Economic Sciences); the Mathematical Association also held its sessions on Saturday. Monday was devoted to a Symposium on Quantum Mechanics, held under the joint auspices of the American Mathematical Society and the American Physical Society.

The joint dinner of the mathematical organizations, held on Friday evening in the Men's Faculty Club of Columbia University, was attended by about one hundred ninety persons. Professor Dunham Jackson was toastmaster.

The mathematicians and their guests were invited to the

various general lectures and receptions held by the A.A.A.S. They enjoyed especially a complimentary concert given to the attending scientists in Carnegie Hall on Sunday afternoon by the New York Philharmonic-Symphony Society.

The attendance at this meeting (the largest on record) included the following two hundred eighty-two members of the Society:

C. R. Adams, R. B. Adams, Adshead, A. A. Albert, E. S. Allen, Joseph Allen, Altshiller-Court, R. L. Anderson, L. B. Andrews, R. C. Archibald, R. G. Archibald, C. L. Arnold, H. E. Arnold, Ashcraft, Atchison, R. W. Babcock, Bacon, Bamforth, Barnett, Beetle, A. A. Bennett, Theodore Bennett, A. C. Berry, William Johnston Berry, Birkhoff, Blumenthal, Bowden, Bower, Bradshaw, Brinkmann, Bristol, Burington, Burkett, Cairns, Caldwell, B. H. Camp, A. D. Campbell, Carlen, R. D. Carmichael, E. T. Carroll, Carruth, A. G. Clark, Clarke, L. W. Cohen, Congdon, Coolidge, Cope, A. H. Copeland, L. P. Copeland, Cramlet, Currier, Dadourian, Darkow, J. E. Davis, Decker, Dehn, Denton, Dimick, DoBell, Dorwart, Dorweiler, Dresden, Edmondson, Eisenhart, W. W. Elliott, Engstrom, H. P. Evans, Farnum, Feinler, Fiske, Fite, D. A. Flanders, Focke, Fort, A. L. Foster, M. C. Foster, Philip Franklin, Frink, Frumveller, Fry, C. A. Garabedian, Gehman, Gergen, J. L. Gibson, Gill, D. C. Gillespie, Gilman, Glasgow, J. W. Glover, P. H. Graham, A. A. Grant, Gronwall, C. C. Grove, Hall, Hallett, Hammatt, Hansell, J. G. Hardy, E. R. Hedrick, Robert Henderson, Hickson, Hille, Himwich, Hofmann, Hollcroft, J. C. Hughes, O. M. Hughes, Huntington, Dunham Jackson, Jeffery, M. I. Johnson, R. A. Johnson, Jonah, E. H. Jones, Kasner, Kazarinoff, O. D. Kellogg, Keyser, Kholodovsky, B. F. Kimball, Rose Klein, Kline, Kramer, Lambert, Landers, Landis, Langman, Larew, Lefschetz, D. H. Lehmer, Lehr, Leib, Leinbach, Lepeshkin, Harry Levy, F. P. Lewis, Linehan, Ling, Littauer, Longley, Lotka, Lubben, Lubin, McGiffert, MacColl, MacCreadie, MacInnes, Marden, Mathewson, S. P. Mead, Mears, Meder, J. E. Merrill, E. J. Miles, Norman Miller, H. H. Mitchell, Mode, Molina, C. N. Moore, R. L. Moore, Moriarty, Frank Morley, C. C. Morris, Richard Morris, D. S. Morse, Marston Morse, Morton, Murnaghan, F. H. Murray, Nassau, C. A. Nelson, Newman, Ogburn, Olson, Oppenheim, Ore, Osgood, F. W. Owens, H. B. Owens, Paradiso, Patterson, Pattillo, Paxton, Pell-Wheeler, F. W. Perkins, T. S. Peterson, Pfeiffer, H. B. Phillips, Pierpont, Pixley, Pooler, Poritsky, Post, R. G. Putnam, Rainich, Rambo, Ramsdell, Raudenbush, Rawles, Reddick, C. J. Rees, M. S. Rees, R. G. D. Richardson, D. E. Richmond, Rider, Risselman, Ritt, Robertson, Robison, E. D. Roe, Roever, Roman, Roos, Rorer, J. E. Rowe, Ruger, Rupp, Saurel, Sauté, Schoonmaker, Seely, Seidlin, Serghiesco, Shaub, Sheffer, Shewhart, Siceloff, Silverman, Simons, Singer, Slaught, Slobin, Slotnick, Smail, A. H. Smith, C. E. Smith, W. M. Smith, Stafford, M. E. Stark, Stetson, Stocker, Stokes, M. H. Stone, Struik, Tamarkin, J. M. Thomas, J. E. Thompson, L. T. E. Thompson, Trjitzinsky, B. M. Turner, Tyler, Vallarta, Vandiver, Van Vleck,

Waddell, Wahlert, G. W. Walker, C. M. Walsh, J. L. Walsh, H. E. Webb, Wedderburn, D. W. Weeks, Weida, Weisner, M. E. Wells, Weyl, A. M. Whelan, H. S. White, Whited, Widder, Wiener, C. E. Wilder, F. B. Williams, W. L. G. Williams, John Williamson, A. H. Wilson, E. B. Wilson, E. W. Wilson, W. A. Wilson, J. W. Young, Margaret M. Young, Zariski.

At the meeting of the Council, the following fourteen persons were elected to membership in the Society:

Mr. John Geoffrey Adshead, Dalhousie University;
Professor Herbert Eli Arnold, Wesleyan University;
Professor May Margaret Beenken, State Teachers College, Oshkosh;
Professor Ludwig Bieberbach, University of Berlin;
Mr. Leo Branovan, University of Minnesota;
Professor James Edward Donahue, University of Vermont;
Professor Karl Wilmot Folley, Trinity College;
Miss Rose Klein, Washington Irving High School;
Miss Erma M. Leinbach, Sweet Briar College;
Mr. Allen Shelly McMaster, University of Colorado;
Dr. Edward Aaron Saibel, University of Minnesota;
Mr. Howard Elmer Wahlert, Washington Square College, New York University;
Miss Dorothy Walcott Weeks, Wellesley College;
Professor Hermann Weyl, Princeton University.

The following were elected to membership as nominees of Allyn and Bacon:

Mr. Hubert Banks Huntley, Brown University;
Mr. William Clemmer Mitchell, University of Cincinnati;
Mr. George Sauté, Harvard University;

as nominees of the National Life Insurance Company of the United States of America:

Mr. Aubrey Wilfred Landers, Brown University;
Dr. William Maier, University of Chicago;

as nominees of the University of Michigan:

Mr. Ruel V. Churchill, Mr. Shirley E. Field, Mr. Nevin C. Fisk, Mr. Walter O. Menge, Mr. Theodore E. Raiford, all of the University of Michigan;

as nominees of the Westinghouse Electric and Manufacturing Company:

Mr. Charles LeG. Fortescue, Mr. Clinton R. Hanna, Mr. George R. Karelitz, Mr. John F. Peters, Mr. Nicholas Rashevsky, Mr. Morris Stone, Mr. Demetry I. Vinogradoff, all of the Westinghouse Company's Engineering Department at East Pittsburgh.

The ordinary membership in the Society is now 1796, in-

cluding 178 nominees of sustaining members and 80 life members. There are also 38 sustaining members. The total attendance of members at all meetings during the past year was 769; the number of papers read was 439. The number of members attending at least one meeting was 560. At the annual election 265 votes were cast.

The reports of the Treasurer and of the auditors (Professors H. W. Reddick and Paul Saurel) showed a balance of $4709.55 exclusive of the balances in the Bulletin, Transactions, Colloquium, Journal and special funds, and of the life membership reserve. The Society's Endowment Fund now has securities of par value $75,000 yielding an annual income of $3515; sustaining membership fees for the year amounted to $4810. The amount received from sales of the Society's publications was $6776.38. The trustees adopted a budget for 1929 showing estimated expenditures and receipts as $32,245.42 and $30,855.42 respectively. During 1928 members made special contributions, amounting, approximately, to twelve hundred dollars, which wiped out the deficit on the year's business; it is hoped that these contributions will be increased this year so that the deficit anticipated for 1929 may be covered.

The Librarian reported that the Library of the Society now contains 7773 volumes.

At the annual election, which closed on Friday morning, the following trustees and officers and other members of the Council were chosen:

Board of Trustees, Professors J. L. Coolidge, W. B. Fite, E. R. Hedrick, R. G. D. Richardson, Oswald Veblen.

President, Professor E. R. Hedrick.

Vice-President, Professor Solomon Lefschetz.

Associate Secretary, Professor Arnold Dresden.

Member of the Editorial Committee of the Bulletin, Professor W. R. Longley.

Member of the Editorial Committee of the Transactions, Professor Dunham Jackson.

Members of the Editorial Committee of the Colloquium Publi-

cations, Professors G. D. Birkhoff, R. L. Moore, J. W. Young.

Members of the Council, Professors H. F. Blichfeldt, Daniel Buchanan, B. H. Camp, J. R. Kline, E. P. Lane.

The tellers appointed to count the ballots were Professors C. R. Adams and R. A. Johnson.

At the annual business meeting of the Society on Friday morning Vice-President Kellogg presented the following resolution, which was unanimously adopted: The American Mathematical Society desires to express to the President and Trustees of Columbia University, its thanks for their hospitality in connection with the present meeting, and its appreciation of the effective measures taken to make the meeting successful.

The following resolution in memory of the late Dean Fine, also presented by Vice-President Kellogg, was unanimously adopted by the Society:

The members of the American Mathematical Society have learned with profound sorrow of the death of Dean Henry Burchard Fine on Saturday last, December 22. Dean Fine was a member of this Society from its earliest days; he was its president in 1911 and 1912, and a member of its Council for many years thereafter. His whole life was devoted with rare singleness of purpose to the cause to which this Society is dedicated, the building up of mathematical science in America. He was an inspiring teacher; his influence was undeviatingly exerted for rigorous thinking and for the highest standards of scientific honesty and quality of production. These things he fostered not only by the force of his personality but also by his keen appreciation of talent in others and his selfless furthering of the opportunities and careers of those so endowed. The department of mathematics at Princeton is a monument to the forty-nine years of patient and wise planning which Dean Fine devoted to its building. The Society desires to express its sense of great loss in the death of its honored member, and order that this minute be entered in its records, also that copies be sent, through the President, to Princeton University, and to his daughter, his brother, and to his sisters.

The following appointments were reported: to represent the Society at the inauguration of President H. N. Davis, of the Stevens Institute of Technology, on November 23, 1928, Vice-President O. D. Kellogg; to represent the Society at the American Mining Congress on December 5–8, 1928, Dr. C. E. Van Orstrand.

At the meeting of the Board of Trustees it was decided to raise the prices of back volumes of the Transactions and the Bulletin and, effective January 1, 1930, to raise the price of current volumes of the Transactions to $8.

At the meeting of the Council, Professor James Pierpont was named to succeed President Virgil Snyder as one of the three representatives of the Society in the Division of Physical Sciences of the National Research Council. Professors E. V. Huntington, chairman, Dunham Jackson and H. E. Slaught were appointed as a committee to advise the printers as to forms of mathematical type.

It was voted to thank Professor Tomlinson Fort for his services on the Committee on Sustaining Memberships.

The Council decided to grant the request of the San Francisco Section that the meetings on the Pacific Slope be hereafter designated as regular meetings of the Society.

Professors E. B. Stouffer, chairman, M. H. Ingraham and H. L. Rietz were appointed a Committee on Program for the Western Meetings of the Society.

It was announced that the seventh Josiah Willard Gibbs Lecture would be held at Des Moines in December 1929 in connection with the Western meeting of the Society and the sessions of the American Association for the Advancement of Science and that a committee consisting of Professors O. D. Kellogg, chairman, E. B. Stouffer and Warren Weaver had been appointed to recommend a lecturer to the Council.

It was announced that the title of Professor R. L. Moore's colloquium lectures, to be delivered in Boulder at the Summer Meeting of 1929, will be *Point set theory*, and that Professor Lefschetz will deliver the colloquium lectures, at

Brown University at the summer meeting of 1930, the title to be announced later.

The general session on Friday morning opened with an address by Professor W. F. Osgood on *Maxime Bôcher*. Immediately afterwards the chairman, Professor E. B. Van Vleck announced that the Council, upon recommendation of a committee consisting of Professors Bell, Carmichael and Lefschetz, had made the third award of the Bôcher Memorial Prize to Professor J. W. Alexander, of Princeton University, for his memoir entitled *Combinatorial analysis situs*, published in volume 28 of the Transactions of this Society. Professor Alexander was unfortunately prevented by illness from being present to receive the award. The session closed with an invited address by Professor James Pierpont, entitled *On the motion of a rigid body in a space of constant curvature.*

At the joint session of the Society, the Mathematical Association, and Section A, held on Friday afternoon, the following papers were read:

I. *The relation of statistics to modern mathematical research*, by Professor Dunham Jackson, retiring Vice-President of Section A. This paper was published in Science for January 18, 1929.

II. *The heroic age of geometry*, by Professor J. L. Coolidge. This address, delivered upon invitation of the Society and the Mathematical Association, was published in this Bulletin, for January-February, 1929.

The sixth Josiah Willard Gibbs Lecture, entitled *An introduction to the theory of numbers*, was to have been delivered by Professor G. H. Hardy, of the University of Oxford. Shortly before the hour announced for this lecture it was learned that illness would make it impossible for Professor Hardy to be present. Professor H. W. Brinkmann, of Harvard University, very graciously consented to present the material of the lecture. The lecture will be printed in full in an early issue of this Bulletin.

The attendance at the joint session and at the Gibbs Lecture ran well over 300.

On Monday was held a symposium on quantum mechanics, under the joint auspices of the American Mathematical Society and the American Physical Society. This was in the form of invited addresses, each followed by discussion. The program was as follows:

Physical meaning of wave mechanics, by Professor J. C. Slater. Discussion led by Dr. W. F. G. Swann and Professor E. U. Condon.

Statistical interpretation of various formulations of quantum mechanics, by Professor J. H. Van Vleck. Discussion led by Professors E. H. Kennard, D. J. Struik, and E. C. Kemble.

The problem of symmetry in quantum mechanics, by Professor Hermann Weyl. Discussion led by Professors G. Y. Rainich, S. A. Goudsmit, and H. W. Brinkmann.

Harmonic analysis and quantum mechanics, by Professor Norbert Wiener. Discussion led by Dr. T. H. Gronwall and Professors H. P. Robertson and G. E. Uhlenbeck.

More than 400 persons, a much larger number than had been anticipated, including chemists as well as mathematicians and physicists, were present at the sessions of this symposium.

Titles and abstracts of the papers read at the regular sessions of the Society follow below. The presiding officers were as follows: Vice-President Kellogg over Section I (Analysis) and (relieved by Professor Lefschetz) over Section III (Analysis and Applications); Vice-President Young over Section II (Geometry) and Section IV (Algebra and Analysis Situs); ex-President Van Vleck at the general session on Friday morning; Professor R. C. Archibald at the joint session with the Mathematical Association and Section A; ex-President Birkhoff during the Gibbs Lecture and (relieved by Dr. T. C. Fry) at the afternoon session of the Symposium on Quantum Mechanics; Professor E. V. Huntington at the joint session with Section K; and Professor K. T. Compton at the morning session of the Symposium. The papers numbered 1–21 were read before Section

I, Nos. 22–39 before Section II, Nos. 40–63 before Section III, Nos. 64–92 before Section IV, No. 93 at the general session, and Nos. 94–99 at the joint session with Section K. The papers of the following authors were read by title: Albert (first, second, and fourth papers), Altshiller-Court (second paper), Ayres, Barrow, Berry, Brinkmann (second and third papers), Douglas, Ettlinger, Frink (third paper), Garver, Harkin, Hickey, Keller, Littauer, Lubben (first paper), McGiffert, Poritsky (last three papers), Powers, Reynolds, Roberts, Rutledge, Schwatt, Sheffer (first paper), Stone (first three papers), Tamarkin (first and second papers), Sullivan, G. T. Whyburn, W. M. Whyburn, and Widder. Professor Hagihara was introduced by Professor Birkhoff, Mr. Muckenhoupt by Professor Wiener, and Dr. Vasilesco by Professor Kellogg.

1. Professor H. J. Ettlinger: *Expansion problems for general second-order systems.*

Expansion problems are considered for the system $y'=K(x,n)z$, $z'=G(x,n)y$, with self-adjoint boundary conditions. The method used is that of transforming to polar coordinates, $y=u\cos v$, $z=u\sin v$. Asymptotic expansions are obtained for characteristic functions and characteristic numbers. Expansion theorems are obtained for a pair of arbitrary functions. The results contain the work of Camp, Hurwitz, and others as special cases.

2. Dr. I. M. Sheffer (National Research Fellow): *On the properties of systems of polynomials which are functionally orthogonal.*

Let there be given a linear operator L which assigns a number to each member $u(x)$ of a suitable class of functions defined over a certain range R of points: $L[u(x)]=c$. Two simple examples of L are (1) $L[u]=\int_a^b p(x)u(x)dx$, where $p(x)$ is a given function; (2) $L[u]=\sum_{k=0}^{\infty}p_k u(x_k)$, where the sequence $\{p_k\}$ is given. In (1) the range R is the set of points $a\leqq x\leqq b$; in (2) R is the set of points $x_0, x_1, x_2, \cdots$. The functions $u(x)$ and $v(x)$ are said to be L-orthogonal if $L[u(x)\cdot v(x)]=0$. We consider a system of polynomials $\{T_n(x)\}$ such that $L[T_m(x)T_n(x)]=0$, $m\neq n$, and examine the properties of this system. These properties are found to be essentially inherent in the *linear character of the operator.*

3. Dr. I. M. Sheffer: *On systems of polynomials which are permutable.*

The Appell polynomials have the property of permutability: that is, if $\{P_n(x)\}$, $\{Q_n(x)\}$ are any two sets of Appell polynomials, they are

permutable. Appell polynomials form, then, a class of permutable sets of polynomials. But this is not the only class having this property. We obtain further such classes, and examine the general theory. To each set of polynomials we assign a characteristic equation which the set satisfies. The theory of permutability links up, on one side, with the theory of summability by the method of Toeplitz and Hurwitz-Silverman, and, on the other side, with the condition that certain linear operators be themselves permutable.

4. Dr. Irwin Roman: *A formula for the summation of series.*

Taylor's theorem leads to the formula $f(n)=Z(n+h)-Z(n-h)$, where $Z(x)=(2h)^{-1}[\int f(x)dx+\sum_{u=1}^{\infty}b_{2u}h^{2u}f^{(2u-1)}(x)]$. Hence we may write the formula $\sum_{n=t}^{s}f(k+2hn)=Z[k+(2s+1)h]-Z[k+(2t-1)h]$, and also the formula $\int_{n-n}^{n+h}f(x)dx=2hf(n)-\sum_{u=1}^{\infty}b_{2u}h^{2u}[f^{(2u-1)}(n+h)-f^{(2u-1)}(n-h)]$. The constants b_{2u} are determined by a recursion formula. They are expressible in terms of the Bernoulli numbers and conversely. Illustrative examples indicate the wide applicability of the formulas, but no attempt is made to discuss the conditions necessary for the formulas to be applicable to a particular problem.

5. Professor J. D. Tamarkin: *On Volterra-Stieltjes integral equations.*

In this paper a system of n integral equations is considered of the type $Y(x)=F(x)+\int_a^x d\xi K(x,\xi)Y(\xi)$, where (in matrix notation) the vector $F(x)$ and the matrix $K(x,\xi)$ are given. It is assumed that $F(x)$ is bounded and Borel measurable and $K(x,\xi)$ is bounded on $a\leqq\xi\leqq x\leqq b$, Borel measurable in x for fixed ξ and continuous and of bounded variation in ξ for fixed x, the total variation being uniformly bounded. The uniqueness and existence of a solution (bounded and Borel measurable) is proved. Certain boundary problems of Volterra-Stieltjes type are considered, and the notion of the Green's function for such problems introduced. The results obtained are applied to the discussion of the generalized adjoint problem.

6. Professor J. D. Tamarkin: *Expansion of an arbitrary function in fundamental functions of a boundary value problem of integral type.*

It is shown, by means of some results previously obtained by the author, that if the "boundary" conditions contain integral terms, then the set of fundamental functions in general is not closed, and a set of supplementary conditions must be satisfied by the function which has to be expanded. However, the set of fundamental functions can always be extended to a closed one by addition of a finite number of functions which are easily determined.

7. Professors Einar Hille and J. D. Tamarkin: *On the summability of Fourier series.* Note 2.

The authors have continued their work on the summability of Fourier series and have found sufficient conditions for the applicability of Hausdorff

means for almost all values of the variable. A more detailed outline of the results will appear in a forthcoming number of the Proceedings of the National Academy of Sciences.

8. Dr. F. W. Perkins: *On the theory of mediation.*

By the mediant of a function $F_0(P)$ (where P is a point of a bounded open region T in a 3-dimensional euclidean space) is meant the function $F_1(P)$ whose value at P is equal to the average value of $F_0(P)$ on the surface of the largest sphere with center at P containing in its interior only interior points of T. Another form of mediation may be obtained by using the average with respect to volume. The definition may also be generalized for a region T in an n-dimensional euclidean space. In this paper a number of general properties of mediation are obtained, and especial attention is given to the discussion of the sequence $F_0(P)$, $F_1(P)$, $F_2(P)$, $\cdots$, obtained by successive application of the process. The study of this sequence shows that mediation is closely connected with the theory of harmonic functions, particularly with the first boundary value problem of potential theory. This phase of the theory has been investigated by Lebesgue (Comptes Rendus, vol. 154 (1912), p. 355), and the present author (Comptes Rendus, vol. 184 (1927), p. 182).

9. Professor I. J. Schwatt: *The summation of series the terms of which alternate in sign, not term-by-term but by groups of terms.*

The author has considered the sum of like powers of the first n natural numbers, the terms alternating in sign by groups of terms, so that, say, h positive terms are followed by h negative terms, and also the case where th numbers of terms in the positive and negative groups are unequal. The results involve the Bernoulli numbers.

10. Professor I. A. Barnett: *A characteristic property of conformal transformations in function space.*

Let R_x denote the totality of all continuous functions $f(x)$; and R_x' the totality of all functions $[f(x), \sigma]$, where σ is a real continuous variable. The projective group in the space R_x' is defined and explicit formulas obtained for all the infinitesimal transformations in this space. The author then finds the subgroup all of whose transformations leave the manifold $\int f^2(x)dx+\sigma^2=1$ invariant. Calling these transformations S, he proves that if S are subjected to a stereographic projection in the space R_x on the linear spread $\sigma=$const., the resulting transformations are the same as those that leave invariant the angle between any two curves in R_x, viz., the conformal group in R_x. This is a generalization of a well known property in spaces of a finite number of dimensions.

11. Professor M. H. Stone: *Expansions associated with self-adjoint elliptic partial differential equations.*

In the present paper the classical problem of the flow of heat in a non-homogeneous conducting medium with assigned initial temperature and

permanently assigned boundary temperature is considered from a modern point of view. The solution of the differential equation of parabolic type governing the flow is studied as a functional of the boundary conditions. There results an important theorem concerning the representation of an arbitrary bounded measurable function in terms of the solutions of an elliptic partial differential equation involving a parameter.

12. Professor M. H. Stone: *Theory of Hermitian transforms.*

In a previous communication to the Society, the author has considered the general theory of Hermitian symmetric operators in abstract Hilbert space. Operators of this type which satisfy the condition that some iterate is the identity are studied in the present paper. It is found that if T is such an operator either T or T^2 is the identity. The second alternative is given special consideration. A useful sufficient condition that an operator be of this character is obtained, and is then applied to yield a unified treatment of certain familiar operators of analysis, including the Fourier and Hankel transforms.

13. Professor M. H. Stone: *Gauss' law of error in several variables.*

In the Mathematische Zeitschrift for 1923, Pólya gave a derivation of Gauss' law of error from a functional equation. The present paper extends the derivation to the law of error for several variables. The method is similar to that of Pólya, and the results differ very little from his. With the increase of the number of variables from one to several, it is necessary to increase the number of hypotheses concerning the law of error in order to determine it uniquely. The relation of these hypotheses to the classical assumptions of Laplace and of Gauss is discussed.

14. Professor M. H. Stone: *Unitary transformations in abstract Hilbert space.*

In this paper the following theorem is proved: every one-parameter continuous group of unitary transformations in abstract Hilbert space possesses a generating infinitesimal transformation of the form iT, where T is an Hermitian symmetric operator which is maximal, and conversely.

15. Professor G. M. Robison: *The consistency and equivalence of certain definitions of summability of divergent double series.*

This paper deals with definitions of summability of divergent double sequences which can be expressed by means of a linear transformation. Let $x_{m,n}$ be the given sequence. A new sequence $y_{m,n}$ is defined by the relation $y_{m,n}=\sum_{k=1,\ l=1}^{m,\ n} a_{m,n,k,l}x_{k,l}$, where the $a_{m,n,k,l}$'s form a set of constants defining the transformation, which we denote by (A). If $\lim_{m,n\to\infty} y_{m,n}$ exists and equals λ, we say that λ is the generalized value of the sequence $(x_{m,n})$ by the transformation (A). This paper shows that all regular transformations of a certain class are consistent and gives a criterion for the equivalence of two regular definitions of summability.

16. Dr. Stefan Serghiesco: *On the number of multiple common roots of a system of n equations.*

In several earlier papers, the author has introduced a new method for obtaining the number of simple common zeros of a set of n functions, and has given two new general formulas. In the present paper, the author extends this method of the case of multiple common zeros, stating the conditions of possibility with the corresponding formulas.

17. Dr. T. H. Rawles: *On minimizing a function of two definite integrals.*

The problem discussed in this paper is that of minimizing a function of two definite integrals of the form $U(\int_{x_1}^{x_2} f(x, y, y')dx, \int_{x_1}^{x_2} g(x, y, y')dx)$. Besides obtaining the extremals, further necessary conditions are developed analogous to those of Legendre and Jacobi in the simplest problem in the calculus of variations. Sufficient conditions are obtained by means of an invariant integral.

18. Professor George Rutledge: *Limiting values of Lagrangean coefficients.*

The coefficients, C_j, of the Lagrange polynomial, of degree $2n$, $P^{[2n]}(\lambda) = C_{-n}y_{-n} + \cdots + C_0y_0 + \cdots + C_ny_n$, useful for interpolation without differences, approach limiting values as n becomes infinite, as follows: $\lim C_j = (-1)^{j+1}[\lambda/(j-\lambda)](\sin \lambda\pi)/(\lambda\pi)$.

19. Professor George Rutledge: *Fundamental tables for Lagrangean coefficients.*

This paper gives tables with fifteen decimal places from which the coefficients of the Lagrange polynomial are readily derived for any range of λ and for any even degree up to and including sixteen.

20. Professor D. F. Barrow: *Functions of iteration and their relation to non-linear difference equations.*

The symbol $f^n(t)$ denotes n iterations of the given function $f(t)$, and defines a function of two arguments; t, a continuous variable, and n, an integer. The author proposes to define this *function of iteration* for all values of n; and this is, by definition, made tantamount to finding analytic solutions of the difference equation $\phi(x+1) = f[\phi(x)]$. The solution of the problem is embodied in a group of existence theorems and approximation formulas for numerical computation. The chief interest centers about the *points of equilibrium* which satisfy $A = f(A)$. The paper closes with a number of examples, including the analytic iteration of the sine and exponential functions, that is, the solution of the equations $\phi(x+1) = e^{\phi(x)}$, and $\phi(x+1) = \sin \phi(x)$.

21. Professor D. V. Widder: *An inequality related to one of Hilbert.*

In the Mathematische Zeitschrift of 1921, L. Fejér and F. Riesz established a fundamental inequality, and applied it to the proof of a familiar inequality of Hilbert. They proved that if $f(z)$ is analytic in a circle $|z| \leqq 1$, then $\int_{-1}^{1} |f(z)|^2 dz \leqq \frac{1}{2}\int_{-\pi}^{\pi} |f(e^{i\vartheta})|^2 d\theta$. In the present paper, the circle is replaced by a half plane, and then the inequality $\int_0^\infty |f(z)|^2 dz \leqq \frac{1}{2}\int_{-\infty}^{\infty} |f(iy)|^2 dy$ is established under suitable conditions. This result is used to prove the inequality $\sum_{m,n=0}^{n} a_m a_n/(m+n+1) \leqq \pi \sum_{m,n=0}^{N} (m+n)!\, a_m a_n \cdot (m!n!2^{m+n+1})^{-1}$, where the a_m are any real numbers. This inequality includes that of Hilbert, for it is shown that the right hand member is not greater than $\pi \sum_{n=0}^{N} a_n^2$.

22. Professor C. A. Nelson: *On surfaces possessing a net of plane isothermally-conjugate curves.*

It is well known that the Laplace transforms of a surface relative to a conjugate system of plane curves are developable. This paper studies the effect of the isothermal property upon such surfaces and their projective properties are determined. The problem of integrating the defining system of partial differential equations is investigated, and, in certain cases, the complete integration is effected, so that explicit formulas are obtained for the coordinates of the surface.

23. Professor C. T. Sullivan: *The determination of plane nets characterized by certain properties of their Laplace transforms.*

This paper will appear in full in an early issue of this Bulletin.

24. Dr. M. M. Slotnick: *A conjugate net of special type.*

A conjugate net for which each tangent to the v-curve of its first Laplace transform intersects the tangent to the corresponding u-curve of its minus first Laplace transform will be called a net Δ. The point of intersection of these tangents is the pivotal point of the axis. A net whose ray curves or whose axis curves are indeterminate is a net Δ. A characteristic property of nets Δ is the following: the tangents to the axis curves and to the ray curves separate the tangents to the curves of the net harmonically. If a net Δ has equal point [tangential] invariants, the ray [axis] curves constitute the associate conjugate net of the given net.

25. Professor Edward Kasner: *Higher partial derivatives of polygenic functions of several variables.*

The higher partial derivatives of polygenic functions of two or more complex variables depend, not only on the curves of approach for each of the independent variables (slope, curvature, etc.), but also on the correspondence between the curves. Also, as indicated in earlier papers (see Transactions of this Society, vol. 30, p. 818) we must take into account the order of the variables.

26. Professor T. R. Hollcroft: *The variety of order n in n dimensions.*

There is a one-to-one correspondence between varieties of order n in i dimensions and varieties of order i in n dimensions, since the number of conditions necessary to determine each is the same. In this correspondence, the variety for which $n=i$ is self corresponding. It is proved that every general variety of order n in n dimensions, $n\geqq 3$, contains the following: an $(n-3)$-fold infinity of lines; a triple infinity of plane elliptic cubics; a single infinity of conics; a finite number of sets of three coplanar lines; for $n\geqq 4$, a $(d+1)(d-3)$-fold infinity of sets of $d+1$ lines, each set contained in a linear d-space, $d<n$; an $(n-1)$-fold infinity of space cubics; a triple infinity of rational space quartics; a fourfold infinity of elliptic space quartics. The general variety of order four in four dimensions contains a finite number of sets of four coplanar lines, and a finite number of sets of four non-intersecting lines, each latter set contained in a linear three-space. The number of these sets is found.

27. Dr. Oscar Zariski: *The existence problem for algebraic functions of two variables.*

The determination of the algebraic functions of two complex variables x, y, which possess a given branch curve D (a two-dimensional Riemann surface) in the real space S_4 of the variables x and y, depends upon the determination of the fundamental group G of the residual space S_4-D. The following theorems are stated: (1) If D, of order n, is irreducible and possesses double nodal points only, then G is the cyclic group of order n. (2) If the order n of the irreducible curve D is <6, then G is always the cyclic group of order n. (3) The first example of non-cyclic fundamental group is furnished by sextics with 6 cusps on a conic. Here G possesses two generators g_1, g_2, the generating relations being $g_1g_2g_1=g_2g_1g_2$, $(g_1g_2g_1)^2=1$. (4) If D possesses double nodal points only, and breaks up into k curves of order $n_1, n_2, \cdots, n_k$, respectively, then G is an abelian group generated by k generators $g_1, g_2, \cdots, g_k$, subjected to one generating relation: $g_1^{\rho}=1$, where ρ is the h.c.d. of $n_1, n_2, \cdots, n_k$. The following questions are raised: (a) What is the minimum number of cusps which an irreducible curve of given order and genus must possess, in order that G should not be cyclic? (b) Does the fundamental group depend upon the number of the cusps only, or does it also depend upon the position of the cusps?

28. Professor Malcolm Foster: *Note on cyclic systems.*

It is shown in this paper that the four focal surfaces of a cyclic system whose planes envelope a non-developable surface are the envelopes of the two families of spheres which contain the circles of the system and have their centers at the focal points of the associated cyclic congruence, and that the focal planes of this cyclic congruence bisect the arcs between corresponding focal points. If the planes of the circles of the system envelope the unit sphere, the associated cyclic congruence is normal; and the circles under an inversion go over into a second family of circles whose axes also

constitute a normal congruence. The paper also considers various cyclic systems lying in the XZ-plane of the moving trihedral of the surface of reference.

29. Mr. H. L. Dorwart: *The discussion of a construction of the rational plane quintic.*

The rational plane quintic, R_2^5, has a perspective conic, C, and if 2 of the 6 nodes of the R_2^5 are in a quadratic involution, it has 15 involution contact conics. The pairs in the involution are given by the extra intersections of conics on the remaining 4 nodes. Two of these conics will touch R_2^5, and will have a line, L, in common with C. Since L is perspective with C, and since each conic on the 4 nodes cuts out 2 points on L which are in (1, 1) correspondence with the extra 2 intersections of the conic and R_2^5, this gives a method for constructing the curve; it is mentioned by Morley in *The contact conics of the plane quintic curve* (The Johns Hopkins Circular, 1912). If two of the four nodes are at I and J, the pencil of conics is a pencil of circles, and the construction is simplified. We show a number of quintics, obtained by taking the different conic sections for the perspective conic, and by varying the relative positions of conic, line, and nodes.

30. Professor C. A. Rupp: *A geometric interpretation of linear dependence in n-space.*

We consider the geometrical implications of the algebraic statement that the set of homogeneous Grassmann-Plücker coordinates of an S_k in S_n is a linear combination of the sets of coordinates of several S_k's in the same S_n. The author has previously shown that $n+1$ lines (or S_{n-2}'s) which are linearly dependent are in the position of Schläfli, but here gives a simpler proof. A set of Schläfli or linearly dependent spaces determine and lie upon a variety of order and dimension $n-1$; Segre discussed this variety synthetically, as does Wong in a recent paper. By using the concept of linear dependence as central in the study of this variety, the author has found a method of obtaining its equation, and displays several examples in spaces of low dimensions. Another of Segre's synthetic discoveries was that of systems of "associated" lines and planes in S_4 and S_5. They are examples of a more general type of configuration, consisting of linearly dependent flat spaces; several of these configurations are discussed, and in particular it is shown how many spaces of such a set may be chosen arbitrarily.

31. Professor Nathan Altshiller-Court: *On nodal cubics circumscribed about a triangle.* First paper.

Consider the pencil of nodal cubics Σ circumscribed about the triangle $A_1A_2A_3$ and having in common the double point O and the two tangents at this point (see this Bulletin, vol. 26, pp. 203 and 114). Any point $P_{1,n}$ of the side $a_1 \equiv A_2A_3$ of the triangle $A_1A_2A_3$ determines a cubic Σ_n of the pencil Σ. (1) As the point $P_{1,n}$ describes the line a_1 the tangents to the respective cubics envelope a tricuspidal quartic (Δ_1) tangent to a_1 in A_2 and A_3, and to the line OA_1 in the points A_1. (2) The tangents drawn from $P_{1,n}$ to the respective cubics pass through two fixed points, namely the

points of intersection of the line OA_1 with (Δ_1). (3) The points of contact of these tangents lie on two fixed straight lines passing through O. (4) The line joining the points of contact of these two tangents passes through a fixed point. (5) The locus of the tangential of $P_{1,n}$ is a quartic curve circumscribed about the triangle $A_1A_2A_3$ and having a triple point at O. (6) The locus of the corresponding point of $P_{1,n}$ is a conic passing through the points O, A_1, A_2, A_3. (7) The line joining $P_{1,n}$ to its corresponding point envelopes a tricuspidal quartic, the line a_1 being doubly tangent to the curve.

32. Professor Nathan Altshiller-Court: *The Longchamps sphere. A definition.*

The usual definition of the Longchamps circle of a triangle (American Mathematical Monthly, 1926, p. 368) may be modified as follows (Mathesis, 1928, p. 156): Consider the three circles each having for center the vertex of a triangle and for the square of its radius the sum of the powers of the remaining two vertices with respect to the conjugate circle of the triangle. The orthogonal circle of these three circles is, by definition, the Longchamps circle of the triangle. This new form of the definition suggests an immediately obvious extension to space. It is shown that the sphere so defined has another characteristic property, analogous to one of the circle, namely, that it is the conjugate sphere of the anticomplementary tetrahedron of the given tetrahedron.

33. Professor A. D. Campbell: *Pencils of quadrics in the Galois fields of order 2^n.*

In the Galois fields of order 2^n the quadric $Q\equiv ax^2+by^2+cz^2+dw^2+fyz+gzx+hxy+lxw+myw+nzw=0$ has the discriminant $\Delta\equiv f^2l^2+g^2m^2+h^2n^2\equiv(fl+gm+hn)^2$. Hence the discriminant Δ of a pencil of quadrics $P\equiv 2Q_1+\mu Q_2=0$ shows that P has one or two degenerate quadrics if $\Delta\not\equiv 0$, but if $\Delta\equiv 0$, P consists entirely of degenerate quadrics. In this paper are derived the classes of pencils of quadrics in the Galois fields of order 2^n and a typical pencil is given for each class. There occur pencils without degenerate quadrics. The most difficult but most interesting part of the problem is the discussion of the equivalence or nonequivalence of typical pencils.

34. Professor C. N. Reynolds: *On polyhedra.*

Given a finite set of integers, $a_n(n\geqq 3)$, when is it possible to construct a convex simply connected closed polyhedron bounded by a_n n-sided polygons with three polygons meeting at each vertex? It is known that Euler's theorem implies the necessity of the condition (1): $\Sigma_3(n-6)a_n+12=0$. This equation has 610 solutions for which $a\equiv\Sigma a_n<11$, whereas only 140 of them are characteristic of known polyhedra. The most important propositions of the present paper yield a group of inequalities of the form (2): $\Sigma_3(n^2+pn+q)a_n\leqq f(a)$, which eliminate 347 of the 470 "extraneous solutions" of (1) mentioned above. As a increases beyond ten, the number of solutions of (1) eliminated by (2) increases numerically, but the

ratio of this number to the total number of extraneous solutions of (1) decreases.

35. Professor C. N. Reynolds: *On the problem of coloring maps in four colors.* Third paper.

Having considered (Annals of Mathematics, (2), vol. 28, pp. 1, 447) the geometry of maps which are irreducible with respect to reductions dealing with the immediate neighborhoods of regions, the author in the present paper deals with the gross structure of irreducible maps. The results of the preceding paper on polyhedra are found to be of value in describing such maps.

36. Miss Deborah M. Hickey: *Isometric circles on the sphere.*

Let the surface of the sphere be transformed into itself in a one-to-one and directly conformal manner. In this paper it is proved that, except for a rotation of the sphere, the areas of magnification and of diminution constitute respectively the smaller and larger spherical caps formed by a small circle on the sphere. Certain of the properties of this isometric circle are derived.

37. Dr. Jesse Douglas (National Research Fellow): *On the problem of Lagrange in the calculus of variations.*

The simplest form of this problem is to minimize the integral $\int F(x, y, z, y', z')dx$ under the auxiliary condition $\phi(x, y, z, y', z')=0$. A singular or so-called *anormal* case arises of which various interpretations have been given by Mayer (Mathematische Annalen, vol. 26 (1886)), von Escherich (Wiener Sitzungsberichte, vols. 107 (1898), 108 (1899), 110 (1901)), and Hahn (Mathematische Annalen, vol. 58 (1903–04)). The principal result of the present paper is a new interpretation of the anormal case to the effect that the characteristics of the auxiliary Monge equation are then extremals. Various other aspects of the problem are considered.

38. Dr. Jesse Douglas: *The affine, voluminar, and descriptive equivalence of spaces of K-spreads.*

(See the author's abstract *The geometry of systems of K-spreads*, this Bulletin, Jan.-Feb., 1928.) Two spaces of K-spreads are (1) affinely equivalent if they can be converted into one another by an arbitrary analytic transformation of coordinates, combined with a linear transformation of parameters, (2) voluminarly equivalent if they can be converted into one another by an arbitrary analytic transformation of coordinates combined with a transformation of parameters with constant jacobian, (3) descriptively equivalent if they can be converted into one another by an arbitrary analytic transformation of coordinates combined with an arbitrary analytic transformation of parameters. Necessary and sufficient conditions are established for the three types of equivalence. In particular, conditions are derived for affine, voluminar, and descriptive flatness, that is, for the

respective types of equivalence to the space $\partial^2 x^i/\partial u^\alpha \partial u^\beta = 0$. The methods and results are in the order of ideas of Christoffel's criteria for the equivalence of two quadratic differential forms, the criteria of Veblen and J. M. Thomas (Annals of Mathematics, Dec., 1925) for the projective (descriptive) equivalence of two restricted spaces of paths, and the author's criteria (abstract in this Bulletin, July-Aug., 1928) for the affine and descriptive equivalence of two general spaces of paths.

39. Dr. Jesse Douglas: *The Weyl tensor in the general geometry of paths.*

In the author's paper *The general geometry of paths* (Annals of Mathematics, (2), vol. 29, No. 2) a formula (7.11) is given for the Weyl tensor, analogous to the one given by J. M. Thomas (Proceedings of the National Academy of Sciences, vol. 11, pp. 207–209) for the restricted geometry of paths. In the present note a second formula is established for the Weyl tensor like the one given for the restricted theory by Veblen and T. Y. Thomas (Transactions of this Society, vol. 25, p. 560), being based on considering the effect on the curvature tensor of an arbitrary change of parameter on the paths.

40. Dr. Morris Marden (National Research Fellow): *On the zeros of linear partial fractions.*

This paper determines the locus of the zeros of the partial fraction $F=\sum_1^p m_j/(z-z_j)$, when each z_j varies over a given circular region C_j. Its results are applied to the location of the zeros of the derivative of a rational function, especially of a polynomial, to the location of the zeros of the Jacobian of two binary forms, and to a certain cross-ratio problem. Its theorems are essentially generalizations obtained analytically of theorems previously obtained by Lucas, Bôcher, and Walsh.

41. Professor W. W. Elliott: *Note on generalized Green's functions.*

In a previous paper (American Journal of Mathematics, vol. 50, p. 243) the existence of generalized Green's functions for compatible differential systems has been demonstrated, and the characteristic properties of certain types of symmetrical generalized Green's functions have been studied. In the present paper other types of symmetrical generalized Green's functions are discussed.

42. Dr. W. J. Trjitzinsky: *On quasi-analytic functions.*

Borel's monogenic functions, whose discovery antedates that of quasi-analytic functions of a real variable (Denjoy), are quasi-analytic in a complex domain C from which a point set of measure zero has been excluded. In this paper it is shown that Borel's functions belong to the first Denjoy class with certain restrictions on the domain of existence. Further, series of the form $\sum c_n/(x-a_n)$, which are quasi-analytic when the c_n satisfy conditions given by Borel and Carleman, are considered. With less re-

strictive specifications of the c_n, conditions are found under which such series belong to the first Denjoy class. Carleman's representation when initial values are given is extended. Necessary and sufficient conditions are found securing convergence of a sequence of polynomials to a quasi-analytic function. A new representation of a class of functions is given.

43. Professor R. L. Jeffery: *The uniform approximation by means of Riemann sums of a function defined by a Lebesgue integral.*

Let $f(x, y)$ defined on a bounded domain be summable in x for each y. The chief concern of this paper is to determine conditions on $f(x, y)$ under which $\int_a^b f(x, y)dx$ can be approximated by a Riemann sum $\sum f(\xi_i, y)(x_i - x_{i-1})$ uniformly in y. Extending a result due to Lebesgue, concerning the simultaneous approximation of p functions $g_1(x), \cdots, g_p(x)$ (Annales de Toulouse, (3), vol. 1, p. 33), the author shows that if $f(x, y)$ is bounded and continuous in y for each x, then for any subdivision with norm sufficiently small, there exists ξ_i on (x_{i-1}, x_i) such that $\left|\int_a^b f(x, y)dx - \sum f(\xi_i, y)(x_i - x_{i-1})\right| < \epsilon$, independent of y. Lebesgue remarks that his result holds for an infinite sequence $g_1(x), g_2(x), \cdots$. It seems that this statement needs some qualification, even when $g_n(x)$ is bounded in x and n.

44. Dr. Hillel Poritsky (National Research Fellow): *The billiard-ball problem on a convex table.*

The problem considered is the motion of a point in the plane inside a simple closed convex curve in a straight line, with reflection at the boundary so that the angle of incidence is equal to the angle of reflection. This problem, because of its lack of formal complication, is well suited for illustrating various methods in dynamics, and has been used by Birkhoff in connection with Poincaré's geometric theorem and his own extension of it (Acta Mathematica, vol. 50). The results obtained in the present paper are based largely on the existence of an invariant integral for the transformation associated with the reflection of paths at the boundary. Thus, the existence and distribution of certain periodic orbits as well as of orbits passing through two points or satisfying other conditions is inferred directly from the existence of the invariant integral, by simple geometric methods. The integrability of the problem is studied from various points of view leading to results of interest in projective and differential geometry. Morse's theory of extremals is found to apply, in spite of the discontinuity in the calculus of variations problem whose extremals constitute the above orbits.

45. Dr. Hillel Poritsky: *Cauchy-Green expansions.*

Under this title are considered certain expansions of analytic functions over points P inside a surface S, that express u as a sum of integrals over S involving the boundary values of $u, \nabla^2 u, \nabla^4 u, \cdots$, multiplied by certain simple functions of the distance from P and its normal derivative. These expansions form the analogue, for analytic functions of several variables, of the Cauchy integral for a function of a complex variable, and, more

precisely, of the Green "double-layer" integral for harmonic functions. A finite series of this type with a remainder is obtained by successive application of Green's theorem. It is then shown that if the number of terms be allowed to become infinite the remainder, under proper restrictions on S, will approach zero. The restrictions on S may be materially lightened if u satisfies certain partial differential equations.

46. Dr. Hillel Poritsky: *On certain integrals over spheres.*

It is shown that the linear operation on a function u that consists in replacing u over each of a family of concentric spherical surfaces by the function of the radius $\int udS/\int dS$ or the "average" of u over the spherical surface is permutable with the Laplacian operator. This property is further extended to certain other operators that may be described as leading from u to the various terms in the expansion of u in a series of spherical surface harmonics. From these simple facts are deduced a great many results, some familiar, some new, in the theory of harmonic and related functions. Among these are various laws of the spherical mean, integral representations for spherical harmonics and Bessel functions, and the reduction of certain expansions to Fourier expansions. The motif of the paper is geometric throughout.

47. Dr. Hillel Poritsky: *On certain oscillation theorems.*

It is shown that a homogeneous harmonic polynomial of degree n that does not vanish along a certain ray, l, through the origin, must change sign within the conical region about l in which the harmonic polynomial $p_n = r^n P_n(\cos\theta)$ stays positive, where r is the distance from the origin and θ is the angle that rays through the origin form with l. Analogous results are established for bi-harmonic, tri-harmonic, $\cdots$, polynomials, as well as for solutions of certain partial differential equations. In this connection a necessary and sufficient condition is derived and utilized for the existence of a linear combination of several given functions of a single variable that does not vanish throughout a certain interval. It is then shown how some of the former results may be extended to regions of arbitrary shape by the use of the characteristic parameter values associated with these regions.

48. Professor E. R. Hedrick: *Integrals along given curves in the complex domain.*

In this paper the author discusses integrals taken along given curves in the complex domain, for functions that are non-analytic. It is shown that certain types of theorems are easily extended from the traditional theory to include integrals of non-analytic functions whenever the paths are fixed in advance. Thus the theorems recently stated by Pfeiffer (this Bulletin, vol. 34 (1928), pp. 656–664) can be thus extended.

49. Professor Marston Morse: *A generalization of Sturm's separation theorem in n dimensions.*

The author considers a set of n self-adjoint, second-order, linear, homogeneous differential equations in n dependent variables (y) and an in-

dependent variable x. The matrix of the coefficients of the second derivatives is assumed to have a characteristic determinant all of whose roots are positive. Any such system of differential equations are shown to be the Jacobi differential equations of an easily determined integral in the calculus of variations. With this point of view, the author considers the determinants $D(x)$ of systems of n linearly independent solutions which are mutually conjugate in the sense of von Escherich. He shows that if one of these determinants vanishes $n+r$ times on a closed interval of the x axis, any other of these determinants will vanish at least r times on the same closed interval. This theorem and others appeared as corollaries of a fundamental theorem on the type number of a quadratic form discovered and used in a development of the calculus of variations in the large.

50. Professor J. L. Walsh: *On approximation by rational functions to an arbitrary function of a complex variable.*

In the plane of the complex variable z, let M be a closed point set whose boundary consists of a finite number of Jordan arcs which do not separate the plane into more than a finite number of regions. Then an arbitrary function $f(z)$, continuous on M and analytic in the interior points of M can be uniformly approximated on M as closely as desired by a rational function of z.

51. Professor Orrin Frink: *Functions defined over an abstract set.*

In this paper the theorems that a function continuous over a connected set assumes all values between its maximum and minimum, and that a function continuous over a compact-in-itself set is bounded and assumes its upper and lower bounds, the Heine-Borel theorem, and other standard theorems of real function theory, are proved either for the general topological space or for a space subject to one weak postulate. The theorems are more general than the corresponding ones proved by Fréchet for neighborhood spaces.

52. Professor James McGiffert: *Two closed forms of particular solutions of the linear differential equation of second order* $(A+Bx^2)y''+Cxy'+Dy=0$.

We assume a solution in series form, $y=\sum_{m=0}^{\infty}a_m x^m$. Substituting this value for y, and its two derivatives for y' and y'', and equating the combined coefficients of x^m to zero, we obtain the recurrence relation $a_{m+2}=-(B/A)[(m^2+(C-B)m/B+D/B)/((m+1)(m+2))]a_m$. This rather complicated recurrence relation does not in general lead to useful forms of particular solutions. We may, however, so choose the coefficients, that the numerator of the fraction in brackets shall have $m+1$ or $m+2$ as a factor. Then the recurrence relation will be much simpler, and will reduce to two binomial forms, depending on the value of D in terms of B and C, as follows: (1) when $D=C-2B$, (2) when $D=2C-6B$. In the first case we obtain the solution $y=(A+Bx^2)^{1-c/(2B)}$, and in the second case, $y=x(A+Bx^2)^{1-c/(2B)}$.

Legendre's equation is obtained by making $A=1$, $B=-1$, $C=-2$, and $D=k(k+1)$.

53. Professor W. M. Whyburn: *Non-isolated critical points of functions.*

A function $f(x_1, \cdots, x_n)$ is studied, where the first partial derivatives of f are assumed continuous in a closed and connected portion R of the space of the n real variables $x_1, \cdots, x_n$. Critical points of f are considered under hypotheses which are not sufficient to cause these points to be isolated, and the notion of "critical sets" is developed. (For the case of isolated critical points, see Marston Morse, Transactions of this Society, vol. 27 (1925), pp. 345–396.) In particular, it is shown that if f has two distinct "minimal sets" in R it must have at least one other critical point in R. Upon specialization, this result yields Bieberbach's form of Birkhoff's "minimax principle" (see Bieberbach, *Differentialgleichungen*, Berlin, 1927, p. 140).

54. Professor Philip Franklin: *Equimomental systems.*

In this note we investigate certain simple configurations of mass points which are equimomental to any given rigid system, that is, have the same moment of inertia about all axes in space. It is shown that any system is equimomental to four mass points, which may be selected in a triple infinity of ways. The ratio of the masses at the points may be arbitrarily assigned, and in general uniquely determines the equivalent set. The tetrahedron with vertices at the four points has its opposite edges conjugate with respect to the homogeneous solid ellipsoid equimomental to the system. Some additional geometrical relations are given.

55. Mr. C. F. Muckenhoupt: *Vibrating systems and almost periodic functions.*

A vibrating system such as a vibrating string or membrane has its energy invariant under proper assumptions as to boundary conditions. By means of the finiteness of this invariant, together with similar invariants of higher order, it is proved that a vibrating membrane or string returns arbitrarily close to its initial position within some finite time. It follows from this with another definition of distance that the membrane or string not merely returns to the neighborhood of its initial position but that the difference between the displacement of the membrane or string and its original displacement has an energy which is arbitrarily small. From this the almost periodic character of the vibration may be proved. It is possible on this basis to discuss the closure of the set of characteristic functions of a vibration problem.

56. Professor Yusuke Hagihara: *Complete stability of a dynamical system with long-period variations.*

The author deals with a dynamical system with long-period variations superposed on short-period variations, from the stand-point of complete

stability as defined by Birkhoff. Then a certain number of parameters are introduced in the differential equations of the dynamical system, and the results compared with Poincaré's work in the problem of three bodies.

57. Professor C. A. Garabedian: *Concerning two-dimensional solutions of the problem of the thick circular plate loaded at its center.*

The author's corrections of de Saint-Venant's solutions (Journal de l'Ecole Polytechnique, (2), vol. 26 (1927)) followed the notation of de Saint-Venant and employed only devices found in the "Note du paragraphe 45." The present paper complements the foregoing in that it examines critically contributions subsequent to de Saint-Venant. Only by thus completing the study of the problem does it seem possible effectively to remove the obscurities which have pervaded the two-dimensional solutions. By including also some consideration of recently developed three-dimensional solutions, there is secured, on this whole question of the thick circular plate centrally loaded, a perspective hitherto lacking.

58. Professor E. G. Keller: *Harmonic functions for an anchor ring of elliptical meridian section.* Preliminary communication.

In this paper a solution of Laplace's equation in series in elliptical toroidal coordinates is obtained. Circular symmetry is assumed. Each term of the series is a particular solution of the equation. Successive changes of independent variables and the same change of dependent variable as that employed by C. Neumann with reference to a circular tore (*Theorie der Elektricitäts- und Wärme-Vertheilung in einem Ringe*) yield the equation $\xi^2 U_{\xi\xi}+2\xi\eta U_{\xi\eta}+\eta^2 U_{\eta\eta}+U_{\zeta\zeta}+\frac{1}{4}U=0$. A solution of this equation is $T_n(r)\cdot Y_n(\zeta)$, where T_n and Y_n are solutions of ordinary differential equations. The latter may be written as a series of Bessel coefficients of increasing order. The function T_1 is reducible to the Bessel function of order zero and of purely imaginary argument. Recurrence formulas for T_n are obtainable. If an expansion of a function $f(r)$, in either of the series $\sum_{n=0}^{\infty} A_n T_n$ or $\sum_{m=0}^{\infty} B_m T_n(a_m r)$, exists, the coefficients are readily expressed as definite integrals.

59. Dr. L. T. E. Thompson: *Note on empirical summaries in engineering ballistics.*

It is essential to facilitate extension (in scale) of experimental results bearing on certain ballistic problems. The study of wear or erosion phenomena, an aspect of the general uniformity problem of the gun cycle, may be developed by means of the ratio $f(C/C_0,\ l/d)\equiv P(C/C_0)/Z(l/d) = E/(kd^n)$; rate of bore enlargement E, powder charge C (datum value C_0), length and diameter of bore l and d; k and n are constants. Introducing appropriate assumptions with respect to maximum permissible enlargement, a relative life-of-gun formula is obtained. For the representation of plate punching data, it is practicable to identify a numerical (penetration)

coefficient $F(e/d, \theta) = K_1 M^{1/2} V \cos \theta / e^{1/2} d$; projectile mass M, diameter d, limit velocity V, plate thickness e, obliquity of impact θ (see the author's paper in Mémorial de l'Artillérie Française, vol. 6 (1927), p. 1255, note 3); by subsequent test this form is justified, especially in case application to moderate departure from similarity of projectiles is contemplated. It is required to determine one set of experimental F curves for each class of projectiles and of plate characteristics. A laminated or divided armor coefficient can be identified by means of these and a corresponding absorption summary.

60. Professor Norbert Wiener: *Hermitian polynomials and Fourier analysis.*

The nth Hermitian function differs from its Fourier transform by the factor i^n. The present paper is devoted to the study of the transformation which multiplies the nth Hermitian function by ω^n, where ω lies on the unit circle.

61. Mr. A. C. Berry: *Spectral intensity of an n-dimensional function.*

A proof of the complex n-dimensional form of the fundamental theorem in the Plancherel theory of the Fourier transform is given in the first part of the present paper. Although it has seemed necessary to use cartesian coordinates in giving the proof, the theorem itself is vectorial. The remainder of the paper discusses the intensity of the spectrum of a complex point function of a point in real n-dimensional space. A sufficient condition that the total spectral intensity be finite is given. This discussion is the vectorial, n-dimensional generalization of the first half of a recent article by N. Wiener (*The spectrum of an arbitrary function*, Proceedings of the London Mathematical Society, (2), vol. 27, No. 7).

62. Mr. S. B. Littauer: *On a theorem of M. Jacob.*

The author proves a theorem of Jacob concerning the equivalents of two forms of mean of a function by reducing this theorem to a particular case of a type of Tauberian theorem treated by Wiener. The reduction in one direction is immediate; in the other, a slight extension of Wiener's methods is necessary.

63. Professor O. D. Kellogg and Dr. F. Vasilesco: *A contribution to the study of a certain lemma in potential theory.*

The lemma is as follows: every closed bounded set of points of positive capacity contains a regular point. It has been proved in the case of the logarithmic potential, but in space it is still in doubt. In order to make progress with the problem, it is necessary to have further knowledge of the properties of regular and exceptional points. In the present paper, the authors derive several new forms of criteria for regular and exceptional points. They show that the capacity of the portion of a set E in a sphere of fixed radius about P is a function of P which is upper semicontinuous,

and that a certain modified form of the series of Wiener is a function of P which is of the second class, in the sense of Baire. In certain cases this function may reduce to one of the first class.

64. Professor H. S. Vandiver: *On Fermat's last theorem.*

In various papers published during the last fifteen years the writer has obtained a number of results concerning Fermat's last theorem. In the present paper all these results are obtained from two general methods of approaching the problem. Some new criteria are also derived by the use of these methods. When applied to special exponents they yield the result that if $n>2$, $x^n+y^n+z^n=0$ is impossible in rational integers x, y, and z, none zero, for every n less than 211.

65. Professors H. S. Vandiver and Elizabeth T. Stafford: *Determination of some properly irregular cyclotomic fields.*

The integer which expresses the number of classes of ideals in a cyclotomic field defined by $e^{2\pi i/l}$, l an odd prime, is generally written as the product of two factors which are also rational integers, called the first and the second factor of the class number respectively, that is, $h=h_1h_2$. A necessary but not a sufficient condition that h_2 be divisible by l is that the first factor h_1 be divisible by l. In the present paper it is found by extensive computations that h_1 is divisible by l for l less than 211 only when $l=37$, 59, 67, 101, 103, 131, 149, 157, and that h_2 is not divisible by l in any of these cases. The paper includes proofs of the formulas employed, and descriptions of the many devices to shorten the computations.

66. Mr. H. T. Engstrom: *Theorems on common index divisors of algebraic fields.*

This paper gives the determination of common index divisors for algebraic fields of third, fourth, and fifth degrees. For these fields the common index divisor is defined by the prime ideal decomposition of the corresponding primes. For fields of ninth degree a special example is constructed showing that this is not the case in fields of ninth degree or higher.

67. Professor W. L. G. Williams: *On absolutely isobaric formal modular concomitants.*

The object of the present paper is to show that certain formal modular concomitants of binary forms are congruent to algebraic concomitants and to point out some applications of this fact in the theory of numbers. As a typical theorem the following may be cited: An absolutely isobaric formal modular invariant, mod p, a prime, of a binary form with binomial coefficients, no one of which is divisible by p, is congruent, mod p, to an algebraic invariant of this form. In this and other cases the author finds the algebraic concomitant to which a given formal modular concomitant is congruent. The following identical congruence will illustrate the applications to the theory of numbers. The coefficient of $x^{p-1}y^{p-1}$ in the expansion of $(ax^2+2bxy+cy^2)^k(x\xi-y\eta)^{2l}$ is congruent, mod p, to

$K(ac-b^2)^{k-(p-1)/2}(a\xi^2+2b\xi\eta+c\eta^2)^l$, where k, K, and l have the values $(p-1)/2 \leqq k \leqq p-1$, $K=(-1)^{(p-1)/2}2^{2k}\binom{2l}{l}$, and $l=p-1-k$.

68. Professor Orrin Frink: *The structure of finite algebras.*

This paper deals with algebras with a finite number of elements, the addition group being abelian, but multiplication not necessarily commutative. Such algebras are important in the number theory of linear algebras. It is shown that such an algebra is reducible unless of prime power order. Most of the standard theorems of linear algebras go through, since their proofs do not depend on the existence of a scalar field, but on idempotents and the general theory of complexes. A special study is made of the important case of associative algebras of prime power order, which can be represented as algebras of matrices whose elements are residue classes of integers modulo a prime power.

69. Professor Orrin Frink: *A theorem in boolean algebra.*

The theorem states that the result of substituting expressions which are in the completely expanded form in a function of several variables is given by multiplying the expressions together symbolically, preserving order, and interpreting any symbolic product which occurs as a coefficient in the result as meaning the result of substituting the factors of this product in the given function in the order in which they occur. It is shown how the theorem is used to simplify computation.

70. Dr. John Williamson: *The complete system of two quadratic quaternary forms.*

This paper obtains by a new method the system determined by H. W. Turnbull (Proceedings of the London Mathematical Society, (2), vol. 18 (1919)). Starting from the complete system, already known (Turnbull and Williamson, Proceedings of the Royal Society of Edinburgh, vol. 45 (1925)), which involves three cogredient sets of point coordinates x, y, and z, it is shown how the variables x, y, and z may be convolved to give a complete system of concomitants expressed in terms of point, line, and plane coordinates; that is, in terms of the variables x, $p=(xy)$, and $u=(xyz)$. While the results obtained are not new, the method has the advantage that it can be extended to finding, if not the complete system, at least a prepared system, for two quadratics in five or six variables. Incidentally, the number of irreducible concomitants obtained by Turnbull is diminished from 125 to 122.

71. Dr. A. A. Albert (National Research Fellow): *On the structure of normal division algebras.*

It is first shown that in a division algebra if y satisfies the minimum equation of x then y is a transform of x. The grade of an element is defined as the degree of its minimum equation. The type of an equation is defined as a function of the number of its ordinary complex roots rationally expressible in terms of one of them. The type of an element and that of an algebra are defined, and the structure of algebras of type S_k is considered.

The concept of "known algebras" is broadened and a necessary and sufficient condition that an algebra be a known algebra over a given subalgebra is found. It is demonstrated that if the direct product of two normal division algebras, each of the maximum type, is a division algebra, then it is a normal division algebra of the maximum type.

72. Dr. A. A. Albert: *On the group of the rank function of a normal division algebra.*

A new theorem in the Hilbert theory of irreducible polynomials whose coefficients are rational functions with coefficients in a field K of several parameters is proved in this paper. The theorem states that, for any complex values of the parameters, the group of the resulting equation with respect to the field obtained by adjunction to K of the resulting coefficients is a sub-group of the group of the original polynomial. This theorem is applied to the rank function of any normal division algebra, and it is shown that this function has the symmetric group. As a consequence of this it is seen that every normal division algebra in n^2 units contains an element whose minimum equation has degree n and the symmetric group.

73. Dr. A. A. Albert: *Normal division algebras in* $4p^2$ *units,* p *an odd prime.*

Normal division algebras in $4p^2$ units, p an odd prime, are considered. Algebras of type R_3 are shown to be known algebras in the sense previously defined. A study is made of conditions that an algebra A in $4p^2$ units contain an element x of grade $2p$ whose minimum equation is $\phi(\omega^2)=0$. It is shown that A is a known algebra over the Σ algebra obtained from x if and only if the equation $\varphi(\rho)=0$ is cyclic. This is the case considered by L. E. Dickson in his note in the September, 1928, number of this Bulletin. It is proved that these algebras are of type R_{2p} and hence not new.

74. Dr. A. A. Albert: *Normal division algebras in* 36 *units.*

In the present paper all normal division algebras in 36 units of type R_2 are shown by an elaborate argument to be of type R_3. The algebras of type R_3 were shown, in another paper, to be known algebras. A necessary and sufficient condition that the new algebras thus found be actually not equivalent to algebras previously constructed is given. Since the assumption of type R_2 is the weakest assumption of this kind that it is possible to make, the discussion is quite general.

75. Professor Oystein Ore: *An explicit case of "Zerlegungs"-groups.*

Several of the general properties of the "Verzweigungs"-groups introduced by Hilbert are known, but except in the case of abelian fields, their structure is unknown. The author has shown in a previous paper that the prime ideals of a Galois field can be divided into two types, binomial and trinomial. For the binomial types the complete structure of the corresponding non-abelian Hilbert group is determined.

76. Professor H. W. Brinkmann: *On the group characteristics of multiply transitive permutation groups.*

A permutation group on n letters is, of course, a sub-group of the symmetric group on those letters. An irreducible homogeneous linear group isomorphic to the symmetric group thus furnishes a homogeneous linear group isomorphic to the given group. But this group is usually reducible. The present paper investigates the irreducible constituents of these groups. For example, it is shown that for a t-fold transitive group certain of these groups are irreducible. The irreducible ones at once give characteristics of the group in question.

77. Professor H. W. Brinkmann: *Imaginary quadratic fields in which each genus contains but one class.* Second paper.

In a previous paper (read at the summer meeting of the Society, 1928) fields containing two genera, each of which contains one class, were discussed. The case of fields with four such genera is taken up in the present paper. The conditions that a field be of this type are derived and applications are made to elementary number theory.

78. Professor H. W. Brinkmann: *Note on algebraic equations not solvable by radicals.*

It is well known that irreducible equations (with rational coefficients) exist for every degree which cannot be solved by radicals. It is the purpose of this note to give explicitly examples of such equations for prime degree. Thus we find that $x^2-px+p=0$ is irreducible and cannot be solved by radicals for any prime $p\geqq 5$.

79. Professor Raymond Garver: *On the removal of four terms from an equation by means of a Tschirnhaus transformation.*

This paper has appeared in full in the January-February number of this Bulletin.

80. Professor Raymond Garver: *Concerning certain functions of Brioschi.*

A paper of Brioschi's (see his Opere, vol. 3, pp. 293–299) proves the existence, for an equation of odd degree n, of $(n-1)/2$ polynomial functions possessing certain properties which make them of some value in connection with Tschirnhaus transformations. The present paper gives a simpler proof of the theorem, and considers its extension to equations of even degree.

81. Mr. R. E. Powers: *Note on the Pellian equation.*

In 1877 Artemas Martin found that the smallest integral value of x which satisfies the Pellian equation $x^2-9817y^2=1$ contains 97 digits (L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 383). I have verified this result, and find that the corresponding value of y contains 95 digits.

82. Professor D. C. Harkin: *The abstract identity of modular systems and ideals.*

By the method of abstraction and transformation by formal equivalence, Dedekind's theory of ideals (1871 and later) and Kronecker's theory of modular systems, as developed in his famous *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* of 1882, are shown to be abstractly identical. A logically consistent theory of forms (or ideals) is built up, leading finally to the unique decomposition of forms into prime forms. The terminology and external form of the general theory developed agrees closely with Kronecker, but the root idea and fundamental meaning is taken almost bodily from Dedekind. Various possible interpretations of the abstract general theory are indicated: the theory of algebraic forms, with either the Gaussian transformation or the Kummer composition of forms as fundamental; geometrically, the correspondence of an ideal to a system of points on a fixed fundamental curve, a principal ideal to the complete intersection system of a fundamental curve with another algebraic curve, with equivalence translated into coresiduality.

83. Professor R. L. Moore: *On the structure of a continuum.*

It is shown that if M is a bounded plane continuum and no upper semi-continuous collection of mutually exclusive subcontinua of M is homeomorphic with a domain, then there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that G is a dendron with respect to its elements and such that if H is another collection satisfying these conditions then every element of H is made up of elements of G. The elements of G will be called *dendratomic* subsets of M. For each non cut point P of the bounded plane continuum M let M_P denote the set of all points X of M such that X is not separated from P in M by any one point. For each cut point P of M let M_P denote P. If M is a continuous curve every M_P is a cyclic element of M in the sense of G. T. Whyburn and if Q is an upper semi-continuous collection of mutually exclusive continua filling up M and no M_P contains points of two different continua of Q then every continuum of Q is made up of continua of the collection G_1 defined on page 700 of volume 34 of this Bulletin.

84. Professor R. G. Lubben: *The metrization problem and complete spaces.*

The author shows that certain well known conditions for the metrization of a topological space may be strengthened so as to give necessary and sufficient conditions that such a space be complete. It follows from this result that a locally compact, metric space is complete. A *universal complete enclosure* of a metric space M is by definition a complete space N which contains M and admits a metric d such that if T is any complete space whatever containing M, and d' is a metric with reference to which T is complete, then if a sequence of points in M is a fundamental sequence with reference to d' it is a fundamental sequence with reference to d. In order that a metric space admit a universal complete enclosure, it is necessary

and sufficient that it be separable; in order that it admit a minimal universal complete enclosure, it is necessary and sufficient that it be locally compact and separable.

85. Professor R. G. Lubben: *Concerning the distributive property.*

The author establishes the following results. (1) If a space L has the distributive property (see Transactions of this Society, vol. 30 (1928), p. 669), then every point set in it having the power of the continuum has a limit point. (2) In a space S having the distributive property, every point set having the power of the continuum *contains* a limit point of itself. (3) If a space S has the distributive property, then every regular, locally compact sub-space of it has this property. (4) Every space S which has the distributive property and satisfies the first countability axiom is regular and locally compact. (5) A space S which is regular, is locally compact, and has the Lindelöf property, has the distributive property.

86. Mr. J. H. Roberts: *On a problem of C. Kuratowski's concerning upper semi-continuous collections.*

In a letter to R. L. Moore dated April 9, 1927, C. Kuratowski raised the question as to whether or not there exists an upper semi-continuous collection X of mutually exclusive continua no one of which is a point such that (1) the sum of the continua of the collection X fills a square plus its interior, and (2) if each continuum of the collection X is regarded as a point, the space so obtained is in continuous one-to-one correspondence with a square plus its interior. In the present paper this question is answered in the affirmative.

87. Mr. J. H. Roberts: *Upper semi-continuous collections of continua some of which are unbounded.*

R. L. Moore has proved that if, in a plane S, G is an upper semi-continuous collection of mutually exclusive *bounded* continua no one of which separates S and such that every point of S belongs to some continuum of G, then, if the continua of G are regarded as points, the space so obtained is in continuous one-to-one correspondence with the plane S. In the present paper it is shown that if G is an upper semi-continuous collection of mutually exclusive continua such that (1) every point of the plane S belongs to some continuum of G, (2) at least two continua of G are unbounded, and (3) the space of elements of G is metric, then the space of elements of G is in continuous one-to-one correspondence with a subset of the plane S.

88. Dr. W. L. Ayres (National Research Fellow): *Continuous curves in which every arc may be extended.*

An arc α of a set K is said to be extendible in both directions in K if K contains an arc β such that every point of α is an interior point of β. It is shown in this paper that a plane continuous curve M has the property that every arc is extendible in both directions in M if and only if one of the two following conditions holds: (a) no points of M may be ϵ-separated by an acyclic continuous curve; (b) the sum B of the boundaries of the

complementary domains of M is closed and every point of B is either a cut point of M or belongs to the boundaries of at least two complementary domains of M. If either (a) or (b) is true, then every point of M is a point of B. For any point P of M and any positive number ϵ, there exists a simple closed curve J of diameter less than ϵ and enclosing P and such that the set $M \cdot J$ consists of a finite number of components.

89. Dr. W. L. Ayres: *On generalizations of continuous curves.*

If $x_i (i = 1, 2, \cdots, n)$, is a continuous function of t, the set of all points $(x_1, x_2, \cdots, x_n)$ for $0 \leqq t \leqq 1$ is called a continuous curve. Two possible generalizations of this notion may be had by allowing t to take on all values or all values $t \geqq 0$. In this note it is shown that these two generalizations are equivalent and the type is characterized.

90. Professor G. T. Whyburn: *Cut points of connected sets and of continua.*

Let M be any continuum (in a locally compact, metric, and separable space), let A and B be any two mutually exclusive closed subsets of M, let K be the set of all those points of M which separate A and B in M, and let G denote the set of all cut points of M. In this paper it is shown that (1) $K+A+B$ contains every point of $\overline{K}$ which is a Menger regular point of $M+P$; (2) K is the sum of a G_δ set and a countable set; (3) G is a $G_{\delta\sigma}$, that is, the sum of a countable number of G_δ sets; (4) if K is uncountable, it contains a perfect set; (6) if M is compact and irreducible between A and B, then K contains every point of $M-(A+B)$ at which M is connected im kleinen. Result (1) is true for any *connected set* M. Some other theorems of a similar character are also obtained.

91. Professor G. T. Whyburn: *On the structure of connected and connected im kleinen point sets.*

It is shown in this paper that all connected and connected im kleinen point sets M in a locally compact, metric, and separable space admit of decomposition into elements highly analogous to the cyclic elements of a continuous curve. Elements (a) of such a set M are connected subsets of M which have no cut point and which are saturated with respect to this property; elements (b) of M are cut points of M; and elements (c) of M, called end points, are points P of M such that a sequence V of points of M exists converging to P and such that each point of M distinct from P is separated in M from P by all save a finite number of points of V. These correspond respectively to the maximal cyclic curves, cut points, and end points of a continuous curve; and it is shown that the structure of any such set M with respect to its elements (a), (b), and (c) is very similar to the structure of a continuous curve with respect to its cyclic elements (see American Journal of Mathematics, vol. 50, pp. 167-194). One interesting theorem proved is that every connected subset of the set of all elements of classes (b) and (c) is arcwise connected.

92. Professor G. T. Whyburn: *Continuous curves and arc sums.*

Menger has suggested the problem of characterizing a continuous curve which is the sum of a countable number of simple continuous arcs. In this connection the following theorems are proved. (1) The continuous curve M is the sum of a countable number of arcs if and only if (a) the end points of M are countable, and (b) each maximal cyclic curve of M is the sum of a countable number of arcs. (2) The boundary N of a complementary domain of a plane continuous curve is the sum of a countable number of arcs if and only if the end points of N are countable. (3) The set K of all the local cut points of any continuous curve M is a subset of the sum of a countable number of arcs which lie in M; and if $M-K$ is countable, M is the sum of a countable number of arcs.

93. Professor James Pierpont: *On the motion of a rigid body about a fixed point in space of constant curvature.*

The first to consider the motion of a solid in elliptic space was Clifford, in a paper read before the London Mathematical Society in 1874, but not published until 1882. This was followed by a paper by R. S. Heath published in 1884 in the Transactions of the Royal Society of London. Finally, de Francesco treated this problem in two papers in the Atti of the Academy of Science of Naples in 1900. Clifford employs biquaternions, Heath and de Francesco use the six homogeneous coordinates of a right line tied down by two biquadratic relations; de Francesco considers hyperbolic as well as elliptic space, and he also considers in detail special cases, which Heath does not do. The present author treats another special case, viz., the Maxwell top in space of constant curvature. In this important case he shows that homogeneous line coordinates are quite unnecessary, from which follows a great gain on the analytical side of the problem. With the same degree of accuracy as employed in classical mechanics we have the notion of center of mass. From this it results that the motion of the top in elliptic (hyperbolic) space would appear to an observer in euclidean space as that of a top with certain assigned moments of inertia and a certain center of mass.

94. Dr. A. J. Lotka: *Notes on Volterra's mathematical theory of biological associations.*

Volterra has set up a system of differential equations to represent the course of events in a system comprising two biological species competing for a common food supply. He has discussed certain properties of the functions defined by these equations, but has not offered any general solution. Such a solution in form of an exponential series is now given, the case discussed by Volterra being a special case of the general theory of the growth of n populations living together in a common environment, as developed in prior publications by the present author. The case discussed by Volterra may also be regarded as the Verhulst-Pearl law of growth for two competing populations. The independent variable t (time) can be

eliminated from the system of differential equations, which then contains only two variables. The *isoclines* of the integral curves of the system are investigated, and a map of the family of integral curves is prepared by their aid. The discussion of some other cases treated by Volterra is reserved for a future occasion. Application of the method of analysis to economic problems (for example, competition of two or more producers supplying the same market) is suggested.

95. Dr. P. R. Rider: *On the moments of the frequency distribution of third moments.*

This paper develops formulas for the moments of the frequency distribution of third moments of samples of n from an infinite population. The first and second moments are given for any type of population, the third and fourth moments for a normal population.

96. Dr. P. R. Rider: *The effect of skewness in the sampled population on the frequency distribution of the ratio of mean to standard deviation in small samples.* Preliminary communication.

The frequency distribution of the ratio of mean to standard deviation in samples of n from a normal population has been rather thoroughly discussed by "Student" and others. In samples from a normal population the mean and the square of the standard deviation are uncorrelated; for other types of population correlation between them exists. Shewhart and Winters have made an experimental study of the distribution of the ratio of mean to standard deviation in samples from certain non-normal populations, and have given their results in a recent number of the Journal of the American Statistical Association. The present paper endeavors to shed more light on the effect of skewness in the sampled population on the distribution of this ratio.

97. Professor A. H. Copeland: *A proof that almost every number is admissible and is associated with the probability one half.*

In this paper it is proved that in a certain sense almost every event satisfies the fundamental assumptions of the theory of probability, and moreover almost every event has the probability one-half. The sense in which this statement is true is as follows. If an event behaves in accordance with the assumptions of the theory of probability, then the history of successes and failures of this event must be represented by an admissible number. The admissible numbers all lie in the interval from zero to one, and the measure of the set of admissible numbers is equal to one. That is, almost every number in the unit interval is admissible. Furthermore it is proved that almost every admissible number is associated with the probability one-half.

98. Professor C. F. Roos: *A mathematical theory of the business cycle.*

Many economists have empirically investigated the changes in prices over a period of time, and have evolved an empirical theory generally known as the theory of the business cycle, which, in brief, states that prices are periodic functions of the time. Other equally competent investigators have branded this theory as a myth, and have exhibited empirical curves to support their contentions. The present paper gives a purely theoretical discussion of prices, and shows that these views, rather than being contradictory, are both only partial manifestations of an underlying theory. The hypothesis that a producer endeavors to determine his price so that he obtains a maximum profit over a period of time requires that the price satisfy a certain differential equation. This differential equation has ten distinct types of solutions, all of which seem to have economic significance. A truly periodic solution supporting the theory of the business cycle can result only if interest is neglected. Conditions are obtained that will lead to the ten different types of solutions; all the results are stated in a form that will allow experimental investigation.

99. Mr. H. H. Pixley: *A theory of discontinuous price and production curves.*

In this paper the author considers a generalization of Roos's problem of depreciation and replacement in the mathematical theory of economics, and removes the restrictions of continuity of price-time and rate of production-time curves at the time of replacement. Some applications are made of the general mathematical theory given in the author's earlier paper *Discontinuous solutions in the problem of depreciation and replacement,* in considering special problems believed to have actual economic significance. The present paper consists essentially of applications.

R. G. D. Richardson,
Secretary

THE CHRISTMAS MEETING IN CHICAGO

The two hundred sixty-sixth, thirty-first regular Western, meeting of the Society was held at the University of Chicago on Monday, December 31, 1928. About sixty persons attended the meeting, among whom were the following fifty members of the Society:

R. W. Babcock, Bartky, Beenken, O. E. Brown, L. H. Bunyan, G. N. Carmichael, R. D. Carmichael, Chittenden, A. G. Clark, H. H. Conwell, C. F. Craig, Crathorne, Curtiss, Dickson, Escott, Everett, Feltges, Gouwens, L. M. Graves, Griffiths, V. G. Grove, Holgate, Ingraham, B. W. Jones, B. F. Kimball, E. P. Lane, Langer, LaPaz, Leavens, Logsdon, McCoy, March, Marshall, E. H. Moore, T. W. Moore, Pepper, Pettit, H. L. Rietz, Ross, Roth, Sanger, Schottenfels, Simmons, Wall, Warren Weaver, Weiss, F. E. Wood, Frederick Wood, Roscoe Woods, Worth.

In the evening thirty-seven members and guests attended dinner in the Del Prado Hotel. Professor Lane acted as toastmaster and, before the dinner, called upon Professor E. H. Moore, who could not remain for the dinner. Professor Moore spoke for a few minutes, welcoming the members and guests to Chicago. After the dinner the toastmaster called upon Professor R. D. Carmichael, who told of the New York meeting, Professor Chittenden, who talked of the coming Des Moines meeting, Professor Ingraham, who spoke of the problems of the location of meetings in the Mid-West and suggestions that had been made towards increasing the interest of the programs, and Miss Weiss, who spoke for the ladies.

The papers whose abstracts appear below were read as follows: papers 1–9 in the morning, and papers 10–19 in the afternoon. Papers 6, 8, 9 and 15–19 were read by title, the rest in person. Professor Marshall presided in the morning and Professor Curtiss in the afternoon.

1. Dr. B. W. Jones: *The number of representations by certain positive ternary quadratic forms.*

The number of solutions of $x^2+y^2+z^2=n$ in integers for any positive integer n is a known function of the class number of n. But the number of

integral solutions of $ax^2+by^2+cz^2=n$, where a, b and c are positive integers, is not, in general, known. In this paper, by means of a transformation taking the form $x^2+2y^2+3z^2$ into $x^2+y^2+z^2$, the number of representations of n by the former is shown to be a simple function of the number of representations by the latter of $6n$ and, in some cases, other multiples of n. From these results can be found the number of representations of n by the former in terms of the class numbers of certain multiples of n. This type of relationship has been considered only incidentally by Nazimoff and Uspensky and only fragmentary results have been obtained for this form. It is demonstrated that not only the form mentioned but many other new forms, seemingly unrelated to the sum of three squares, yield to similar treatment.

2. Dr. Lois W. Griffiths: *A generalization of the theorem of Fermat and Cauchy on polygonal numbers.*

This theorem, that every positive integer is the sum of $m+2$ polygonal numbers of order $m+2$ and that all but four of these may be taken to be 0 or 1, has been extensively generalized by Dickson. Another generalization is suggested by the following restatement of the theorem. Let $n, a_1, \cdots, a_n$ be positive integers and $p_1, \cdots, p_n$ be polygonal numbers; then the function $f=a_1p_1+\cdots+a_np_n$ is universal when $n=m+2$ and $a_1=\cdots=a_n=1$, and $p_5, \cdots, p_{m+2}$ may be taken to be 0 or 1 if $m>2$. In this paper the universal functions f are determined for $m\geqq 3$, $a_1+\cdots+a_n\leqq m+2$; the cases $m=1$ and 2 have been treated for any n by Liouville and Dickson.

3. Professor L. E. Dickson: *All universal ternary quadratic forms are null forms.*

A positive or indefinite form f is called universal if it represents all positive integers or all positive and negative integers respectively. For three variables every universal f is proved to be a null form, that is, f vanishes for integral values not all zero of the variables. Since all universal Null ternary forms were found in another paper presented by the writer at this meeting, this completes the theory of all universal ternary quadratic forms.

4. Professor L. E. Dickson: *All indefinite quadratic forms in 3 or more variables which represent every integer.*

Of ternary quadratic forms, no positive form represents every positive integer, but infinitely many indefinite forms are universal, that is, represent every positive, negative, and zero integer. All universal $ax^2+by^2+cz^2$ are determined explicitly. A method is developed to find all Null universal quadratic forms in n variables. The results are complete for $n=3$ and essentially complete for $n=4$. For $n\geqq 5$, the theory is sufficient but involves more computation.

5. Professor F. E. Wood: *Cubics whose (Hessian)^n are themselves.*

This paper appears in full in the present issue of this Bulletin.

6. Professor K. E. Rosinger: *A note concerning the symbols ϕx and $\phi\hat{x}$.*

It is the purpose of this paper to examine the basis of the distinction made in *Principia Mathematica* between the symbols ϕx and $\phi\hat{x}$. The problem is considered in the light of Frege's studies in the logical analysis of functions. It is found that the distinguishing characteristic of propositional functions recognized by Whitehead and Russell, namely "ambiguity," is derived from a more fundamental notion, which Frege calle Ergänzungsbedürftigkeit. This notion is represented by Principia's symbol ϕx. On the other hand, $\phi\hat{x}$ expresses Frege's concept Zusammengehörigkeit. It appears that neither Frege nor Principia have recognized that these two concepts are independent of each other. Hence the distinction between ϕx and $\phi\hat{x}$ is fundamental. The fact that this has not been detected by many mathematical logicians has produced considerable confusion. From this fundamental difference, the paper deduces a number of properties of the notions here considered, thereby bringing to explicit statement the meanings and purposes of ϕx and $\phi\hat{x}$.

7. Dr. H. S. Wall: *On extended Stieltjes series.*

This paper refers to the author's previous article which appears in the Transactions of this Society, January 1929, on Stieltjes series and continued fractions. The writer considers the problems of extending the Stieltjes series

$$\sum_{i=0}^{i=\infty} c_i(-z)^i \tag{1}$$

to the left by prefixing successively terms of the form $(-1)^k c_{-k} z^{-k}$, $k=1, 2, 3, \cdots, n$, in such a way that all of the series

$$\sum_{r=0}^{r=\infty} c_{-k+r}(-z)^r, \quad (k=1, 2, 3, \cdots, n),$$

shall be Stieltjes series. A necessary and sufficient condition for a first extension, given in the preceding article, is replaced by the following. Let the limit of the sequence of even convergents of the Stieltjes continued fraction corresponding to (1) be

$$F(z)=\int_0^\infty \frac{d\phi_1(u)}{1+zu} \tag{2}$$

Then (1) admits a first extension if and only if

$$\lim_{a\to 0}\int_a^\infty \frac{d\phi_1(u)}{u}, \quad (a>0)$$

exists and has a finite value k; and $c_{-1} \geqq k$. From this it follows that for a large class of Stieltjes series, including among others those for which the corresponding Stieltjes continued fractions are divergent, extensions of infinite order exist in which

$$c_{-k}=\int_0^\infty \frac{d\phi_1(u)}{u^k}.$$

The Stieltjes continued fraction corresponding to the Stieltjes series

$$\sum_{i=1}^{i=\infty}(-1)^{i+1}\frac{C_i}{z^i} \qquad (3)$$

converges for all z not belonging to the interval $(0, -R)$, where R is the distance from the origin to the most remote pole of (2), to $F(z)$, and the series (3) converges for all $|z|>R$ to the function $F(z)$.

8. Professor R. L. Wilder: *A topological definition of simple closed surface.*

The following characterization of simple closed surface is obtained. In a metric separable space let M be a compact set containing at least one simple closed curve, and satisfying the following conditions: (1) If t is an arc of M, then $M-t$ is connected; (2) if J is a simple closed curve of M, then $M-J$ is the sum of two uniformly regular components. Then M is homeomorphic with the sphere $x^2+y^2+z^2=1$.

9. Professor R. L. Wilder: *Concerning the Phragmen-Brouwer Theorem.*

Let S be a topological space. Consider the following properties: I. If F_1 and F_2 are mutually exclusive closed subsets of S, neither of which separates S, then F_1+F_2 does not separate S. II. If N is a continuum in S and R a component of $S-N$, the boundary of R is a continuum. Either of these properties serves as a statement of the Phragmen-Brouwer Theorem if S is a euclidean space. It is the purpose of the present paper to study the equivalence between these properties, and certain related properties, as well as to determine certain spaces in which they hold true. Examples are given of spaces where I holds but II does not hold, and conversely. It is shown, however, that in a certain class of general spaces, of which euclidean spaces form a proper subset, the two properties are equivalent. The following generalization of the theorem of Kuratowski (Fundamenta Mathematicae, vol. 8, p. 149) is obtained: If, in a euclidean plane E_2, M is a bounded continuum containing more than one prime part and such that no one of its prime parts separates E_2, then in order that M may be considered as a space having property II it is necessary and sufficient that M should not separate E_2.

10. Professor V. G. Grove: *A canonical form of Green's projective analogue of the Gauss differential equations.*

In this paper the author develops a canonical form of the projective analogue of the Gauss differential equations. The vertices of the tetrahedron of reference are covariants of simple geometric significance. Each invariant appearing in the canonical form has a geometric significance easily read from the equations. By specializing the parametric net to be conjugate, one obtains the equivalent of Lane's canonical form for conjugate nets. By assuming the parametric net to be non-conjugate, a canonical

form for Green's differential equations of a non-conjugate net is obtained. In this case also the geometrical significance of the invariant coefficients may be seen directly from the differential equations.

11. Professor E. P. Lane: *On the fundamental transformation of surfaces.*

The author develops the theory of a fundamental transformation of surfaces in ordinary space, treating the subject projectively and using Wilczynski's method based on a completely integrable system of differential equations. The conjugate and harmonic invariants recently used by Slotnick receive new geometric interpretations. Certain quadric surfaces called the associated quadrics of the transformation are introduced into the theory. Various theorems are obtained concerning fundamental transformations with conjugate and harmonic congruences of special types.

12. Professor R. E. Langer: *On the asymptotic forms of the solutions of certain differential equations containing a parameter.*

This paper deals with the equation $y''+\lambda^2\phi(x)y=0$ in which $\phi(x)$ vanishes like x^k at $x=0$. It is shown that a certain approximate equation is solvable in terms of Bessel's functions, and that there exists in any upper or lower half λ-plane a pair of solutions of the given equation which for $0<\alpha\leqq x\leqq\beta$ have asymptotic forms with the same leading terms as the solutions of the approximate equation, and which are also represented asymptotically by the latter at $x=0$. By means of these solutions and a formula expressing the general solution $y(x)$ in terms of an arbitrary pair $y_1(x)y_2(x)$, one obtains the leading terms of the asymptotic expressions representing $y(x)$ for λ in different half-planes, or x in different intervals on opposite sides of the origin.

13. Professor L. E. Dickson: *New results and methods for minima of quadratic forms.*

In 1903 Markoff obtained the first three minima of all real indefinite ternary quadratic forms for integral values of the variables, and emphasized the difficulty of the problem of finding further minima. The fourth mimimum is now found by his method. But new methods are developed to obtain further minima. An application is to the tabulation of representatives of all classes of a given determinant. Such a table is used to obtain general theorems on the representation of all integers by such quadratic forms.

14. Professor M. H. Ingraham: *The reducibility of algebras without a finite base.*

Systems are considered which satisfy the usual postulates for a linear associative algebra except they may not contain a finite base. From the definition of a linear set as one which is closed under addition and scalar

multiplication, we find that the definition of the sum of a class of linear sets, as the minimal linear set containing every linear set of the class, has content. As Wedderburn has shown under slightly stronger hypotheses, most of the elementary theorems on the direct sum of linear sets follow with, at most, a slight change in the usual proof. Algebras exist containing no irreducible sub-algebras. If an algebra can be reduced into irreducible sub-algebras, each with a principal unit, the reduction apart from order is unique. An algebra with a principal unit cannot be expressed as the direct sum of more than a finite number of algebras. This last fact is partly due to restrictions in the definition of a sum rather than being inherent in the problem.

15. Professor E. T. Bell: *A partial isomorphism between the functions of Lucas and Weierstrass.*

This paper will appear in full in an early issue of this Bulletin.

16. Professor E. T. Bell: *A class of polynomials and rational functions in four variables.*

The polynomials and functions are those arising as coefficients of $1, i, j, k$ in the nth power, n an integer, of the quaternion whose coordinates are w, x, y, z. These functions have a number of interesting properties, of which the following will suffice as specimens. With respect to one and only one of the independent variables, the polynomials are Appell polynomials. With respect to a properly chosen argument, the polynomials are Tchebycheff polynomials. With respect to n, both the polynomials and the functions are Lucas functions. Both the polynomials and the functions, with respect to w, x, y, z, satisfy an extremely simple set of 4 self-adjoint differential equations of the second order. The polynomials are rationally resolvable into factors of degrees $\leqq 2$ in an elegant way. With respect to n they have addition and subtraction theorems abstractly identical with those for the circular, hyperbolic and Lucas functions. If the norm of the generating quaternion is unity, the polynomials and functions, in their algebraic aspects, are simply isomorphic with the sigma, zeta, and pe functions of Weierstrass.

17. Professor E. T. Bell: *A class of polynomials and rational functions in eight variables.*

These are the generalization of the preceding four-variable polynomials and functions to the like arising from the eight-square theorem of Degen and Cayley. The preceding abstract may be repeated verbatim with respect to these. No further generalization in the same direction is possible.

18. Professor W. A. Manning: *The degree and class of doubly transitive groups.*

In this paper it is proved that if μ is the class and n the degree of a doubly transitive group which contains a substitution of degree μ and of

prime order $p(>3)$, then $\mu>(n/2)(1-1/p)-(n^{1/2}/2)(1-1/p^2)^{1/2}-1$. The same limit holds when $p=3(\mu>3)$ if n is sufficiently large. This result complements the limit $\mu>n/2-n^{1/2}/2-1$, for $p=2$, published by the author in 1914.

19. Professor T. H. Hildebrandt: *Remarks on Carathéodory's theory of measure.*

Carathéodory bases his theory of measure on an upper measure function satisfying five postulates. This paper shows how from an upper measure function satisfying the first three postulates it is possible to construct an upper measure function satisfying the fifth, and obtains conditions under which the property of measurability is unaltered. The same study is undertaken beginning with an upper measure function satisfying only the first postulate.

M. H. Ingraham,
Associate Secretary

A CORRECTION

BY W. J. TRJITZINSKY

In my paper, *Zeros of a function and of its derivative*, (this Bulletin, Vol. 33 (1927), pp. 693-695), 2π should be replaced everywhere by $2\pi n$ (integer $\geqq 1$), except on line 7, page 693, and in expressions $1/(2\pi i)$.

THE INAUGURATION OF THE INSTITUTE HENRI POINCARÉ IN PARIS

BY MAURICE FRÉCHET

In November, 1928, there was formally inaugurated in Paris a new institute of mathematics. It was both the official opening of a new building and the beginning of new courses of lectures, all to be a part of the Faculty of Sciences of the University of Paris.

The building is now ready, but the internal arrangement and the furnishing will not be completed for some time. It was considered wise, however, to hold the ceremony in the building in order to attract public attention to the opening of the lectures and to the foundation of the Institute.

It was desired to express the gratitude of the University of Paris towards those who had provided the necessary means. The history of this Institute is brief. It had been noted by the International Education Board that several opportunities had led them to give very large sums of money to various universities in Europe, but that gifts to French universities had been on a much smaller scale. Noting the importance of the French Mathematical School, it was thought that helping mathematics in France was perhaps one of the best ways to help science all over the world.

The decision was taken after many consultations, in which Professor Trowbridge, then representing in Paris the International Education Board, and Professor Birkhoff as a great mathematician, took decisive part.

It was decided to ask Professor Emile Borel to draw up a plan. The plan, which was approved, creates under the name INSTITUT HENRI POINCARÉ a center for teaching and for research on mathematical physics and on the calculus of probabilities.

The new teaching positions have been given to three men.

The courses on *Physical Theories* will be delivered by Professor Léon Brillouin and M. Louis de Broglie (to be distinguished from physicists of the same names, both members of the Académie des Sciences). Professor Léon Brillouin has made himself known by his fundamental researches on the theory of quanta and its applications. He was called last year to expound them in several universities of the United States and Canada. Dr. Louis de Broglie is the creator of that wave mechanics which, though created only yesterday, now plays a leading part in mathematical physics, and has been the inspiration of many investigations of revolutionary nature.

Although these courses will form an important addition, those interested in theoretical physics will know that there were already important courses on this subject in Paris, among which should be mentioned those of Professor Brillouin and Professor Langevin at the Collège de France, and those of Professor Eugène Bloch and Professor Villat at the Sorbonne.

As for the calculus of probability, it had already its great exponent at the Sorbonne in Professor Emile Borel. His researches on this subject and his personal action have done much to revive in France the interest in this science which owes so much to such French scientists as Pascal, Fermat, Laplace, Poisson, Bienaymé, Cauchy, Cournot, Bertrand, Henri Poincaré.

To Professor Borel's course will now be added a new course by Maurice Fréchet, formerly professor at the University of Strasbourg. His theory of abstract spaces and functions has already made him known in America, where he was called to expound it at the University of Chicago in the summer quarter of 1924. But of late he has devoted much attention to the theory of probability, on which he published (in collaboration with Professor Halbwachs) a work entitled *Le Calcul des Probabilités à la Portée de Tous*.

It should be stated also that the applications of proba-

bilities to social sciences are taught in the already existing Institut de Statistique of the University of Paris.

The activities of the Institute Henri Poincaré will not be confined, however, to the new courses. It aims to be international in scope. The attendance at these courses is very cosmopolitan indeed. But the Institute will also have an international staff of lecturers. In addition to the permanent courses, single lectures or brief series of lectures will be given by distinguished scientists. Professors Vito Volterra of Rome, and de Donder of Bruxelles, have already promised their cooperation. Other engagements will soon be published.

Finally, as the ever-increasing numbers of lecturers and students at the Sorbonne called for new measures, it was decided to seize upon the opportunity and to erect a new building where not only the new courses, but all the advanced courses on mathematics will be given, and where the mathematical library will be housed. The International Education Board is to contribute one hundred thousand dollars to these expenses; Baron Edmond de Rothschild also contributed twenty-five thousand dollars, and the French Ministry for Education 30,000 francs.

It is to be hoped that some of the students and scholars who plan to complete their scientific education or to pursue their researches in Europe, will remember that there has been created in Paris, thanks chiefly to American generosity a great international scientific center for mathematical physics and the calculus of probability.

MAURICE FRÉCHET

REPORT ON THE 1928 INTERNATIONAL CONGRESS OF MATHEMATICIANS*

BY L. TONELLI

At the Toronto Congress of 1924, Professor Salvatore Pincherle was elected president of the International Mathematical Society and the city of Bologna was chosen as the seat of the 1928 congress. A program committee was selected, the presidency being entrusted to Senator G. Albini, Rector of the University of Bologna. An executive sub-committee was also formed with Professor S. Pincherle as president, the other members being: Professor E. Bompiani, Professor P. Burgatti, Professor F. Guarducci, Professor G. Horn d'Arturo, Professor Q. Majorana, Professor U. Puppini, Professor L. Tonelli, Professor D. Zucchini, Comm. G. Borsari (treasurer) and Professor Ett. Bortolotti (general secretary). The congress was held under the distinguished patronage of H. M. the King of Italy, and the honorary presidency of His Excellency, Benito Mussolini, the leader of the government.

The program committee, following the wishes of the majority of the Toronto delegates, invited to the congress all mathematicians, and representatives of the leading universities and academies of the world. The proceedings of the congress were divided into seven sections, and several chairmen were named for each section, their task being the detailed organization of their particular section. The arrangement of sections was as follows:

SECTION I. Arithmetic, Algebra, Analysis (Chairmen: M. Cipolla, G. Fubini, G. Scorza, L. Tonelli).

SECTION II. Geometry (Chairmen: L. Berzolari, E. Bompiani, G. Fubini, F. Severi).

SECTION III. Mechanics, Astronomy, Geodesy, Geophysics,

* Translated by Dr. A. O. Hickson, Brown University.

Mathematical Physics, Theoretical Physics (Chairmen: G. Armellini, P. Burgatti, E. Fermi, T. Levi-Civita, C. Somigliana).

SECTION IV. Statistics, Mathematical Theory of Economics, Calculus of Probability, Actuarial Science (Chairmen: L. Amoroso, P. Cantelli, C. Gini, G. Toja).

SECTION V. Engineering and Industrial Applications (Chairmen: G. Albenga, C. Di Pirro, G. Giorgi, F. Lori, M. Panetti, C. Porro, U. Puppini, L. Silla, G. Vacchelli, G. C. Vallauri).

SECTION VI. Elementary Mathematics, Didactics, Mathematical Logic (Chairmen: U. Amaldi, G. Peano, A. Perna).

SECTION VII. Philosophy, History of Mathematics (Chairmen: E. Bortolotti, F. Enriques, R. Marcolongo, G. Vacca).

Eight hundred active members (those mathematicians who took part in the proceedings of the congress), 287 associate members (those belonging to the families of the delegates), and most of the members of the program committee attended the congress. Thirty-four nations were represented, and the larger delegations, except that of Italy (249 members), were: Germany with 76, Great Britain and Dominions with 64, France with 51, the United States with 43, Switzerland with 33, Poland with 32, Russia and Ukraine with 35, Hungary with 22, etc.

On Sunday evening, September 2, the Italian Mathematical Society held a reception for the delegates in the hall of the Bologna Circolo di Cultura. On Monday morning, September 3, the inaugural meeting was held in the assembly hall of the Archizinnesio in the presence of H. R. H., the Duke of Bergamo; the speakers were: Mayor Arpinati of Bologna, Senator Albini, rector of the University, Professor Pincherle, Professor Birkhoff, representative of all the foreign mathe-

maticians, and S. E. Belluzzo, minister of Public Instruction. The last speaker formally opened the congress.

The proceedings of the congress began on the afternoon of the same day and all meetings were held in the halls of the Royal University (Via Zamboni). At the first session a general assembly was held for the nomination of the president and vice-presidents; the following nominations were made by acclamation: Professor S. Pincherle for president of the congress, and for vice-presidents Professors de la Vallée Poussin, J. Hadamard, D. Hilbert, W. H. Young, J. C. Fields, O. Veblen, E. Terradas, W. Sierpinski, H. Bohr, N. Lusin, S. Kakeya.

The general lectures, all of which were given in the mornings, were successively as follows:

D. Hilbert—*Probleme der mathematischen Logik.*

J. Hadamard—*Le développement et le rôle scientifique du calcul functionnel.*

U. Puppini—*Le bonifiche in Italia.*

E. Borel—*Le calcul des probabilités et les sciences exactes.*

O. Veblen—*Differential invariants and geometry.*

G. Castelnuovo—*La geometria algebrica e la scuola Italiana.*

W. H. Young—*The mathematical method and its limitations.*

V. Volterra—*La teoria dei funzionali applicata ai fenomeni ereditari.*

H. Weyl—*Darstellung kontinuirlichen Gruppen.*

T. von Kármán—*Mathematische Probleme der modernen Aerodynamik.*

L. Tonelli—*Contributo italiano alla teoria delle funzioni di variabili reali.*

M. Fréchet—*L'analyse général et les espaces abstraits.*

R. Marcolongo—*Leonardo da Vinci nella storia della matematica e della meccanica.*

N. Lusin—*Sur les voies de la théorie des ensembles.*

L. Amoroso—*Le equazioni differenziali della dinamica economica.*

The number of sectional papers, all of which were pre-

sented in the afternoons, reached the remarkable total of almost four hundred.

A reception by the Mayor, a reception by the National Government, an orchestral concert of historical Italian music, and a banquet by the program committee of the congress occupied the evenings.

September 7 was devoted to a tour of the cities of Ravenna and Ferrara, and Lake Ledro.

Monday morning, September 10, the last day of the congress, the delegates departed for Florence, where, in the Palazzo Vecchio, Professor Birkhoff spoke on *Quelques éléments mathématiques de l'art.* The final session of the congress was also held here, and it was decided, completing the discussion of the preceding day, that the congress of 1932 should be held in Switzerland.

During the seven day visit of the delegates at Bologna and the closing day of the congress at Florence, everything proceeded in perfect order, and the best of relations existed between the delegates of different nationalities. To the scientific interest, determined by the interchange of ideas, of methods and of results on the most modern tendencies of mathematics, one must add the result, politically important, of bringing the international congress of mathematicians to the universality of the pre-war period.

L. TONELLI

MAXIME BÔCHER*

BY WILLIAM F. OSGOOD

The twelfth of last September marked a decade since Bôcher passed away. He retained the full vigor of his mind till shortly before the end, but the last months of his life were clouded by a wasting disease, to which he finally succumbed.

Maxime Bôcher was born in Boston, August 28, 1867. He prepared for college at the Cambridge Latin School and graduated from Harvard in the Class of '88. He then went to Göttingen, where he spent six semesters and took the degree of Doctor of Philosophy in 1891. He was appointed instructor in mathematics at Harvard and entered on his duties in the fall of that year.

To understand Bôcher as a teacher, we must first look at mathematics in Cambridge in the late eighties. The traditional regard in which mathematics had always been held in New England was enhanced by the influence of Benjamin Peirce, a rugged figure in the scientific life of his time. With Byerly and B. O. Peirce came not only more effective instruction, but a systematic organization of basal courses in the calculus, in geometry, in mechanics, and in mathematical physics. Cole had just brought from Leipzig Klein's geometric treatment of the theory of functions of a complex variable, and his enthusiasm for the mathematics that had been revealed to him in Klein's seminar—for algebra and geometry, even more than for analysis—predisposed Bôcher to seek the guidance of that great leader.

On his return to Harvard as instructor he was entrusted with an elementary course. I use the word *entrusted* advisedly; for although Professor Byerly was willing that any one who was master of his subject should teach graduate stu-

* An address delivered before the Society at the request of the program committee, New York, December 28, 1928.

dents, he well knew that a beginner must be a born teacher if he is to teach beginners effectively. And Bôcher was a born teacher. On his mother's side he was of Puritan ancestry, and it is hard to imagine a Puritan mother who had not the call to teach. His father was professor of French, whose courses, because of the joy Professor Bôcher felt in the beauty of literature, made a wide appeal to the undergraduates of that day. Of the sternest scientific conscience, the younger Bôcher was keenly aware that, though truth may not be relative, the ability to recognize it is. With his broad and deep knowledge of his subject, whether he was teaching analytic geometry to freshmen or linear differential equations to graduates, he sought first of all to understand the student's domain of ideas, and then to give him the best which he could receive. It is generally recognized that interest in the science of mathematics and productive scholarship in that field redound to the good of elementary instruction. Bôcher proved the converse. Because he entered into the intellectual life of the freshman and devised ways and means to guide him in the pursuance of mathematics, he was thereby enabled to make his graduate instruction effective to an astonishing degree. When the Annals of Mathematics came to Cambridge, with Bôcher as the first editor, he sought and secured collections of problems in advanced subjects, results of such teaching as I have indicated in the higher fields. He encouraged, too, the preparation of expository articles.

He understood as few teachers do, the value and the difficulty of formulation. Think what a change would spread over the face of mathematical literature if writers had any adequate conception of formulation and the role it plays in clear thinking, and if they had the tact required to show them the relation of their message to the reader's knowledge. For most mathematicians are not intentionally mystics, either in the class-room or in the written word. They fail in the first requirement which science imposes on its followers, namely, to appreciate the importance of stating the probem definitely

and of seeing clearly the relation of the problem or theory to the rest of mathematics. I say, to *appreciate* the importance; for no improvement can come in presentation, whether it be a question of the author's own discoveries or of a college text-book, till writers *feel* the importance of clear formulation of results and of methods, and of clear insight into the inter-relations of the parts. It was this emotional element in Bôcher that at once made him a master in the class-room and directed his own energies into fruitful fields of research.

His *Algebra* was an outgrowth both of his teaching and of his own scientific work. When he gave his first advanced course in algebra (in 1898) there was no systematic treatment, in any language, of those parts of algebra which are most useful in their applications to geometry, to mathematical physics, and particularly to the calculus and analysis. The need of the student in this direction was clear to Bôcher, for he had himself, as a student, been obliged only recently to provide himself, unaided, with these tools. A paper on *Linear dependence and linear equations*, in the Annals of Mathematics for 1901, was his first published work in this field. It opens with the words:

> The subject treated in the following pages is not only one which, owing to its numerous applications, is of considerable importance, but, while distinctly elementary, illustrates one of the most striking tendencies of modern algebraic and analytic work, namely, the tendency not to be satisfied with results which are merely true "in general," i.e., with more or less numerous exceptions, but to strive for theorems which are *always* true.

And his understanding of the needs of the reader is shown by the footnote which he appends to this section:

> The great importance of this tendency will be apparent if we remember that when we apply a theorem, it is usually to a special case. If we know merely that the theorem is true "in general," we must first consider whether the special case to which we wish to apply it, is not one of the exceptional cases where the theorem fails.

Although his personal needs had not, so far as I am aware, led him to treat the factoring of polynomials in several variables, he recognized the importance of this subject, and his chapter on this topic gives precisely the information

which is needed for the study of the analogous problem of factoring analytic functions of several complex variables in the neighborhood of a point. Out of the systematic study of this latter problem has grown a theory of elimination which makes possible the proof of Weierstrass's second theorem on implicit functions in a single page.

The finish, both in form and substance, with which this book is written, can easily present a danger to the beginner, who has not yet learned that he must write his own book if he is to master the methods. But Bôcher's great skill in devising exercises, small and great, serve as a corrective for ill-founded self-confidence. As a sample of the kind of demand Bôcher encouraged his students to make on themselves, I may mention his paper in the Annals of Mathematics (October, 1905): *A problem in analytic geometry with a moral.*

Bôcher's first book, *Ueber die Reihenentwickelungen der Potentialtheorie*, appeared in 1894, and was a further development of the subject of his doctoral dissertation and prize essay. A perusal of the table of contents shows the fields with which he had become familiar: not projective geometry alone, but the geometry on a quadric manifold in higher spaces, with the method of elementary divisors for purposes of classification; the cyclides of Dupin and curvilinear coordinates, with extension to space of n dimensions; the partial differential equations of mathematical physics, leading him in his later work to consider that which is vital in integral equations and eventually to write a Cambridge Tract on this subject; total differential equations of the second order, linear and homogeneous, considered both in the complex domain and in reals, with reference to the theorems of oscillation of Sturm and Klein; Lamé's equation. It was in his later lectures on this part of the subject that one of his most gifted pupils obtained his start in mathematics. The development of arbitrary functions into series of standard types; systems of orthogonal functions, preparing the way for his later guidance of a student in the field of divergent

series and convergence factors, out of which studies have grown valuable contributions to this important subject.

It may seem that Bôcher imitated Klein, or that he was an adherent of a Göttingen School. But ideals cannot be imitated, and these ideals, in particular, are the spirit of the thought of free men, striving for a higher and deeper sense of scientific values in broad fields of mathematics.

Bôcher gave, at one time or another, nearly every one of the back-bone courses in the Department; but his special interest lay in the comprehensive course on modern geometry, to the development of which he contributed much, and on which, at the time of his death, he had begun to write a book. He gave, furthermore, for a number of years, one or the other of the introductory courses in the theory of functions of a complex variable, and of real variables. After Cole, he was the first at Harvard to give the former course from the geometric standpoint of Klein.

The latter course originated with him, being entitled *Infinite Series and Products*.* In the literature of that time every rigorous proof was given by ϵ's, *ab ovo*. The fundamental theorems of analysis must, of course, be established by means of ϵ-proofs. But the derived theorems stand out more clearly when deduced from the former. It was not a question of logical elegance or of a doctrinal method, but rather of putting first things first, and helping the student to grasp the inherent difficulties of higher analysis. Bôcher had contempt for all that smacked of platform erudition, the cult peculiar to those who had a sense for rigor, of stating every theorem with the maximum of generality, at the expense of incisiveness with reference to its essential meaning.

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* On looking up the date I find that I gave this course for the first time in 1897–98. The error is interesting, for it shows the close touch in which Bôcher and I were in organizing these courses. Bôcher had, for a year or two, brought into a half-course on the theory of equations and invariants a small amount of algebraic analysis, and he suggested the desirability of a half-course on infinite series and products—a suggestion in which I heartily concurred.

His advanced course was frequently on total linear differential equations of the second order, alternating between the complex theory and the real case. He also lectured repeatedly on the partial differential equations of mathematical physics, and on integral equations. Out of these courses grew his address at the Cambridge Congress (1912), and his Paris lectures (1913–14).

Throughout his whole academic career, beginning while he was still an instructor, he was actively engaged in guiding students in their work for the doctorate, and about twenty of these students completed their work under him. He was extraordinarily successful in discerning important topics, with range and depth, yet still within the powers of students at that stage of advancement. He was skillful in his guidance, not giving help that hampered the student's initiative or resourcefulness, nor neglecting him when he was ready to profit from a conference.

As editor of the Transactions he took a similar interest in the work of the younger mathematicians. His instinct for square dealing was shown anew by his impartial consideration of each paper submitted and his insistence on a prompt acceptance or rejection, followed in the former case by prompt publication. No referee could delay action long without hearing from Bôcher, and further delay caused the paper promptly to be recalled. He was no respecter of persons. Each paper was judged on its merits, and an opportunistic attitude toward contributors would have seemed to him sacrilege. He took great pains in helping unskillful writers to improve their presentation, but his first thought was for the genuineness of the scientific results obtained. Thus he contributed in those early days to make the traditions of the Transactions high in relation both to form and to substance, and to give the journal a reputation for fairness alike toward the older and the younger writers.

His work on the Annals had given him a valuable apprenticeship. It would have been easy to make the journal, when it came into the hands of the younger

mathematicians, interested as we all were in making for the Transactions the highest possible place in the mathematical world, a repository for papers not quite good enough for the latter journal. This was not Bôcher's view of the province of the Annals. A paper that had little to recommend it but the novelty of the result and the avoidance of logical errors in the proof would not pass muster. He set the same high standards of scientific merit which characterized the more advanced journal. The scope of the paper might be restricted, but not the genuineness of the results.

Nor were his sympathies and interests confined to his own courses and his special fields of study. At a time when mathematicians were just beginning to turn their attention to postulates, he recognized in the work of Huntington investigations of substantial character and used his influence to secure for Harvard the latter's services. Again, it was with full consciousness of the hearty moral support Bôcher accorded that Bouton built up his two courses on the Lie theory, and developed them to an extraordinarily high degree of effectiveness. He was well informed concerning the character of Coolidge's work, though he had no great technical knowledge of the subject, and I recall the keen interest he felt in it when Coolidge was a beginner. The same is true of his interest in the work of each of the others of his colleagues in the department,—Birkhoff, Jackson, Graustein,* and Gabriel Green. It is due in no small measure to these qualities in Bôcher, to his sense of justice and honesty of thought, to his willingness to accord others, irrespective of age or rank, what it seemed to him reasonable that they should accord him under like conditions, that our department has been able to pass from a patriarchal system to a pure democracy.

The student of mathematics would, indeed, be temerous, who would ignore the past and, like some modern novelists, attempt to build up a civilization out of his inner conscious-

* Professor Graustein was an instructor in the department for the year 1913–14.

ness. True, the seer has visions of what man may accomplish. But the visions which light the way to new achievement are those which spring from an understanding of what man has accomplished. They are not the dreams of a child, just getting its first glimpses of the world.

Bôcher sank his roots deep into the past of his science. He was a student of history, for he found in it an aid toward perspective, toward seeing mathematical truths in their relative meaning and their relative importance, and toward putting first things first. He shunned ill-defined speculations and held rather to the method of research in the natural sciences—that of defining precisely an important problem and bringing the knowledge of the day to bear on its solution. He may sometimes have been too little attracted by the desire to make a brilliant guess; but what he built is a part of the permanent structure of mathematics.

Creating for the mere sake of creating was a thing in which Bôcher did not believe. Productive scholarship he did believe in, and his whole life gave sustained evidence of this cardinal point in his scientific faith, this corner stone in his scientific character. But productive scholarship meant to him something larger and grander than the discovery of new truth. For new truths come to us in unassorted lots, and because a theorem is true, it does not follow that it has value. Productive scholarship meant to him a deeper insight into mathematics. And the deeper insight comes not merely from the new theorem, but chiefly from its contacts. Bôcher was keenly aware that new truth is under suspicion; that the great majority of new theorems which can not stand the test of usefulness, are unimportant, spurious, and will not survive the nurturing care of their discoverers. He strove rather to advance mathematics according to a well-defined program, refusing to be attracted by novelty for novelty's sake, when the science was in the greatest need of coordination of great truths already obtained and of deepening the understanding of important results as yet but imperfectly perceived.

For this reason Bôcher's work was not spectacular. It was basic. For example, in reporting on his scientific contributions, Professor Birkhoff remarks that none of Bôcher's published papers lay properly in the theory of functions of a complex variable. And yet, after the founders of analysis have been cited, no name appears more frequently in the footnotes of the speaker's treatise than that of Bôcher.

Perhaps nowhere in the domain of human thought is a cultural background more important than in mathematics. To train one's mind to recognize the great structures man has built, the great visions human thought has seen, in the realm of the greatest of the sciences, it is helpful to have first-hand knowledge of achievements in other fields of human thought and human action. As an undergraduate Bôcher informed himself of the then recent discoveries of Darwin—not as the philosopher or theologian conceives them, but as the scientist knows them—experiments with well defined boundary conditions; conclusions exhausting the results of the experiments; speculations suggesting further study, but not accepted as truth till rigorously established.

All through his life Bôcher turned to literature for recreation. He was fond of Shakespeare, of the German and the French classics, and of biography. Though not a performer, he had a good appreciation of music and derived much enjoyment from listening to the production of classical music. And what was true of his intellectual life outside the science was eminently true within it. His sympathies were broad and the amount of accurate information he possessed in various fields of mathematics extensive.

So it came about that Bôcher was able to render a great service to mathematics in America through his sense of values. Now, a sense of values is not something that is inherited, like the artistic temperament. It is an expression of character. Character is formed in the determined and sustained struggle for ideals. But if the ideals are not intelligently chosen; if one has not taken the pains to form

consciously a certain philosophy of life; to detach oneself from the present and inquire whence and whither; to estimate that which is great and to plan how one may strive intelligently for the highest which, under the given limitations of temperament and ability, one may hope to attain, the result must be mediocre. I never knew a time in Bôcher's life in which he was not working on the most important problem of those among which he could reasonably expect to make advance.

It was this habit of mind, these standards relating to scientific activity, accompanied as they were by his striving for the highest and the purest beauty in mathematics, that made him helpful to others. He wished to see the influence of the Transactions, particularly on the younger writers, directed into these channels, and in this I am well aware that he had the earnest cooperation of our colleague and President, Van Vleck. What he lacked was the zeal of the reformer, who believes that men can be made to improve their ideals by having the truth thrust at them. He gave freely of himself to all who, in the spirit of truth, sought to discuss the truth with him. He felt no inner urge to make men do what of their own initiative they could not. Among the strongest of his contemporaries, however, his influence was strong, and he found hearty support from some of the best men of his time. His pupils have rendered a good account of themselves, and he is remembered by them as an inspiring teacher, not through emotional appeal, but through the genuineness of his mathematical perceptions and ideals, his sincerity, the rugged honesty of his thought, and his unswerving devotion to his science.

I may seem to have laid too much stress on Bôcher's appreciation of the work of others and his efforts to clarify and unify it, and the impression can easily be created that he absorbed knowledge rather than that he smote from the rock the living waters, that he had imagination of high order, and that he produced original results of high merit. It is not easy to correct this impression by citing specific results readily in-

telligible to the lay mathematician. All of Bôcher's more important work was based on theories which had already attained a high degree of advancement. To understand his precise contribution, it is often necessary to make a comprehensive study of a whole field, or, indeed, of different fields, of mathematics. But then it will be found that he brought an originality of thought and method which secure for him a high place among the mathematicians of his time. His scientific biographer, Professor Birkhoff, closes his estimate with the words:

"In amount and quality, his production exceeds that of any American mathematician of earlier date in the field of pure mathematics."

The contemplation of Bôcher's life leads us to the question: Is mathematics a collection of concepts and relations between these concepts—a *logical system*, in the broadest interpretation of this notion, as formulated by Bôcher himself in his St. Louis address—or is not a truer picture that of the various civilizations through which the human race has passed and is continuing to pass? These civilizations develop gradually, often at great expense, a purer conception of beauty, a deeper insight into truth, a perception of the eternal verities, a recognition of the absolute invariants of the human mind and of the human soul. And when these invariants have once been discerned, they pass over into the traditions of that people. They become a part of the intellectual and the spiritual heritage of the race.

Bôcher valued a new result, but he did not find the satisfaction in a collection of new results, or even new theories, which the child finds in a stamp collection. He looked beyond the new result to its place in the science, and his powers of criticism were such that he saw with remarkable clearness the bearing of that which is new on the development of the science.

Critical is a treacherous word to apply to a scholar, for it suggests destructive criticism and the cavilling of literary men. No greater mistake could be made than to think of

Bôcher as belonging to that category of critics. Constructive in all his thinking, he made his criticism of the work of others productive of new achievement on their part. He knew, too, how to profit from criticism of his work by others, even though given in an unfriendly spirit. It was the content, not the form, to which he chose to direct his attention.

We have witnessed the revolt of youth in society, the tearing of the mask from the conventions of the past, in these days since the War, and some of the older ones of us have felt humbled at the thought of our failure to make clear to those who come after us the eternal verities which are the heritage of the race. But each new generation must find these truths, and if it will not learn from the past, then it must create again the experience on which the wisdom of the past was founded. For build man must, whether for the hour or for eternity.

No less in mathematics have we a rich heritage from the past, and the desire of youth to go forward. The great problem is to stir the soul of youth to understand the former, while retaining in full measure the latter. That which Bôcher has left us has permanent value in this great field of human endeavor.

Bôcher's influence on American mathematics will wax with time, if we believe in progress. His influence is needed today. Not in economics and in ethics alone are the results of the Great War manifest. There is in mathematics an absence of scientific tranquility, a desire to get results without paying the price of scientific productivity. It is with satisfaction and inspiration that one turns to Bôcher's writings and Bôcher's teachings. The younger generation of mathematicians is not lacking in those who understand the message, and to them we may confidently entrust the future of mathematics in America.

We look to Bôcher as an example of one whose highest satisfaction was in service, but whose conception of service was derived from the highest needs of the science and of its disciples. To understand before undertaking; to examine

objectively the products of imagination; to make that which is genuine accessible for those who come after—these are the three great traits of his intellectual being.

A tribute to Bôcher would be imperfect which did not lay emphasis, not only on his high idealism and his intellectual achievements, but also on the character of the man, his passion for truth, his freedom from self-seeking, his high conception of duty and his loyalty to his fellowmen, above all, his faithfulness in little things. To him the words apply:

> How happy is he born or taught
> Who serveth not another's will;
> Whose armor is his honest thought,
> And simple truth his highest skill;
>
> Whose passions not his masters are;
> Whose soul is still prepared for death,
> Untied unto the world by care
> Of prince's ear or vulgar breath;
>
> Who hath his life from rumors freed,
> Whose conscience is his strong retreat,
> Whose state can neither flatterers feed,
> Nor ruin make oppressors great;
>
> Who God doth late and early pray
> More of his grace than goods to lend;
> And walks with man, from day to day,
> As with a brother and a friend.
>
> This man is freed from servile bands
> Of hope to rise, or fear to fall;
> Lord of himself, though not of lands,
> And having nothing, yet hath all.

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ON REGULAR POINTS OF CONTINUA AND REGULAR CURVES OF AT MOST ORDER n*

BY G. T. WHYBURN

1. *Introduction.* In this paper it will be shown, as a consequence of some more general results, that if n is any integer $\neq 2$, the set of all points of order n of any continuum M in a locally compact metric and separable space is punctiform, that is, contains no continuum; and hence there exists no continuum every point of which is of order exactly n.

The ordinary notation and terminology of point set theory will be employed. For example, $\overline{X} = X + X'$, where X' is the set of all limit points of the set X; $K \cdot H$ means the set of points common to K and H; $K \subset H$ means that K is a subset of H and $K \supset H$ that K contains H; $\delta(M)$ denotes the diameter of the set M; $\rho(X, Y)$ denotes the minimum distance between the sets X and Y; if R is an open set, $F(R)$ denotes the boundary of R relative to the whole space, and if R is an open subset of a set M, $F_m(R)$ denotes the boundary of R relative to M, that is, the set of all those points of $M - R$ which are limit points of R. By a continuous curve is meant any connected im kleinen continuum. A neighborhood of a point is an open set containing that point. A point P of a continuum M is called a Menger regular point† of M, or simply a regular point of M, if for each $\epsilon > 0$, P can be ϵ-separated‡ in M by some finite subset of M, that is, a finite subset U of M exists such that $M - U = M_1 + M_2$, where M_1 and M_2 are mutually separated, $M_1 \supset P$, and $\delta(M_1) < \epsilon$. If an integer n exists such that, for each $\epsilon > 0$, the ϵ-separating set U can be chosen of power n, but cannot, (for every ϵ), be

* Presented to the Society, Southwestern Section, December 1, 1928.

† See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306.

‡ P. Urysohn, *Sur la ramification des lignes Cantoriennes*, Comptes Rendus, vol. 175 (1922), p. 481.

chosen of power $<n$, P is said to be a point of order n of M. A continuum all of whose points are regular points is called a Menger regular curve, or simply a regular curve; and a continuum M such that for some integer n, all the points of M are of order $\leqq n$, is called a regular curve of at most order n.*

DEFINITION. The point P of a continuum M will be said to *separate locally* a given subset N of M in M if there exists a compact neighborhood G of P such that if R is any neighborhood of P lying in G, then $M \cdot \overline{R} - P$ is separated between some two points of $N \cdot R$, that is, $M \cdot \overline{R} - P = M_1 + M_2$, where M_1 and M_2 are mutually separated and $M_1 \cdot N \cdot R \neq 0 \neq M_2 \cdot N \cdot R$.

The point P of a continuum M is called a *local separating point*† of M if a compact neighborhood R of P exists such that $M \cdot \overline{R} - P$ is separated between some two points of the component C of $M \cdot \overline{R}$ which contains P. Obviously every local separating point of a continuum M *locally separates* M in M. However, the converse is not necessarily true unless P is a regular point of M.

2. THEOREM 1. *If for the closed subset N of a continuum M there exists an integer n such that the set K of points of N which are points of order $\leqq n$ of M and which locally separate N in M is dense in N, then the points of M of order $\leqq n/2+1$ are dense in N.*

PROOF. Let R be any open subset of N. By hypothesis $K \cdot R$ contains a point P_1. There exists a compact open subset R_1 of M containing P_1 and such that $N \cdot \overline{R}_1 \subset R$, $\delta(R_1) < 1$, $F_m(R_1)$ is of power $\leqq n$, and $\overline{R}_1 - P_1 = R_{11} + R_{12}$, where R_{11} and R_{12} are mutually separated and $R_{11} \cdot N \cdot R_1 \neq 0 \neq R_{12} \cdot N \cdot R_1$. There exists at least one of the sets R_{11} and R_{12}, call one such set D_1, such that $D_1 \cdot F_m(R_1)$ is of power $\leqq n/2$. Then since $F_m(D_1) \subset P + D_1 \cdot F_m(R_1)$, it follows that $F_m(D_1)$

* See K. Menger, loc. cit., and K. Menger, *Zur allgemeinen Kurventheorie*, Fundamenta Mathematica, vol. 10 (1927), pp. 96–115.

† See my paper, *Local separating points of continua*, presented to the Society October 27, 1928, appearing in Monatshefte für Mathematik und Physik, vol. 36 (1929), No. 2.

is of power $\leqq n/2+1$. Clearly $\overline{D}_1 \cdot N \subset R$ and $\delta(D_1) < 1$. Let N_1 denote the set of points $\overline{D}_1 \cdot N$. By hypothesis, there exists a point P_2 in $K \cdot N_1 \cdot [\overline{D}_1 - F_m(D_1)]$. There exists an open subset R_2 of M containing P_2 and such that $\overline{R}_2 \subset \overline{D}_1 - F_m(D_1)$, $\delta(R_2) < 1/2$, $F_m(R_2)$ is of power $\leqq n$, and $\overline{R}_2 - P = R_{21} + R_{22}$, two mutually separated sets with $R_{21} \cdot N \cdot R_2 \neq 0 \neq R_{22} \cdot N \cdot R_2$. For at least one of the sets R_{21} and R_{22}, call one such set D_2, it is true that $D_2 \cdot F_m(R_2)$ is of power $\leqq n/2$. Then, just as in the case of D_1, it follows that $F_m(D_2)$ is of power $\leqq n/2+1$ and that $\overline{D}_2 \subset \overline{D}_1 - F_m(D_1)$ and $\delta(D_2) < 1/2$. Let $N_2 = \overline{D}_2 \cdot N$. Then $N_2 \subset N_1$, and there exists a point P_3 in $K \cdot \overline{N}_2 \cdot [\overline{D}_2 - F_m(D_2)]$. Just as above it follows that there exists a subset D_3 of D_2 containing P_3 and such that $D_3 \cdot N_2 \neq 0$, $\delta(D_3) < 1/3$, $\overline{D}_3 \subset \overline{D}_2 - F_m(D_2)$, and $F_m(D_3)$ is of power $\leqq n/2+1$. Let this process be continued indefinitely, giving a sequence $D_1, D_2, D_3, \cdots$, of subsets of M such that, for each i, $\overline{D}_i \subset \overline{D}_{i-1} - F_m(D_{i-1})$, $\delta(D_i) < 1/i$, $0 \neq D_i \cdot N \subset R$, and $F_m(D_i)$ is of power $\leqq n/2+1$. There exists a point P which belongs to every set D_i. Then clearly P is a point of order $\leqq n/2+1$ of M. And since P is a limit point of N and N is closed, then P belongs to N and hence to R. Thus every open subset of N contains a point of order $\leqq n/2+1$ of M, and our theorem is proved.

COROLLARY 1a. *If for a continuum M there exists an integer n such that the local separating points of M of order $\leqq n$ are dense in M, then the points of order $\leqq n/2+1$ are dense in M.*

THEOREM 2. *If M is a Menger regular curve, and if for some integer n, all the local separating points of M are of order $\leqq n$, then points of M of order $\leqq n/2+1$ are dense in M.*

By a theorem of the author,* the local separating points in a regular curve M are dense in M. Then since every local separating point of M *locally separates* M in M, it follows by hypothesis and Theorem 1 that the points of M of order $\leqq n/2+1$ are dense in M.

* See my paper, *Concerning points of continuous curves defined by certain im kleinen properties*, offered to Mathematische Annalen.

COROLLARY 2a. *If for some integer n, every point of a continuum M is of order $\leqq n$, that is, if M is a regular curve of at most order n, then the points of M of order $\leqq n/2+1$ are dense in M.*

COROLLARY 2b. *There does not exist an integer $n \neq 2$ and a continuum M every point of which is of order exactly n.*

COROLLARY 2c. *The simple closed curve is the only continuum of at most finite order every point of which is of the same order.*

THEOREM 3. *If N is any connected set of regular points of a continuum M, the set K of all those points of N which are local separating points of M and locally separate N in M is dense in N.*

PROOF. Let R be any open subset of N, which we will suppose is $\neq N$, and let P be a point of R. Since P is a regular point of M, there exists a compact open subset D of M containing P and such that $F_m(D)$ is finite and $\bar{D} \cdot N \subset R$. Since N is connected and $D \cdot N \neq 0 \neq N \cdot (M - \bar{D})$, it follows that $F_m(D)$ contains a point X of N which is a limit both of $N \cdot D$ and of $N \cdot (M - \bar{D})$. And since N is connected im kleinen, it follows that there exist compact connected subsets A and B of N such that $A \cdot B = X$, $A - X \subset D$, and $B - X \subset M - \bar{D}$. Let G be a compact neighborhood of X such that $\bar{G} \cdot [F_m(D) - X] = 0$. Then if V is any neighborhood of X lying in G, $M \cdot \bar{V} - X = \bar{V} \cdot D + \bar{V} \cdot (M - \bar{D})$; and since D and $M - \bar{D}$ are mutually separated, it follows that $\bar{V} \cdot D$ and $\bar{V} \cdot (M - \bar{D})$ are mutually separated. Clearly these two sets contain points of A and B respectively. Therefore X is a local separating point of M which locally separates N in M, and hence it belongs to K. Thus, since $X \subset R$, it follows that K is dense in N.

THEOREM 4. *If M is any continuum and n is any integer >2, the set K^n of all points of M of order n is punctiform, that is, contains no continuum.*

PROOF. Suppose, on the contrary, that for some integer $n > 2$, K^n contains a continuum N. Now by Theorem 3, the set K of points of N which locally separate N in M is dense in N. Then, since every point of K is a point of order n of M, it

follows by Theorem 1 that N contains at least one point of order $\leqq n/2+1$ of M. But since $n>2$, the greatest integer which is less than or equal to $n/2+1$ is less than n. Hence N contains a point of order $<n$ of M, contrary to supposition.

NOTE. Menger (loc. cit.) has shown that the set K^1 of all points of order 1 (end points) of any continuum M is null dimensional, that is, each of its points can, for each $\epsilon>0$, be ϵ-separated in K^1 by a null set. Hence, using Theorem 4, if $n\neq 2$, the set K^n of points of order n of any continuum is punctiform. It would be interesting to determine whether or not, for $n\neq 2$, the set K^n is always null dimensional as well as punctiform. I will remark that the set K^w of points of order w (that is, regular points which are of no finite order) of a continuum, or even of a regular curve, is not necessarily punctiform, as can be shown by the curve obtained by making some simple additions to the Sierpinski regular curve.*

THEOREM 5. *In any regular curve M of at most order n, the set H of points of order $>n/2+1$ is null dimensional.*

PROOF. The set H must be punctiform. For suppose it contains a continuum N. Then since every point of N is a regular point of M of order $\leqq n$, by Theorem 3 the set K of points of N which locally separate N in M is dense in N. Hence, by Theorem 1, N contains a point of order $\leqq n/2+1$, contrary to supposition. Thus H is punctiform. And since by a theorem of Menger,† H is an F_σ (that is, the sum of a countable number of closed sets), it follows by a theorem of Mazurkiewicz‡ that H is homeomorphic with some linear set. Therefore, since H is punctiform, it is§ null dimensional.

We note here the fact that while the set H in Theorem 5 is null dimensional, it is not necessarily countable; for an

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* See W. Sierpinski, Comptes Rendus, vol. 160 (1915), p. 302.

† *Grundzüge einer Theorie der Kurven*, loc. cit., Theorem 4.

‡ S. Mazurkiewicz, Bulletin de l'Académie des Sciences de Cracovie, 1913.

§ See W. Sierpinski, *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 89.

example is easily constructed of a regular curve of at most order 3 in which the points of order 3 form a perfect set.

3. *Regular Curves of at Most Order 3.* Knaster and Kuratowski have shown* that the Sierpinski regular curve, which is a curve of at most order 4, contains a connected set which contains no perfect subset. In this section I shall show that no curve of at most order 3 can contain any such set, and indeed, that every connected subset of a continuous curve M, every maximal cyclic curve of which is a regular curve of at most order 3, contains a perfect set of local separating points of M. The continuous curve M is cyclicly connected† if every two of its points lie together on some simple closed curve in M. The cyclicly connected continuous curve C is said to be a maximal cyclic curve of a continuous curve M if C is a subset of M and is not a proper subset of any cyclicly connected continuous curve which is a subset of M.

LEMMA 6a. *In a regular curve of at most order 3, no two maximal cyclic curves have a common point.*

Lemma 6a follows at once from the facts that no two maximal cyclic curves of a continuous curve can have more than one point in common and that every point of a cyclicly connected curve C is of order not greater than 2 of C.

THEOREM 6. *If every maximal cyclic curve of a continuous curve M is a regular curve of at most order 3, then every connected subset H of M contains a perfect set of local separating points of M.*

PROOF. If every point of H is either a cut point or an end point of M, then‡ H is arcwise connected and hence contains

* *A connected and connected im kleinen point set which contains no perfect subset*, this Bulletin, vol. 33 (1927), pp. 106–109.

† See my paper, *Cyclicly connected continuous curves*. Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38;for the extension of the results in this paper to a more general space than the plane, see an abstract by W. L. Ayres in this Bulletin, vol. 33 (1927), p. 661.

‡ See my paper, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194, Theorem 8; see also an abstract by C. M. Cleveland in this Bulletin, vol. 32 (1926), p. 420.

an arc every point of which is a cut point of M. If not, then H contains at least two points of some maximal cyclic curve C of M. Let $W = H \cdot C$. Then* W is connected and contains more than one point. Since, by hypothesis, every point of W is a point of order $\leqq 3$ of C, it follows by Theorem 3 that W contains a point P which locally separates W in C. Then since P obviously is a point of order $\geqq 2$ of W, it follows that a connected open subset R of C exists containing P and such that (1) $F_c(R)$ is of power $\leqq 3$, (2) $\overline{R} \cdot W$ is connected, and (3) $F_c(R)$ contains at least two points A and B of W which are separated in $\overline{R}$ by the point P. Now since $\overline{R} \subset C$, $\overline{R}$ is a regular curve of at most order 3. Hence, by Lemma 6a, no two maximal cyclic curves of $\overline{R}$ have a point in common. And since A and B do not lie in the same maximal cyclic curve of $\overline{R}$, it follows by a theorem of the author† that A and B are separated in $\overline{R}$ by each point of a perfect point set K. Now $K \subset W \cdot \overline{R}$, for $W \cdot \overline{R}$ is connected and contains both A and B. Clearly every point of K is a local separating point of M. Thus H contains a perfect set K of local separating points of M, and our theorem is proved.

It would be interesting to determine whether or not, in a regular curve of at most order 3, every connected subset is arcwise connected, and whether every punctiform subset is null dimensional.

The foregoing results yield immediately the following additional theorem on regular curves of at most order 3.

THEOREM 7. *In a regular curve M of at most order 3 the boundary with respect to M of every connected open subset of M is totally disconnected.*

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* G. T. Whyburn, loc. cit., Theorem 30.

† See my paper, *Concerning the cyclic elements of continuous curves*, presented to the Society September 4, 1928.

CUBICS WHOSE (HESSIAN)^n'S ARE THEMSELVES*

BY F. E. WOOD

The four equianharmonic, four degenerate and six harmonic cubics which belong to the syzygetic pencil $\lambda C+\mu H=0$, where $C=0$ is a general cubic and $H=0$ is its Hessian, have been studied in much detail. We denote them as the 14 *special* cubics of the pencil. It is known that the Hessian of a degenerate or equianharmonic cubic is a degenerate cubic, and that the (Hessian)^2, that is, the Hessian of the Hessian—of a harmonic cubic is itself and that the harmonic cubics are characterized by this property. The more general problem has been considered by Hostinský only, who entirely by calculation has obtained certain results for $n=3$ and $n=4$.† In this note a characterizing property for $n=3$ is obtained and certain general results noted, the method being quite different. The theorems of this note are believed to be new, though a part of Theorem 2 could be obtained from the work of Hostinský.

Denote by γ_n a cubic whose (Hessian)^n is itself and whose (Hessian)^t for every value $t<n$ is not itself. Let m denote the parameter of a cubic, and $m^{(n)}$ the parameter of its (Hessian)^n; then

$$m^{(r+1)} = -\frac{1+2(m^{(r)})^3}{6(m^{(r)})^2}, \qquad (r=0,1,\cdots,n-1),$$

where $m^{(0)}=m$. Then $m^{(n)}=m$ gives an equation in m of degree 3^n, which is also satisfied by any cubic whose (Hessian)^r is itself, for r a divisor of n. Dividing out from the equation the factors corresponding to these cubics, one will have in general an equation whose roots give cubics γ.

* Presented to the Society, December 31, 1928.

† B. Hostinský, *Sur les Hessiennes successives d'une courbe du troisième degré*, Proceedings of the International Congress, Cambridge, 1912, vol. 2, pp. 102–104.

None of the cubics involved can be special, when $n>2$. It can be shown that to every non-special cubic of a syzygetic pencil there are eleven others forming a group of twelve, each projectively equivalent to the others. The Hessians of such a group form another group of twelve projectively equivalent cubics, all distinct unless the group of Hessians is a group of special cubics; in which case for no value of n will the (Hessian)^n of one of the original group be itself.

If the group of Hessians is each time different from all the preceding groups, then the existence of one non-special cubic γ_n implies the existence of $12n$ such cubics. For every value of n the degenerate cubics will appear, and when n is a prime and greater than 2 the parameters corresponding to the degenerate cubics will be the only ones to appear in the equation, mentioned above, of degree 3^n. And one notes that 3^n-3, n a prime, is divisible by $12n$ when $n>3$. Thus *the totality of cubics γ_n form k groups of 12 projectively equivalent cubics each. When n is a prime and greater than 3, k is a multiple of n.*

Consider $n=3$. Let a_1 be a non-special cubic whose (Hessian)^3 is itself; let $a_2, \cdots, a_{12}$ denote the other cubics of the syzygetic pencil projectively equivalent to a_1, forming all together the group A; denote the Hessian of a_i by b_i, the group of b_i by B. Then there are two possible cases; either the group B is not, or is, the group A. In the first case one finds that there must be 36 cubics γ_3, which is impossible since for $n=3$, $3^n-3=24$. In the second case one finds that the continued Hessians of A form the group A, the twelve cubics of A being arranged in four groups of three, such that the continued Hessians of the cubics of any one triple are the triple itself.

Conversely, consider the non-special cubics which have the property that the Hessian of one such is projectively equivalent to it. Let m be the parameter of the cubic, then $-(1+2m^3)/(6m^2)$ is the parameter of its Hessian; hence we have the equation

$$\frac{m^3(1-m^3)^3}{(1-20m^3-8m^6)^2}=\frac{-\left(\frac{1+2m^3}{6m^2}\right)^3\left[1+\left(\frac{1+2m^3}{6m^2}\right)^3\right]^3}{\left[1+20\left(\frac{1+2m^3}{6m^2}\right)^3-8\left(\frac{1+2m^3}{6m^2}\right)^6\right]^2}.$$

Dividing out the factors $(1-20m^3-8m^6)^2(8m^3+1)^3$, to which correspond special cubics, we obtain an equation of degree 24. The 24 cubics corresponding are exactly the 24 previously mentioned and we have proved that *a necessary and sufficient condition that a non-special cubic have its (Hessian)^3 coincide with itself, is that its Hessian be projectively equivalent to itself.*

NORTHWESTERN UNIVERSITY

A PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA*

BY I. M. SHEFFER†

1. *Introduction.* The number of proofs given of the fundamental theorem of algebra is large. Perhaps for that very reason still another proof may not be unacceptable. The one that is offered here is not "elementary," since it makes use of some general results in analysis. Yet it may be termed simple, and may be of interest. It is our hope that this proof, which is believed to be new, may, with no great embarrassment, take its place in the family of proofs that every algebraic equation has a root.

2. *The Proof.* We consider the equation

(1) $a_kx^k+a_{k-1}x^{k-1}+\cdots+a_1x+a_0=0,\ (a_k\neq 0, k>0).$

There is no loss in generality in supposing that‡ $a_1\neq 0$; and,

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* Presented to the Society, September 6, 1928.

† National Research Fellow.

‡ For suppose $a_1=0$. Make the substitution $x=y+\alpha$. We obtain an equation in y, in which the coefficient of y is a polynomial in α of degree

by dividing through by a_1, we can always make the coefficient of x unity. We then take $a_1 = 1$. Let us replace the constant term a_0 by $-\lambda$:

$$a_k x^k + a_{k-1}x^{k-1} + \cdots + x - \lambda = 0. \tag{2}$$

We shall prove that (2) has a root for every λ. For $\lambda = 0$ we have a root $x = 0$. Let us assume that x, considered as a function of λ, can be expanded in a power series about the origin (with no constant term):

$$x(\lambda) = \sum_{n=1}^{\infty} x_n \lambda^n. \tag{3}$$

If we substitute (3) into (2), and equate coefficients of like powers of λ, we find for $x_1, x_2, \cdots$, the series of equations

$$x_1 = 1, \quad x_2 = P_2(x_1), \quad x_3 = P_3(x_1, x_2), \cdots,$$
$$x_n = P_n(x_1, \cdots, x_{n-1}), \cdots,$$

where the P's are polynomials in their respective variables; and these permit us to determine, step by step, *and uniquely*, $x_1, x_2, \cdots, x_n, \cdots$.

The series (3) has a non-zero radius of convergence. One method of proof is to determine suitable inequalities for $|x_n|$. But it is simpler to appeal to the theory of implicit functions.

Denote the left hand member of (2) by $F(\lambda; x)$. We have $F(0; 0) = 0$, $\partial F(0; 0)/\partial x = 1 \neq 0$; consequently, in a sufficiently small neighborhood of $\lambda = 0$ (in the λ-plane), there exists a unique analytic function $x(\lambda)$ with $x(0) = 0$, which when put into (2) makes (2) an identity in λ. This function will have a convergent power series expansion, which by the uniqueness of the coefficients x_n, must coincide with (3).

For every λ inside the circle of convergence of (3), $x(\lambda)$ is a root of (2). If (3) converges for all* (finite)λ, then (2)

$k-1$, and since we know (by elementary algebra) that an equation of degree $k-1$ cannot have more than $k-1$ roots, it follows that α can be chosen so that this coefficient is not zero.

* This will be the case when $k = 1$.

has a root for every λ, and the desired theorem is established. Let us then consider the contrary case, where the radius of convergence, r, is finite; and let C denote the circle of convergence. On C, $x(\lambda)$ has at least one singular point. Let $\lambda=\lambda'$ be such a singularity.

Interior to C, $x(\lambda)$ *is bounded.* For suppose the contrary. Then there will exist an infinite sequence of values λ_n, with $|\lambda_n|<r$, such that $\lim |x(\lambda_n)|=\infty$. But $|\lambda|$ being bounded, for $|x|$ sufficiently large the dominant term in the left hand member of (2) is $|a_k x^k|$, so that for n sufficiently large, $x(\lambda_n)$ cannot satisfy (2); a contradiction.

It follows that there exists an infinite sequence λ_n, with $|\lambda_n|<r$, such that $\lim \lambda_n=\lambda'$ and such that $\lim x(\lambda_n)$ *exists.** Call this limiting value x'. Since $\lambda=\lambda_n$, $x=x(\lambda_n)$ satisfy (2), therefore x' is a root of (2) for $\lambda=\lambda'$.

We must then have $\partial F(\lambda;\ x)/\partial x=0$ for $(\lambda;\ x)=(\lambda';\ x')$. For if not, then the implicit function argument tells us that in a sufficiently small neighborhood of $\lambda=\lambda'$ there is a unique analytic function $\bar{x}(\lambda)$ with $\bar{x}(\lambda')=x'$, such that $x=\bar{x}(\lambda)$ satisfies (2) identically. This function must be an analytic continuation of $x(\lambda)$.† But this means that λ' is not a singular point of $x(\lambda)$; a contradiction.

Now the condition $\partial F(\lambda;\ x)/\partial x=0$ does not involve λ; it is in fact an algebraic equation in x of degree $k-1$. Such

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* We use Weierstrass' Theorem on an infinite set which is bounded.

† A detailed argument is as follows: By the implicit function theorem there exist two positive numbers δ, ϵ such that for every λ in the λ'-region $|\lambda'-\lambda|<\delta$ there is one and only one x in the x'-region $|x'-x|<\epsilon$ which satisfies (2) (for $\lambda=\lambda'$ the value of x being x'); and this correspondence between λ and x in the regions considered constitutes an analytic function $\bar{x}(\lambda)$. Now the λ'-region overlaps with the region of convergence of (3). Hence for n sufficiently large we shall have λ_n in the λ'-region and $x(\lambda_n)$ in the x'-region, and this requires that $x(\lambda_n)=\bar{x}(\lambda_n)$. Now about this point $\lambda=\lambda_n$ (and for the value $x=x(\lambda_n)$), we can reapply the implicit function theorem. In a sufficiently small neighborhood of $\lambda=\lambda_n$ we find a unique analytic function of λ taking on the value $x(\lambda_n)$ at λ_n; and this function of course coincides with $\bar{x}(\lambda)$. But $x(\lambda)$ is analytic in a sufficiently small neighborhood of λ_n, and has there the value $x(\lambda_n)$, so that we must have $x(\lambda)\equiv\bar{x}(\lambda)$ in the neighborhood of λ_n.

an equation cannot be satisfied by more than $k-1$ values of x, so that (as we see from (2)) there cannot be more than $k-1$ singular points λ on C. We may then by analytic continuation follow the function $x(\lambda)$ across C; and for every λ in the circle of convergence of each continuation, $x(\lambda)$ will, by the principle of permanence, be a root of (2).

The preceding arguments apply to these new circles of convergence, so that we can extend the function $x(\lambda)$ throughout the entire (finite) λ-plane, with the exception of a *finite number* of singular points. And at each non-singular point λ, $x(\lambda)$ is a root of (2). But also at each singular point we have a root.* The theorem is thus established.

3. *Remark.* A previous argument can be used to show that in every bounded λ-region, $x(\lambda)$ is bounded. This is true in particular in the neighborhood of the singularities of $x(\lambda)$. In consequence, *the singularities*† *of* $x(\lambda)$ *are branch points at which the function remains finite.* For, the only other possibilities are that a singularity is either a pole or an essential singularity (whether or not also a branch point); but in either case there would exist an infinite sequence of points λ approaching the singularity, for which $|x(\lambda)|$ would approach infinity, and this contradicts the boundedness of $x(\lambda)$.

PRINCETON UNIVERSITY

* The argument used for $\lambda=\lambda'$ on C applies.

† That is, in the finite plane.

AN APPLICATION OF TENSOR ANALYSIS TO THE FIRST VARIATION OF AN INTEGRAL*

BY J. H. TAYLOR

The method which J. L. Synge† has used to develop the first and second variations of an integral of a restricted type is readily applicable so far as the first variation is concerned, to a general integral of a *regular* calculus of variations problem. Let C be a curve in an n-dimensional space

$$(1) \qquad x^\alpha = x^\alpha(u), \qquad (u_1 \leqq u \leqq u_2), \qquad (\alpha = 1, 2, \cdots, n).$$

The value of the integral

$$(2) \qquad t = \int_{u_1}^{u_2} F(x, \dot{x}) du$$

will be called the *arc length* of the curve.‡ Let the curve C be imbedded in a one-parameter family of comparison curves

$$(3) \qquad x^\alpha = x^\alpha(u, v), \qquad (u_1 \leqq u \leqq u_2, v_1 \leqq v \leqq v_2),$$

such that C is given by $v = 0$, that is, $x^\alpha(u, 0) = x^\alpha(u)$ identically in u. We suppose the limits of integration to be unvaried and so the value of the integral t taken along any one of the curves $v = \text{constant}$ is a function of v alone. The equations (3) define a surface in terms of the parameters u and v. Let the tangent vectors to the u and v-curves be designated by ξ and η respectively,

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* Presented to the Society, December 31, 1926.

† *The first and second variations of the length-integral in Riemannian space*, Proceedings of the London Mathematical Society, vol. 25 (1925), pp. 247–264.

‡ For the conditions imposed on the function F and the notation here employed see the author's paper, *Parallelism and transversality in a subspace of a general (Finsler) space*, Annals of Mathematics, vol. 28 (1927), pp. 620–628.

$$\xi^\alpha = \frac{\partial x^\alpha}{\partial u}, \qquad \eta^\alpha = \frac{\partial x^\alpha}{\partial v}. \tag{4}$$

We take as corresponding points on the curve C and its comparison curves the points lying on the curves $u = \text{const.}$ The vector η will be called the *variation* or *displacement* vector. Since the limits of integration are independent of v, we have

$$\frac{\partial t}{\partial v} = \int_{u_1}^{u_2} \frac{\partial F}{\partial v} \cdot du$$

the arguments of F being x and ξ.* On account of the homogeneity property of F we may write $F = F_\alpha \xi^\alpha$, where α is to be summed from 1 to n, and where $F_\alpha = \partial F / \partial \dot{x}^\alpha$. Hence we have $\partial F / \partial v = \theta F = \theta(F_\alpha \xi^\alpha)$, where the differential operator θ is defined below. Let a symbol $V_\mu^\alpha(x, \xi)$ be introduced by

$$V_\mu^\alpha(x,\xi) \equiv \Gamma_{\mu\beta}^{\alpha} \xi^\beta + \frac{1}{2} \frac{\partial \xi^\gamma}{\partial u} f_{\mu\beta\gamma} f^{\beta\alpha}, \tag{5}$$

where the arguments of $\Gamma_{\mu\beta}^{\alpha}$, $f_{\mu\beta\gamma}$ and $f^{\beta\alpha}$ are x, ξ.† Then the θ-process with respect to the curve C when applied to a contravariant vector X^α is defined by‡

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* Any functions employed will be understood to have the arguments x, ξ unless otherwise indicated.

† The quantities in (5) are defined as follows:

$$f = \frac{1}{2} F^2, \qquad f_{\alpha\beta} = \frac{\partial^2 f}{\partial \dot{x}^\alpha \partial \dot{x}^\beta}, \qquad f_{\alpha\beta\gamma} = \frac{\partial f_{\alpha\beta}}{\partial \dot{x}^\gamma},$$

$f^{\beta\alpha}$ is the element of the reciprocal matrix of $f_{\lambda\mu}$ corresponding to the term $f_{\alpha\beta}$; $\Gamma_\mu^\alpha = f^{\alpha\sigma}[\mu\beta,\sigma]$ where

$$[\mu\beta,\sigma] = \frac{1}{2}\left(\frac{\partial f_{\mu\sigma}}{\partial x^\beta} + \frac{\partial f_{\beta\sigma}}{\partial x^\mu} - \frac{\partial f_{\mu\beta}}{\partial x^\sigma}\right).$$

‡ Further properties of the operator θ and the resulting theory of parallelism cannot be given here in detail. Reference may be made to J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas*, Transactions of this Society, vol. 27 (1925), pp. 252–259. This differentiation process was developed about the same time by J. L. Synge, *A generalization of the Riemannian line-element*, Transactions of this Society, vol. 27 (1925), pp. 61–67. The θ-process and resulting parallelism are distinct from the corresponding processes as developed by E. Noether, *Invarianten beliebiger Differentialausdrücke*, Göttinger Nachrichten, 1918,

$$\theta X^\alpha \equiv \frac{dX^\alpha}{du} + V^\alpha_\beta(x,\xi)X^\beta. \tag{6}$$

We adopt the notation of Synge in this connection and designate θX with respect to a v-curve by X and with respect to a u-curve by $\overline{X}$, where X is any tensor.

With this notation, we have

$$\frac{\partial F}{\partial v} = \hat{F} = \hat{F}_\alpha \xi^\alpha + F_\alpha \hat{\xi}^\alpha.$$

If we assume that

$$\frac{\partial^2 x^\alpha}{\partial u \partial v} = \frac{\partial^2 x^\alpha}{\partial v \partial u},$$

it follows that $\xi^\alpha - \bar{\eta}^\alpha = V^\alpha_\lambda(x,\eta)\xi^\lambda - V^\alpha_\lambda(x,\xi)\eta^\lambda$. By means of this relation one obtains

$$\frac{\partial F}{\partial v} = F_\alpha \bar{\eta}^{\alpha\prime} + Q, \tag{7}$$

where Q is an abbreviation for the expression

$$\hat{F}_\alpha \xi^\alpha + F_\alpha(V^\alpha_\lambda(x,\eta)\xi^\lambda - V^\alpha_\lambda(x,\xi)\eta^\lambda).$$

On expanding these terms and making use of the homogeneity condition, it results that Q is identically zero. Hence, we have

$$\delta t = \delta v \int_{u_1}^{u_2} F_\alpha \bar{\eta}^\alpha du, \tag{8}$$

where the arguments of F are x, ξ of the curve C defined by (3) for $v=0$, and where the directional derivative $\bar{\eta}^\alpha = \theta\eta^\alpha$ of the variation vector η is with respect to the curve C.

From (8) we see that the first variation certainly vanishes if $\bar{\eta}^\alpha = 0$ along the curve. However, if this condition holds

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pp. 37–44, and L. Berwald, *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Mathematische Zeitschrift, vol. 25 (1926), pp. 40–73, except when the θ-process is carried out with respect to a curve which satisfies the equations

$$x'^i = -\phi^i(x,x')$$

of Berwald, p. 44.

at every point of C, the original and varied curve cannot have a point in common unless they coincide. For $\bar{\eta}^\alpha = 0$ is the condition for *parallel displacement*, and under such a displacement the length of a vector when measured with respect to the tangent vector to the curve, as is the case here, remains constant.*

By the homogeneity condition, δt may be written in the form

$$\delta t = \delta v \int_{u_1}^{u_2} \frac{f_{\alpha\beta}\xi^\alpha \bar{\eta}^\beta}{F} du.$$

We now change the form of this expression by means of the relation

$$\frac{\partial}{\partial u}[f_{\alpha\beta}\xi^\alpha\eta^\beta] = \theta[f_{\alpha\beta}\xi^\alpha\eta^\beta] = f_{\alpha\beta}\bar{\xi}^\alpha\eta^\beta + f_{\alpha\beta}\xi^\alpha\bar{\eta}^\beta,$$

and the choice of a parameter such that $F = 1$ along the curve C. Then an integration gives

$$\delta t = \delta v\left\{ f_{\alpha\beta}\xi^\alpha\eta^\beta \Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} f_{\alpha\beta}\bar{\xi}^\alpha\eta^\beta du \right\},$$

or

$$(9)\quad \delta t = \delta v\left\{ |\eta|_\xi \cos(\xi,\eta)_\xi \Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} |\bar{\xi}|_\xi |\eta|_\xi \cos(\bar{\xi},\eta)_\xi \, du \right\},$$

where $|\eta|_\xi$ denotes the *length* of the vector η, and $\cos(\xi, \eta)_\xi$ is the cosine of the angle between ξ and η measured with respect to the tangent vector ξ. Here $\bar{\xi}^\alpha$ is the *principal normal* of the curve C and $|\bar{\xi}|_\xi$ is the *first curvature* of the curve.† A curve whose first curvature vanishes will be called a *geodesic*. From (9) we have the following well known result.

THEOREM 1. *A necessary and sufficient condition that the first variation of the length integral of a curve C shall be zero for arbitrary variations except that at the end points the varia-*

* J. H. Taylor, Transactions of this Society, loc. cit., p. 259.

† J. H. Taylor, Transactions of this Society, loc. cit., pp. 261, 263.

tions shall vanish or be orthogonal (with respect to the line elements of C) to the curve, is that C shall be a geodesic. The equations of a geodesic referred to the arc length t as parameter are*†

$$\bar{\xi}^\alpha \equiv \frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\lambda\mu}\left(x, \frac{dx}{dt}\right)\frac{dx^\lambda}{dt}\frac{dx^\mu}{dt} = 0, \quad (\alpha = 1, 2, \cdots, n).$$

The quantity $\delta v|\eta|_\xi \cos(\xi, \eta)_\xi$ represents the projection or component of the displacement vector η on the principal normal $\bar{\xi}$ of the curve C. *Hence for variations which cause the integrated term of* (9) *to vanish at the limits of integration, the first variation of the arc length of a curve depends only on the displacement of the curve in the direction of its principal normal.*‡

Let R be a region of the surface (3) in which the curves $v=$constant constitute a one-parameter family of geodesics of the n-space. We suppose that through each point of R there passes one, and only one, of the geodesics of the family. Furthermore, let each of the geodesics of the family be cut transversally (orthogonally with respect to the line elements of the geodesics) by a curve $u=u_1$. Now if $u=u_2$ is another curve in R also cutting the geodesics transversally we see from (9) that the arc length of these geodesics between these two transversal curves is a constant. Conversely, let a given distance be measured off on each geodesic from its point of intersection with the transversal $u=u_1$. Then for each point determined in this fashion we have the condition

* The orthogonality here used includes transversality as a special case. See L. Berwald, loc. cit., p. 56.

† These equations in this form were explicitly obtained by Synge, Berwald and the author at about the same time. J. L. Synge, Transactions of this Society, loc. cit., L. Berwald, *Über Parallelübertragung in Räumen mit allgemeiner Maszbestimmung*, Jahresbereicht der Vereinigung, 1925, p. 217. J. H. Taylor, *Reduction of Euler's equations to a canonical form*, this Bulletin, vol. 31 (1925), p. 262. One observes that this method of treating the first variation leads directly to the Euler equations in *normal* form.

‡ See W. Blaschke, *Vorlesungen über Differentialgeometrie*, vol. I, 1921, p. 35, where this result is given for a curve in euclidean 3-space.

$$|\eta|_\xi \cos(\xi,\eta)_\xi|^{u_2} = 0 .$$

By hypothesis $|\eta|_\xi \neq 0$, since the region R is simply covered by the geodesics of the family, and hence $\cos(\xi, \eta)_\xi|^{u_2}_{=0}$, that is, η is orthogonal to the geodesic, the angle being measured with respect to the tangent vector to the geodesic. Thus we have a theorem which is a simple extension of Kneser's transversality theorem (Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 339).

THEOREM 2. *Let a one-parameter family of geodesics of an n-space* $(n>2)$ *simply cover a region R of a two-dimensional space, and let* C_1 *be a curve which cuts each of these geodesics transversally (orthogonally with respect to the tangent vector of the geodesic). Then a necessary and sufficient condition that the arc length in the sense of* (2) *measured along these geodesics from the curve* C_1 *to a second curve* C_2 *shall be a constant is that* C_2 *shall also cut the geodesics of the family transversally.*

Let K be a sub-space of k dimensions $(k<n)$. A vector ξ at a point of K will be said to be orthogonal to K if it is orthogonal to every vector in K at that point. A necessary and sufficient condition for this is that ξ shall be orthogonal to k linearly independent vectors in K at that point, say the tangent vectors to the parametric curves. Hence if we consider the first variation of an integral with respect to a class of curves joining a fixed point with a point in a given K-space we have the following result.

THEOREM 3. *A necessary and sufficient condition that the first variation of an integral taken along a curve C in an n-space, one end point of which is fixed while the other end point is required to lie in a given k-dimensional space K,* $(k<n)$, *shall vanish is that the curve C shall be a geodesic and that it meet the space K orthogonally with respect to the tangent vector of C.*

THE UNIVERSITY OF WISCONSIN

ON THE MAXIMUM DEFICIENCY OF r-SPACE CURVES

BY B. C. WONG

The problem of determining the greatest deficiency of an r-space curve of given order seems yet unsolved. This problem is equivalent to that of determining the maximum dimension of a sub-space of S_r in which a curve of given order and genus can lie. Thus the maximum genus of a 4-space sextic curve is 2; but if the sextic curve is of genus 3 or 4, it is necessarily a 3-space curve or a plane curve; and if it is of genus greater than 4, it is a plane curve.

It is well known that the deficiency of an r-space curve C^n of order r is zero. Veronese* has shown that the greatest deficiency of an r-space curve C^{r+s} where $s<r$ is s and that of C^{2r} is $r+1$. It is the purpose of this paper to present a method of calculating the maximum deficiency of curves whose orders are greater than $2r$.

To determine the greatest deficiency P of a curve C^n, it is only necessary to determine the least number, h, of apparent double points of its projection on an S_3, for

$$p = (n-1)(n-2)/2 - h. \tag{1}$$

We shall use the phrase *the apparent double points of* C^n instead of *the apparent double points of the 3-space projection of* C^n. To determine h, we make use of the formula†

$$D = \mu_1\mu_2\cdots\mu_{r-1}(\mu_1\mu_2\cdots\mu_{r-1} - \sum\mu_i + r - 2)/2), \tag{2}$$

for the number D of apparent double points of C^n which is the complete intersection of $r-1$ hypersurfaces of order

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* *Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens*, Mathematische Annalen, vol. 19 (1882), pp. 161–234.

† Loc. cit., p. 205.

$\mu_1, \mu_2, \cdots, \mu_{r-1}$. D will be a minimum if $r-2$ of the hypersurfaces are hyperquadrics. Hence, putting

$$\mu_1 = \mu_2 = \cdots = \mu_{r-2} = 2, \quad \mu_{r-1} = n/2^{r-2},$$

in (2), we have

$$h \geqq \frac{n}{2^{r-1}}[(2^{r-2} - 1)n - 2^{r-2}(r - 2)], \tag{3}$$

and

$$p \leqq \frac{1}{2^{r-1}}[n^2 + 2^{r-2}(r - 5)n + 2^{r-1}]. \tag{4}$$

The equality signs hold when $n = 2^{r-2}\,\mu_{r-1}$, where $\mu_{r-1} > 1$. When $n \neq 2^{r-2}\mu_{r-1}$, (3) gives too low a limit for h and (4) gives too high a limit for P. To find the exact value of h we distinguish two cases: n even and n odd.

First take the case n even and equal to $2m$. Consider an equivalent degenerate curve C^{2m} made up of two $(r-1)$-space curves C^m and C'^m lying in different $(r-1)$-spaces but having m points in common. If h' is the least number of apparent double points on each of C^m and C'^m, then

$$h = m^2 - m + 2h',$$

since $m^2 - m$ is the number of apparent intersections of C^m and C'^m. Replacing m by $n/2$ in the above, we have

$$h = n(n - 2)/4 + 2h' \tag{5}$$

and

$$p = (n - 2)^2/4 - 2h'. \tag{6}$$

Now take the case n odd and equal to $2m-1$. Consider an equivalent degenerate curve C^{2m-1} composed of two $(r-1)$-space curves C^m and C^{m-1} lying in different $(r-1)$-spaces but having $m-1$ points in common. Letting h'' be the least number of apparent double points on C^m and h''' be that of apparent double points on C^{m-1}, we have

$$h = (m - 1)^2 + h'' + h''',$$

since $(m-1)^2$ is the number of apparent intersections of C^m and C^{m-1}. If we replace m by $(n+1)/2$, the result is

$$h = (n-1)^2/4 + h'' + h''' \tag{7}$$

and hence

$$P = (n-1)(n-3)/4 - h'' - h'''. \tag{8}$$

To find h' or h'' and h''' we repeat the process. We soon arrive at the component curves whose orders are equal to or less than twice the dimensions of the several sub-spaces in which the component curves lie, and then apply Veronese's rule. Or, in case n is very large, we arrive at the component curves which are all plane and have no double points.

It is to be noticed that $h'=h''=h'''=0$ if $r=3$. Hence the greatest deficiency of any 3-space curve C^n is

$$p = (n-2)^2/4, \qquad \text{for } n \text{ even};$$

and

$$p = (n-1)(n-3)/4, \qquad \text{for } n \text{ odd}.$$

THE UNIVERSITY OF CALIFORNIA

A NOTE ON CERTAIN CONTINUOUS NON-DIFFERENTIABLE FUNCTIONS*

BY F. W. PERKINS

This note gives a treatment of some phases of the theory of a class of functions of which a particular example has already been studied by the author in a note entitled *An elementary example of a continuous non-differentiable function*, in the American Mathematical Monthly (vol. 34 (1927), pp. 476–478). The method there used for the construction of a function with the desired properties bears some resemblance to that used by Brodén, Köpke, and Steinitz for the

* Presented to the Society, September 6, 1928.

construction and study of functions with notable properties with respect to differentiability.* The example consists of a function belonging to a class investigated by Steinitz, but the discussion is somewhat more elementary in form than that given previously in the more general treatment. The method consists, briefly, in assigning functional values at an everywhere dense set of points in such a way that the function can be extended so as to be continuous on the whole interval and to have no derivative. The principle involved enables us not only to construct non-differentiable functions but also to give in simple form a sufficient condition that a continuous function be convertible into a non-differentiable continuous function by a certain change of variable.

First let us examine a few of the more immediate consequences of non-differentiability. We note that such a function cannot be of bounded variation on any subinterval, for by a well known theorem, a function of bounded variation has a derivative almost everywhere, that is, except possibly at the points of a set of measure zero in the sense of Lebesgue.† It follows that a function which has no derivative must have maxima and minima‡ on every subinterval.

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* E. Steinitz, *Stetigkeit und Differentialquotient*, Mathematische Annalen, vol. 52 (1899), pp. 58–69. For a description of the methods of Brodén and Köpke, see Hobson, *The Theory of Functions of a Real Variable*, 3d ed., 1927, vol. 1, pp. 387 ff. See also a paper by E. H. Moore, Transactions of this Society, vol. 1 (1900), pp. 81 ff; and one by Hahn, Jahresbericht der Vereinigung, vol. 26 (1918), pp. 281 ff.

† See, for example, Carathéodory, *Vorlesungen über Reelle Funktionen*, 2d ed., 1927, p. 188, Satz 9, and p. 563 ff.

‡ By a maximum value of a function we shall mean a value attained at a point x', not exceeded at any point in the neighborhood of x', and not maintained constantly in any interval having x' as an interior or end point. The corresponding definition for a minimum is of course also used. It may be noted that a function may have an extremum in this sense which is not an extremum in the "strict" sense in which it is required that a maximum value, for example, be actually greater than all other functional values in the neighborhood of the point at which it is attained. Consider, for instance, the behavior at $x=0$ of the function defined below:

$$\begin{cases} f(x)=0 \text{ when } x=0, \\ f(x)=-x^2 \sin^2(1/x) \text{ when } 0<|x|<1. \end{cases}$$

This condition may be expressed analytically in the following manner: Let $f(x)$ be any function continuous on the interval I: $a \leqq x \leqq b$, but not constant on any subinterval of I. A necessary and sufficient condition that $f(x)$ have maxima and minima which are everywhere dense on I is that every subinterval $x_0 \leqq x \leqq x_3$ at the ends of which $f(x)$ assumes distinct values contains two interior points x_1 and x_2 such that $x_1 < x_2$ and

$$\frac{f(x_1) - f(x_2)}{f(x_3) - f(x_0)} > 0.$$

For if $f(x)$ has extrema which are everywhere dense we may obviously choose x_1 and x_2 so that the numerator has the same sign as the denominator. On the other hand, if the fraction is positive, the function $f(x)$ has an extremum between x_0 and x_3. In establishing this fact, we may assume, without loss of generality, that the denominator is positive. In the case in which $f(x_0) \leqq f(x_2)$, we see that $f(x)$ has a maximum between x_0 and x_2; if $f(x_0) > f(x_2)$, we infer that $f(x)$ has a minimum between x_1 and x_3. Moreover, on any subinterval at the ends of which $f(x)$ assumes equal values, $f(x)$ must have an extremum at an interior point. It follows, then, that $f(x)$ has maxima and minima which are everywhere dense on I.

We note that this property in regard to extrema is one which is preserved if we replace the variable x by a variable ξ connected with it by the relation $x = \phi(\xi)$, where the function $\phi(\xi)$ is continuous on the interval $a \leqq \xi \leqq b$ and satisfies the conditions $\phi(\xi') < \phi(\xi'')$ if $\xi' < \xi''$, $\phi(a) = a$, $\phi(b) = b$. The property of non-differentiability is obviously not always preserved by such a change of variable. Hence we see that although the condition that the extrema of $f(x)$ be everywhere dense on I is necessary for non-differentiability, it is not sufficient. Moreover there exist functions with extrema which are everywhere dense and which cannot be rendered non-differentiable by such a change of variable.

We propose to show now that a slight modification of the condition given above for everywhere dense extrema leads to a condition which is sufficient to insure non-differentiability after a suitable change of variable.

THEOREM. *If $f(x)$ is continuous on the interval I: $a \leqq x \leqq b$, but is not constant on any subinterval, and if there exists a positive quantity δ such that every subinterval $x_0 \leqq x \leqq x_3$ at the ends of which $f(x)$ assumes distinct values contains two points x_1 and x_2 such that $x_1 < x_2$ and*

$$\frac{f(x_1) - f(x_2)}{f(x_3) - f(x_0)} > \delta,$$

then there exists a function $\phi(\xi)$, continuous and monotonically increasing on the interval $a \leqq \xi \leqq b$, such that $F(\xi) = f[\phi(\xi)]$ fails to have a derivative with respect to ξ at every point of the interval $a \leqq \xi \leqq b$.

We shall first treat the case in which $f(x)$ attains its least and greatest values on I at the points $x = a$ and $x = b$ respectively: $f(a) \leqq f(x) \leqq f(b)$ or $f(b) \leqq f(x) \leqq f(a)$ when $a \leqq x \leqq b$.

We consider first the value of the fraction

$$\frac{f(x_1) - f(x_2)}{f(b) - f(a)}$$

for values of x_1 and x_2 such that $a \leqq x_1 \leqq x_2 \leqq b$. Since this is a particular form of the fraction given in the hypothesis of the theorem, we know that its greatest value must be greater than δ. We note that to attain this maximum value we may have neither $x_1 = a$ nor $x_2 = b$. The fraction may conceivably attain its maximum value at more than one pair of points x_1 and x_2. However, the set of all points which may be used as x_1 is closed, and therefore contains a point farthest to the left. We shall call this point X_1. All the points which are possible choices for an x_2 corresponding to $x_1 = X_1$ form a closed set, and therefore contain a point farthest to the right, which we shall call X_2. We now repeat the same process for each of the intervals $a \leqq x \leqq X_1$,

$X_1 \leqq x \leqq X_2$ and $X_2 \leqq x \leqq b$. It will be noted that the function $f(x)$ attains its extreme values on any one of these intervals at the ends of that interval. The three new pairs of points thus obtained, taken together with the points $x = X_1$ and $x = X_2$, divide the interval I into nine parts, to each of which we apply the process again, and so on indefinitely.

The set $\{X\}$ of all points thus obtained is everywhere dense on I. For if this is not true there exists a subinterval γ containing none of these points. Let μ be the difference between the greatest value and the least value of $f(x)$ on γ. Since $f(x)$ is not constant on any interval we know that $\mu > 0$. Let I_1 be that subinterval obtained by the first subdivision (by the points X_1 and X_2) which contains γ; let I_2 be that subinterval obtained by the second subdivision which contains γ, and so on. In this way we get an infinite sequence of subintervals $I_1, I_2, I_3, \cdots$, each of which contains all the following, and also contains γ. Let $X_1, X_1', X_1'', \cdots$, be the left hand end points of these intervals. Each of these points lies to the left of the next following point of the sequence, and also to the left of γ. Consequently they have a unique limit point, X, on I. Moreover, at any two successive points of the sequence $X_1, X_1', X_1'', \cdots$, the function $f(x)$ has values which, it is readily seen, differ by at least μ. This implies that $f(x)$ is discontinuous at $x = X$, contrary to our hypothesis. This contradiction establishes the desired result. It may be noted, incidentally, that the points of the everywhere dense set which we have just described yield extrema of the function $f(x)$.

We now proceed to the definition of the function $\phi(\xi)$. Since this function is to be continuous it is possible to define it completely by assigning suitable functional values at the points of an everywhere dense set. First we set $\phi(a) = a$, $\phi(b) = b$, and $\phi(\xi_1) = X_1$ and $\phi(\xi_2) = X_2$, where*

* Since $F(\xi) = f[\phi(\xi)]$, while $\phi(a) = a$ and $\phi(b) = b$, it follows that $F(a) = f(a)$, and $F(b) = f(b)$. However in analogous work a little later it will prove necessary to distinguish carefully between the functions F and f.

$$\begin{cases} \xi_1 = \dfrac{aF(b) - bF(a) + f(X_2)[b - a]}{F(b) - F(a)}, \\ \xi_2 = \dfrac{aF(b) - bF(a) + f(X_1)[b - a]}{F(b) - F(a)}. \end{cases}$$

These formulas may be given a simple geometric interpretation: ξ_1 and ξ_2 are the abscissae of the points of intersection of the lines joining $[a, F(a)]$ and $[b, F(b)]$ and the lines $\eta = f(X_2)$ and $\eta = f(X_1)$, respectively, in the ξ, η plane.

Each of the subintervals into which X_1 and X_2 divide I contains two points standing in the same relation to that subinterval as X_1 and X_2 do to I. By the method given above we determine the corresponding values of ξ. We repeat this process indefinitely, obtaining an infinite set of points at each of which the function $\phi(\xi)$ is defined. The order relations between these points is the same as the order relations between the corresponding points X, and so $\phi(\xi)$, in so far as it is now defined, is a monotonically increasing function.

The straight line joining $[a, F(a)]$ and $[b, F(b)]$ is the graph of a function $F_0(\xi)$ which is a first approximation to the function $F(\xi)$. The broken line joining the points $[a, F(a)]$, $[\xi_1, F(\xi_1)]$, $[\xi_2, F(\xi_2)]$ and $[b, F(b)]$ yields a second approximation, $F_1(\xi)$; a third approximation may be obtained by replacing each segment of this line by a broken line of three pieces related to the segment in the same way as the graph of the second approximation is to that of the first, and so on.*

Denoting by λ_i the slope of a line segment of the graph of an approximating function $F_i(\xi)$, we see, by a simple computation, that the slope of the second of the line segments by which it is replaced in the graph of $F_{i+1}(\xi)$ is $-\lambda_i$, while the slope of each of the other two segments is at least as great numerically as $|\lambda_i|(1+\delta)$. This enables us to prove that the

* The sequence $F_0(\xi)$, $F_1(\xi)$, $F_2(\xi)$, $\cdots$ converges uniformly, but this is of no importance for our proof, as we shall obtain $F(\xi)$ by another method. We shall have occasion, however, to use these functions in another connection.

point set on which we have already defined the function $\phi(\xi)$ is everywhere dense. For if such is not the case there exists an interval throughout which each approximating function is linear with a slope at least as great numerically as the positive quantity $|\lambda_0|$. Consequently there exists a number $c>0$ and pairs of points of the set $\{X\}$, as near together as we choose to stipulate in advance, at which the function $f(x)$ assumes values differing by at least c. But this contradicts the hypothesis of continuity, and so shows that $\phi(\xi)$ is defined on an everywhere dense set of points.

At any point $\xi=\xi'$ at which $\phi(\xi)$ is not already defined we set $\phi(\xi)$ equal to the least upper bound of all its known values on the interval $a\leqq\xi<\xi'$. The function $\phi(\xi)$ is now defined and monotonically increasing on the interval $a\leqq\xi\leqq b$. It is continuous there; for a discontinuity could only be a finite jump, which would imply the existence of a subinterval of I free from points of the set $\{X\}$, contrary to the known fact that this set is everywhere dense.

We shall now show that the continuous function $F(\xi)=f[\phi(\xi)]$ has no derivative at any point of the interval $a\leqq\xi\leqq b$. We understand this statement to mean that at no point $\xi=\xi'$ of this interval does the difference quotient $[F(\xi'+h)-F(\xi')]/h$ approach a limit as h approaches zero.*

If ξ' corresponds to one of the points of the set $\{X\}$, then it is clear that $F(\xi)$ has no derivative at $\xi=\xi'$, because, for sufficiently large values of n, the forward derivative of $F_{n+1}(\xi)$ at that point is at least $1+\delta$ times as great numerically as the forward derivative of $F_n(\xi)$ at $\xi=\xi'$, and so the forward derivative of $F_{n+k}(\xi)$ at $\xi=\xi'$ can be made as large numerically as we like by choosing k sufficiently large. Since the forward derivatives of $F_n(\xi)$, $F_{n+1}(\xi)$, $F_{n+2}(\xi)$, $\cdots$ at $\xi=\xi'$ give us the values of the difference quotient $[F(\xi'+h)-F(\xi')]/h$ for a sequence of values of h approaching zero as a limit, we see that the difference

* Some writers use a definition of differentiability which allows the difference quotient either to approach a limit or to become positively or negatively infinite.

quotient cannot approach a limit in this case. Backward derivatives of the approximating functions may be used equally well. The forward and backward derivatives have opposite signs, and so we see that for a certain sequence of positive values of h tending toward zero the difference quotient becomes infinite with one sign, while for a certain sequence of negative values of h tending toward zero the difference quotient becomes infinite with the other sign. It should be noted, however, that this does not necessarily mean in either case that the difference quotient becomes infinite independently of the manner in which h approaches zero from one side.*

It may also be noted that $F(\xi)$ has no forward derivative at the point $x=a$ and no backward derivative at $x=b$, though the difference quotient may conceivably become positively or negatively infinite at these points.

It remains to show that $F(\xi)$ has no derivative at any point ξ' of the open interval $a<\xi<b$ which does not correspond to a point of the set $\{X\}$. At such a point each of the functions $F_n(\xi)$ has a derivative. Let $\xi=\xi_n'' <\xi'$ and $\xi=\xi_n''' >\xi'$ be points corresponding to points of the set $\{X\}$ between which $F_n(\xi)$ has a constant derivative. If we set $h=h_n''=\xi_n''-\xi'$ or $h=h_n'''=\xi_n'''-\xi'$, the value obtained for the difference quotient $[F(\xi'+h)-F(\xi')]/h$ is in one case not greater algebraically than $D_\xi F_n(\xi')$ and in the other case not less algebraically than $D_\xi F_n(\xi')$. Since $\lim h_n''=\lim h_n'''=0$, it

* In special cases, however, this may happen. In the case of the function previously studied by the author, cited above, it can readily be shown that at each of the everywhere dense set of maxima (given by $x=p/3^n$, where n is any positive integer and p any odd positive integer less than 3^n) the difference quotient becomes positively infinite on the left and negatively infinite on the right, while at the minima (given by the remaining values of x of the form $p/3^n$ in the open interval $0<x<1$) the difference quotient becomes negatively infinite on the left and positively infinite on the right. For $x=0$ and $x=1$ the difference quotient becomes positively infinite. The coordinates of the extrema are here given for the function as defined in the earlier paper. It is readily verified that the properties stated regarding the behavior of the difference quotient are preserved (in this case) by the transformation described in the present paper.

follows that this difference quotient can approach a limit only if $D_\xi F_n(\xi')$ approaches a limit as n becomes infinite. This, however, is never the case. In the ratio $D_\xi F_{n+1}(\xi')/D_\xi F_n(\xi')$ neither the numerator nor the denominator is zero and neither approaches zero as a limit. Moreover for any n this ratio has either the value -1 or else is greater than $1+\delta$, and so cannot approach the limit unity, as it would if $D_\xi F_n(\xi')$ approached a limit. Hence the difference quotient $[F(\xi'+h)-F(\xi')]/h$ cannot approach a limit.

This completes the proof of the theorem for the case in which either $f(a) \leqq f(x) \leqq f(b)$ or $f(b) \leqq f(x) \leqq f(a)$. The proof can be extended to the general case immediately by dividing the interval I into a finite or denumerably infinite set of subintervals, each of which satisfies the above condition, and then applying the above reasoning to each such subinterval. This subdivision of I can be accomplished by choosing as the first subinterval the interval between points at which $f(x)$ attains its extreme values. In case these values are attained at more than one pair of points we may choose one particular pair by means of the device used in the similar situation for the choice of X_1 and X_2. We then apply this method to the remaining interval or intervals and continue the process indefinitely, or until we have exhausted I.

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A NEW PROOF OF CERTAIN RELATIONS OF MORSE IN THE CALCULUS OF VARIATIONS IN THE LARGE*

BY D. E. RICHMOND

1. *Introduction.* In a recent paper,† M. Morse has developed a theory of calculus of variations "in the large," in which he has made use of the results of his paper‡ on critical points of functions of n variables. He establishes a parallelism between types of critical points and types of extremal arcs in which the type of an extremal arc whose end points are not conjugate is completely determined by the number of mutual conjugate points within the arc. As a first application of the critical point theory, he obtains a set of necessary relations between the numbers of extremal arcs of different type, which join two given points. He also discusses periodic extremals as to type, and obtains number relations from the critical point theory.

Morse's relations are obtained for the parametric problem under certain general boundary conditions and under the hypothesis of the existence of a field of extremals covering the region of operation.

The present paper treats only the number relations between extremal arcs which join two points. Using Morse's hypotheses, we derive these relations rather simply from well known continuity properties of the solutions of differential equations, without making use of the critical point theory or n-dimensional analysis situs. It should be stated, however, that Morse has since proved the relations without

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† M. Morse, *The foundations of a theory in the calculus of variations in the large*, Transactions of this Society, vol. 30 (1928), pp. 213–274.

‡ M. Morse, *Relations between the critical points of a real function of n independent variables*, Transactions of this Society, vol. 27 (1925), pp. 345–396.

the field hypothesis* and has also announced results for the n-dimensional calculus of variations problem. It seems improbable that our method can be generalized to treat these problems.

The method here presented is of interest, however, because in this case, it makes possible the statement of the relations in a somewhat more restrictive form than that given by Morse, in fact, in a form which immediately gives a new existence theorem. Moreover, our method gives an insight into the relative positions of the extremal arcs of different type, and is thus supplementary to the critical point method.

2. *Hypotheses and Relations of Morse. Let the given integral be*

$$\int_{t_0}^{t_1} F(x, y, \dot{x}, \dot{y})dt. \tag{1}$$

I. *We assume that the function $F(x, y, \dot{x}, \dot{y})$ is positively homogeneous of the first degree in $\dot{x}$ and $\dot{y}$, and analytic in all its arguments for (x, y) any point interior to an open region S of the (x, y) plane and $\dot{x}$ and $\dot{y}$ any two numbers not both zero. We also assume that for the same arguments, $F_1(x, y, \dot{x}, \dot{y}) > 0$.*

II. *Let there be given a closed region S_1, interior to S, bounded by a simple closed curve β consisting of a finite number of analytic arcs. We assume that β is extremal-convex in the following sense*: (1) *The interior angles at the vertices of β all lie between* 0 *and* π; (2) *An extremal tangent to any arc of β lies outside of S_1 in the neighborhood of the point of contact.*

III. *We assume that S_1 is covered in a one-to-one manner by a proper field of extremals of the form*

$$x = h(u, v), \qquad y = k(u, v), \tag{2}$$

where u is the parameter, and v is the arc length measured along the extremals, and where at every point (u, v) that corresponds to a point (x, y) in S_1, the functions (2) *are single-valued and analytic in u and v, and*

* See Morse, first paper, loc. cit., footnote, p. 223.

$$D = \begin{vmatrix} h_u & h_v \\ k_u & k_v \end{vmatrix} \neq 0.$$

IV. *We assume that the problem is reversible.**

The region S_1 in the (x, y) plane corresponds to a closed, extremal-convex region R in the (u, v) plane, whose boundary γ is a simple closed curve consisting of a finite number of analytic arcs. If $\dot{u} > 0$ and we let $t = u$, the integral (1) can be put into non-parametric form with an integrand, $f(u, v, v')$, where $v' = dv/du$. The non-parametric problem is regular ($f_{v'v'} > 0$). The field extremals become the lines $u =$ constant, which of course are not included among the solutions of the non-parametric problem. All other extremals of the parametric form are included in the non-parametric problem because of the assumption of reversibility.

Morse proves a lemma (p. 224) which we take over. It amounts to the statement that in the simply-connected region R, no line $u =$ constant cuts the boundary γ in more than two points.

In R, to the extremals $u =$ constant, no other extremals can be tangent. Hence, any extremal arc, other than $u =$ constant, is representable in the form $v = M(u)$, where $M(u)$ is analytic for that part of the arc which lies within R.

Let A and B be any two points, within or on the boundary of R, and not on the same line $u =$ constant. We state two preliminary results: (a) *There exists at least one extremal arc which joins A to B in R and which has no points on the boundary of R, with the possible exception of A and B*; (b) *The points A and B can be joined by at most a finite number of extremal arcs in R.* We shall prove (a) in §3, while (b) is proved in Morse's paper (p. 226) without the use of the critical point theory.

We consider the set G of arcs which join A to B in R. For the present, we assume that there exists no arc in G upon which B is conjugate to A. Each arc in G has a type number

* See Morse, loc. cit., p. 223.

k which will be defined to be equal to the number of points on the arc conjugate to A. Let m be the maximum of the type numbers k and $M_k (k = 1, 2, \cdots, m)$ be the number of extremal arcs in G of type k. The relations of Morse are then the following:

$$(3)\quad \begin{cases} M_0 \geqq 1, \\ M_0 - M_1 \leqq 1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ (-1)^{m-1}[M_0 - M_1 + \cdots + (-1)^{m-1} M_{m-1}] \geqq 1, \\ M_0 - M_1 + \cdots + (-1)^m M_m = 1. \end{cases}$$

3. *A Preliminary Theorem.* Morse gives the following consequence of the assumptions of §2. *There exists a positive constant e so small that any two points in R whose abscissae differ in absolute value by less than e, can be joined by a unique extremal arc, all of whose points (except possibly the end points) lie within R.*

We take over this result and show that it follows that, if A and B are any two points in R, not on the same line $u = \text{constant}$, there exists at least one extremal arc E which joins A to B in R, and that no points of E, except possibly A and B, lie on the boundary γ of R. Let A and B be represented by (u_0, v_0) and (u_1, v_1), respectively, $u_0 < u_1$. By virtue of the result just stated, a finite number p of lines of the form $u = \text{constant}$ can be chosen between $u = u_0$ and $u = u_1$ such that, if p arbitrary points are chosen in R, one on each of these lines, there is determined a unique chain of extremal arcs, which joins A to B. The value of the fundamental integral taken along such chains becomes a single-valued continuous function of the p ordinates. The domain of definition is closed in each of the variables. Hence there exists a set of ordinates for which the integral takes on its minimum value. The extremal arcs of the corresponding chain join on to each other so as to form a single extremal arc E from A to B, since in a regular problem a curve with corners cannot minimize the integral. Now E has no points

on the boundary γ with the possible exception of A and B. This statement follows from the extremal-convex boundary and from the impossibility of corners.

We now prove the following theorem.

THEOREM 1. *If there exists in R, joining A to B, an extremal arc E_0 of type $n>0$, there exist in R, joining A to B, at least two other extremal arcs, one of which lies above E_0, the other below E_0.*

Proof. Let E_0 be extended in both senses until it intersects γ, the boundary of R, thus dividing R into two regions, R_1 and R_2. Now R_1 is extremal-convex in a somewhat broader sense than that of §2 II, in that part of the boundary is itself an extremal arc. But as before, A and B can be joined in R_1 by an extremal arc E which minimizes the integral in comparison with all other arcs which join A to B in R_1. The points of E lie within R_1 or on its boundary. But E cannot coincide with E_0 because the latter contains at least one point conjugate to A by hypothesis and hence cannot minimize the integral. In fact, it is readily proved that E has no points except A and B in common with the boundary of R_1. A similar argument applies to R_2.

4. *The Extremals through A.* Let A and B be represented by (u_0, v_0) and (u_1, v_1), respectively. Let E_0 be an extremal arc in G of type $n \geqq 0$. We consider the family of extremals through A. Any such extremal E is defined by the angle α, measured from E_0 to E at A, since in a regular problem there exists* one and only one extremal through a given point with a given slope. Hence the family may be represented by

$$v = v(u, \alpha). \tag{4}$$

Suppose that a particular extremal E of (4) makes an angle of $\alpha_0 \neq 0$ with E_0 at A, and intersects E_0 exactly r times between A and B. The abscissae of these points of

* Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 184–185.

intersection will be denoted by $U_i (i = 1, 2, \cdots, r)$, where for each i, $u_0 < U_i < u_1$. The number of intersections r will depend upon α_0. An arc AP_i of E has no points on the boundary of R, except possibly A. This follows from the regularity of the problem (extremals have no corners) and from the extremal-convex nature of the boundary. For α sufficiently near α_0 and $u_0 < u \leqq U_i$, the extremals (4) lie entirely within R, since $v(u, \alpha)$ is a continuous function* of α. Let E_0 be represented by $v = f(u)$, $u_0 \leqq u \leqq u_1$. Then at P_i we have

$$v(U_i, \alpha_0) = f(U_i), \quad (i = 1, 2, \cdots, r).$$

Since* $v(u, \alpha)$ and $f(u)$ are of class C' and $v_u(U_i, \alpha_0) \neq f'(U_i)$ for $\alpha_0 \neq 0$, the implicit function theorem is applicable and for α near α_0 the abscissa U_i of the ith point of intersection of the extremals (4) with E_0 is a continuous differentiable function of α. In fact, for α near α_0, we have identically

$$v[U_i(\alpha), \alpha] = f[U_i(\alpha)], \quad (i = 1, 2, \cdots, r).$$

Differentiating with respect to α, we have

$$v_u[U_i(\alpha), \alpha] \cdot U_i'(\alpha) + v_\alpha[U_i(\alpha), \alpha] = f'[U_i(\alpha)] \cdot U_i'(\alpha).$$

Rearranging and omitting arguments, we obtain

$$v_\alpha = U_i'(f' - v_u). \tag{5}$$

Here f' and v_u represent the slopes of E_0 and E respectively, at their ith point of intersection.

By definition, any function $U_i(\alpha)$ will be restricted by the inequality $u_0 < U_i(\alpha) \leqq u_1$.

For α sufficiently small, exactly n functions $U_i(\alpha)$ exist (that is, $r = n$) and the abscissae of the conjugate points on E_0 are given† by $U_i(0)$, $(i = 1, 2, \cdots, n)$. For numerically larger values of α, some of these functions may cease to be defined, or on the other hand, functions $U_i(\alpha)$ may exist for $i > n$.

If $U_k(\alpha_0) < u_1$, we have seen that the function $U_k(\alpha)$ is defined for α sufficiently near α_0. The ends of the intervals

* Bolza, loc. cit., p. 73.

† Bolza, loc. cit., p. 79.

of definition of $U_k(\alpha)$ must therefore correspond to values of α for which $U_k(\alpha) = u_1$.

Since the problem is regular, no extremal is tangent to any other in R. The points of intersection of two extremals are therefore discrete. Hence if $U_k(\alpha)$ is defined on a certain α interval, the functions $U_i(\alpha)$, $i<k$, are also defined on this interval and

$$U_i(\alpha) < U_j(\alpha), \quad i < j \leqq k.$$

The necessary and sufficient condition that an extremal E, through A, pass through B is that

$$U_k(\alpha) = u_1, \tag{6}$$

for some integer k. For values of α for which (6) holds, $U_k(\alpha)$ will have in general only a one-sided derivative.

In discussing equation (5), with i replaced by k, it will suffice for our purposes to consider the case $\alpha > 0$. Then, if k is odd, $v_u < f'$ and v_α takes the sign of U_k'. If k is even, $v_u > f'$ so that v_α and U_k' have opposite signs.

Now $v_\alpha(0, \alpha) = 0$ and $v_{\alpha u}(0, \alpha) > 0$. Hence the conjugate points to A on E are given* by those values of u for which $v_\alpha(u, \alpha) = 0$. Then if $v_\alpha > 0$ for $u = u_1$, there exist an even number of points conjugate to A between A and B; if $v_\alpha < 0$, an odd number. From (5) therefore, we obtain the following theorem.

THEOREM 2. *Let the functions $U_i(\alpha)$ be defined relative to an extremal arc E_0, which joins A to B. Suppose E is an extremal arc which joins A to B and which makes an angle $\alpha_0 > 0$ with E_0 at A. Let n be the type of E and k the integer such that $U_k(\alpha_0) = u_1$, the abscissa of B. Then n is odd if $U_k'(\alpha_0) > 0$ and k is even, or if $U_k'(\alpha_0) < 0$ and k is odd; n is even if $U_k'(\alpha_0) > 0$ and k is odd, or if $U_k'(\alpha_0) < 0$ and k is even. Finally, B is conjugate to A if $U_k'(\alpha_0) = 0$, for any k.*

5. *Types of Adjacent Arcs.* Now let the extremal arcs of G be arranged in order of their slopes at A. Two arcs

* Bolza, loc. cit., p. 78.

which are consecutive in this order will be said to be adjacent to each other. We now prove the following theorem.

THEOREM 3. *Adjacent extremal arcs in G differ in type by unity.*

Let E and $\bar{E}$ be the adjacent extremal arcs, E being that arc which has the lesser slope at A. Let n and $\bar{n}$ be the type of E and $\bar{E}$ respectively. Suppose first that the functions $U_i(\alpha)$ are defined relative to E.

Since E is of type n, we know that $U_n(0) < u_1$ and that $U_{n+1}(\alpha)$ is not defined for α sufficiently small. Let α_0 be the angle at A from E to $\bar{E}$. Then $U_k(\alpha_0) = u_1$ for some integer k, which it remains to determine. Now $U_n(\alpha_0)$ is defined. Otherwise we would have $U_n(\alpha) = u_1$ for some positve $\alpha < \alpha_0$, which would imply that $\bar{E}$ was not adjacent to E. There are therefore two cases:

(A) $$U_n(\alpha_0) = u_1,$$

(B) $$U_n(\alpha_0) < u_1,$$

CASE A. $U_n(\alpha_0) = u_1$. We have assumed that B is not conjugate to A on $\bar{E}$. Then by Theorem 2, $U_n'(\alpha_0) \neq 0$. Since $U_n(\alpha) < u_1$ for $0 \leqq \alpha < \alpha_0$, it now follows that $U_n'(\alpha_0) > 0$.

CASE B. $U_n(\alpha_0) < u_1$. Here $U_{n+1}(\alpha_0)$ must exist. If $U_{n+1}(\alpha_0) < u_1$, there exists a positive $\alpha < \alpha_0$ for which $U_{n+1}(\alpha) = u_1$, which implies that $\bar{E}$ and E are not adjacent. Hence $U_{n+1}(\alpha_0) = u_1$. Since $U_{n+1}'(\alpha_0) \neq 0$ and $U_{n+1}(\alpha)$ is not defined for $0 \leqq \alpha < \alpha_0$, it follows that $U_{n+1}'(\alpha_0) < 0$. Thus we must have one of the following possibilities:

(A) $$U_n(\alpha_0) = u_1, \qquad U_n'(\alpha_0) > 0;$$

(B) $$U_{n+1}(\alpha_0) = u_1, \qquad U'_{n+1}(\alpha_0) < 0.$$

Using Theorem 2, with $n = \bar{n}$ and $k = n$ and $n+1$ respectively, we now see that, for either (A) or (B), if n is odd, $\bar{n}$ is even; while if n is even, $\bar{n}$ is odd. Hence in particular, $n \neq \bar{n}$.

We now reverse the roles of E and $\bar{E}$. Let the functions

$\overline{U}_i(\alpha)$ be defined relative to $\overline{E}$ in the same manner that the functions $U_i(\alpha)$ were defined relative to E.

For Case A, we have

$$\overline{U}_n(-\alpha_0) = U_n(\alpha_0) = u_1,$$

which implies that

$$\overline{U}_i(-\alpha_0) < u_1, \qquad (i < n).$$

Since the functions $\overline{U}_i(\alpha)$ are continuous and bounded, it follows that

$$\overline{U}_i(0) < u_1, \qquad (i < n).$$

Perhaps also $\overline{U}_n(0) < u_1$, but certainly $\overline{U}_{n+1}(0)$ is not defined since E is adjacent to $\overline{E}$. Since the numbers $\overline{U}_i(0)$ are the abscissae of the conjugate points to A on $\overline{E}$, the type $\bar{n}$ of $\overline{E}$ is either n or $n-1$. But we proved that $\bar{n} \neq n$, so that for Case A there remains the single possibility $\bar{n} = n-1$.

A similar argument shows that for Case B, $\bar{n} = n+1$. Combining the results of both cases, $\bar{n} = n-1$ or $n+1$, as the theorem states.

6. *Proof of Number Relations.* Let the extremal arcs which join A to B in R, be arranged in order of their slopes at A. The first and last must be of type zero. Otherwise, according to Theorem 1, there would exist extremal arcs of G whose initial slopes were respectively less than and greater than the initial slopes of these arcs.

Since adjacent arcs differ in type by unity, we see that in order that the first and last shall be of type zero, the total number of extremal arcs in G must be odd. Let us remove from G either the first or last arc and group the remaining ones into adjacent pairs. If m is the maximum type number which occurs in G, the integers M_k must be of the following form:

$$(7) \qquad \begin{cases} M_0 = 1 + p_1, \\ M_1 = p_1 + p_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad p_i \geqq 1, (i = 1, 2, \cdots, m), \\ M_{m-1} = p_{m-1} + p_m, \\ M_m = p_m, \end{cases}$$

where the integers p_i represent the total number of pairs, which comprise extremal arcs of types i and $i-1$. It is readily seen that there exists at least one pair for each value of i.

By addition and subtraction of the equations of (7), we obtain the relations (3) of Morse. But a stronger form of these relations follows if we substitute $p_i = q_i + 1$ in (7). Then

$$(7') \qquad \begin{cases} M_0 = 2 + q_1, \\ M_1 = 2 + q_1 + q_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \qquad\qquad q_i \geqq 0, (i = 1, 2, \cdots, m), \\ M_{m-1} = 2 + q_{m-1} + q_m, \\ M_m = 1 + q_m. \end{cases}$$

Now by addition and subtraction, we obtain the following theorem.

THEOREM 4. *If M_k represents the number of extremal arcs of type k which join A to B, and if m is the maximum of the types k, then*

$$(8) \qquad \begin{cases} M_0 \geqq 2, \\ M_0 - M_1 \leqq 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ M_0 - M_1 + \cdots + (-1)^m M_m = 1, \end{cases}$$

where the next to the last relation is

$$M_0 - M_1 + \cdots + (-1)^{m-1} M_{m-1} \geqq 2 \quad \text{or} \quad \leqq 0,$$

according as m is even or odd.

From (7′) we have the following new theorem.

THEOREM 5. *If M_k represents the number of extremal arcs of type k which join A to B and if m is the maximum of the types k, then*

$$(9) \qquad \begin{cases} M_i \geqq 2, \quad (i = 1, 2, \cdots, m-1), \\ M_m \geqq 1. \end{cases}$$

We have as an immediate corollary,

COROLLARY. *If there exists in G an extremal arc of type m, there exist in G at least $2m$ other extremal arcs.*

It is of interest to consider the geodesics, on a torus generated by revolving a circle C of radius a about an axis in its plane which does not intersect it. Let u measure the angle through which C has been revolved from an initial position, and let v represent arc length measured along V from the point farthest from the axis. Let $A(u_0, 0)$ and $B(u_1, 0)$ be the end points of a segment of the geodesic $v=0$ of type m. The geodesics which join A to B and lie in the uv plane within the rectangle bounded by $u=u_0$, $u=u_1$, $v=\pi a$, $v=-\pi a$ are then exactly $2m+1$ in number and the integers M_i satisfy the equalities in (9). By choosing the two points sufficiently far apart, m may be made arbitrarily large.

7. *A Restriction Removed.* It is now necessary to remove the restriction so far imposed, namely, that on no arc in G is B conjugate to A. Here we adopt the procedure of Morse (pp. 231–232), merely stating the results. A point B' may be chosen near B and not on an envelope of the extremals through A. For the extremal arcs which join A to B', relations (8) hold. Now allow B' to approach B continuously in such a manner as to avoid envelopes. It will be found *that relations* (8) *continue to hold for $B=B'$*, provided that certain conventions are adopted as to the types of the extremal arcs in G. These conventions are as follows. *Let g be an extremal arc joining A to B on which there are m points conjugate to A prior to B. If B is not conjugate to A, g is to be counted as one arc of type $k=m$. If B is conjugate to A and B is an ordinary point on the $(m+1)$st envelope T of the extremals through A, g is to be counted as two extremal arcs of types $k=m+1$ and $k=m$, respectively. If B is conjugate to A and T has a cusp at B which opens toward A, g is to be counted as one extremal arc of type $k=m+1$. If T has a cusp at B which opens away from A, g is to be counted as one arc of type $k=m$.*

WILLIAMS COLLEGE

ALEXANDER ZIWET—IN MEMORIAM

In the death on November 18, 1928, of Alexander Ziwet, Professor Emeritus of Mathematics in the College of Engineering of the University of Michigan, this Society lost one of its most distinguished scholars.

Professor Ziwet was born in Breslau, Germany, February 8, 1853, of a Polish father and a German mother. His father was a landed proprietor, and his early education was at home under private tutors. At the age of twelve he entered the gymnasium at Glogau, German Silesia, and after five years attendance was graduated in 1870. His father having died in the meantime, he joined his mother and sisters who had gone to Warsaw, and there he learned Polish and Russian and entered the University. In 1873 he transferred to the University of Moscow, which he left in 1874 to study engineering in Germany. After some time in an architect's office, he entered the Polytechnic School at Karlsruhe, Baden, and in 1880 was graduated as a civil engineer. He had served meanwhile the required year in the army, and had passed his examination for a commission as a lieutenant in the reserve artillery corps.

In the fall of 1880 he came to Detroit, and, after about two years in the Lake Survey and five years in the computing division of the United States Coast and Geodetic Survey, he was appointed an instructor in mathematics at the University of Michigan. After several promotions in rank, he became professor of mathematics and head of the Department of Mathematics in the College of Engineering. He was also head of the Department of Modern Languages in the College of Engineering for several years. He had a command of seven modern languages. He retired in 1925, becoming Professor Emeritus of Mathematics. The University conferred on him the honorary degree of Doctor of Science in 1927.

Professor Ziwet has been prominently connected with the development of mathematics in the United States. He was a member of the editorial board of this Bulletin from 1892 to 1920, and served as vice-president of the American Mathematical Society during 1903. His numerous book reviews showed not only a broad knowledge of mathematics, but also a keen critical sense. He helped in the establishment of the Chicago section of the Society in 1897, was a member of its first program committee, and gave it his unqualified support from the beginning by regular attendance and discussion. He took a keen interest in the Mathematical Congress at Chicago, and in the first colloquium of the Society at Evanston, in the latter to the extent of writing up for publication by the Society the lectures delivered by Klein at that time. He was vice-president and chairman of Section A of the American Association for the Advancement of Science during 1905. His retiring address on *The relation of mechanics to physics* was published in Science, January 12th, 1906. He was also a member of the executive council which organized the Mathematical Association of America in 1916.

To his colleagues and to serious students Professor Ziwet gave most

generously of his time and thought. His familiarity with the literature of mathematics was amazing, covering long periods of time and journals in half a dozen languages. He was one of the founders of the University of Michigan Mathematical Club. For many years the club met at his rooms. To its activities, he devoted much time and attention. He was also active in promoting the general scholarly interests of the University, being one of the influential members of the University of Michigan Research Club.

Professor Ziwet left a library of over 5000 volumes which he presented to the University of Michigan. In the collection there were about 400 volumes on the history of mechanics, including many notable works such as a first edition of Newton's *Principia*. The library owes much to him for constant attention over a long period of years. Professor Ziwet's library, together with Professor Beman's, which was also large, and which was presented to the University at the time of Professor Beman's death, are valuable additions to the Mathematical Library of the University.

The following is a list of Professor Ziwet's publications.

An Elementary Treatise on Theoretical Mechanics, Macmillan, 1894, 591 pp.

Elements of Theoretical Mechanics, Macmillan, 1904, 494 pp.

Introduction to Analytical Mechanics (with P. Field), Macmillan, 1912, 378 pp.

Analytic Geometry and Principles of Algebra (with L. A. Hopkins), Macmillan, 1913, 369 pp.

Analytic Geometry (with L. A. Hopkins), Macmillan, 1916, 280 pp.

Note on the composition of finite rotations about parallel axes, this Bulletin, vol. 14 (1908), pp. 208–212.

A note on plane kinematics (with P. Field), American Mathematical Monthly, vol. 21 (1914), pp. 105–113.

The accelerations of the points of a rigid body (with P. Field), American Mathematical Monthly, vol. 23 (1916), pp. 371–381.

His translations are as follows.

Somoff, *Theoretische Mechanik* (translated from Russian into German). Volume I, *Kinematics*, Teubner, 1879; Volume II, *Statics*, 1879.

P. Duhem, *Émile Mathieu, His Life and Works*, this Bulletin, (1), vol. 1 (1891), pp. 156–168.

F. Klein, *Riemann and his significance for the development of modern mathematics*, this Bulletin, (2), vol. 1 (1894), pp. 165–180.

V. Schlegel, *On the problem of the minimum sum of the distances of a point from given points*, this Bulletin, (2), vol. 1 (1894), pp. 33–52.

F. Klein, *Evanston Colloquium Lectures*, 1894. Published by American Mathematical Society. Reported by A. Ziwet.

He also published numerous book reviews in this Bulletin.

The University of Michigan

L. C. Karpinski
J. W. Bradshaw
T. H. Hildebrandt
Peter Field

GERMAN EDITION OF DICKSON ON ALGEBRAS

Algebren und ihre Zahlentheorie. By L. E. Dickson. *Mit einem Kapitel über Idealtheorie von A. Speiser.* Zürich und Leipzig, Orell Füssli Verlag, 1927. 308 pp.

Another notable advance in the development of the subject of linear algebras occurred with the appearance of this book. The mathematical world naturally expects a high standard in the case of treatises by Professor Dickson and the present one will not be found disappointing. While to a certain extent a revision and translation into the German language of his earlier book, *Algebras and their Arithmetics* (reviewed in this Bulletin, vol. 30 (1924), pp. 263–270, by Professor Olive C. Hazlett), it contains much material that is new and interesting, particularly along the lines of extension of the theory of division algebras and of the theory of numbers. Included in the latter category is the final chapter on ideal theory by Professor A. Speiser, under whose supervision the translation was made and who has written the preface.

On the basis of the new results contained in this book the first award of the Cole Prize by this Society was made at its meeting in Chicago in April, 1928. It is worth noting also that some years earlier the annual prize of the A. A. A. S. was awarded to Professor Dickson for a paper of unusual merit on the same subject.

It will be sufficient here to discuss the chief ways in which this book differs from the earlier one. Aside from a rearrangement of the material the first thing that strikes the reader's attention is the extension of the theory of linear associative algebras contained in sections 34–44. While based on an article by the author in volume 28 of the Transactions of this Society, it nevertheless differs considerably from that in that it avoids the use of the theory of finite groups. Whether this is really an advantage may perhaps be debatable.

The author deals here with a division algebra A over a general field K that contains a principal unit and that has the four properties: (I) A is of order n^2; (II) it contains an element i that satisfies an equation, $f(\omega)=0$, irreducible in K; (III) the only elements in A that are commutative with i are polynomials in i with coefficients in K; (IV) all the roots of $f(\omega)=0$ are rational functions of i with coefficients in K. In view of a theorem proved later, however, the only condition, as the author points out, that really restricts in any way the generality of the attack on the determination of associative division algebras is the last.

For the case where the irreducible equation $f(\omega)=0$ is "cyclic," to use the terminology of finite groups, algebras of this sort had been determined previously by the author and had been discussed in the English edition. The notable advance here consists in the use of non-cyclic equations. The chief difficulty of the problem lies in the imposition of the associative law, which leads to some conditions that are not particularly simple. Any doubt

that may arise as to the existence of algebras of this type is dispelled however by detailed consideration of special cases that has been carried out. A recent contribution along this line is a Chicago dissertation by A. A. Albert, in which the division algebras of order 16 are determined.

In a later chapter the author discusses generalized quaternion algebras over the field of rational numbers, these being defined by elements i, j that satisfy the conditions $i^2=-1$, $j^2=\tau$, $ji=-ij$, where τ is restricted to certain rational integral values. As in the case $\tau=-1$, which has been treated at considerable length by A. Hurwitz in his book *Vorlesungen über die Zahlentheorie der Quaternionen*, he develops a theory of integral elements of such algebras that parallels closely the corresponding theory for quadratic algebraic fields, allowance being made for the failure of the commutative law. For certain values of τ the analog of the euclidean algorithm for rational integers is shown to hold and a theory of representation of integers by certain quaternary quadratic forms is obtained. The results established here have very likely furnished the inspiration for the recent work by the author and some of his students on the representation of integers by more general forms of this sort. A result of particular interest concerns the solution of diophantine equations of the type $f=y_1y_2\cdots y_k$, where f represents certain quaternary quadratic forms.

The following chapter on the general number theory of algebras has been to a large extent rewritten and supplemented by new and interesting material. This theory, restricted in the English edition to algebras over a rational field, is here extended to algebras over a general algebraic field, being based on independent work in this direction by the author and by Professor Olive C. Hazlett. "Integral elements" of an algebra are here defined as those that satisfy equations with coefficients that are integers in the field over which the algebra is defined and with leading coefficient unity, and that belong to a "maximal integral domain" of the algebra, that is, a set of elements that reproduce themselves under addition, subtraction, and multiplication, and that are not included in a larger domain of the same sort. The second condition is not a consequence of the first, as is the case with algebraic integers, and hence all integral elements of an algebra may not be included in one integral domain. The definition given here seems simpler than that in the English edition where the "rank equation" was employed. As a justification for it there is proved the important theorem that integral elements exist in every algebra.

As in the earlier edition, the theory of integral elements for a general algebra is shown to be associated with that for a semi-simple subalgebra, and the latter to be dependent on the corresponding theory for simple algebras. This in turn leads to the consideration of matrices whose elements are integral quantities of a division algebra. In particular, some interesting theorems are obtained for the case where this division algebra possesses a euclidean algorithm. The theory of the factorization of such matrices leads to a remarkable result concerning the solution, in integers belonging to algebraic fields that possess this property, of an equation of the form $d=y_1y_2\cdots y_k$, where d represents a determinant whose n^2 elements are independent variables. An application is given to the solution of the

equation $x^2+y^2+z^2=u^2+v^2+w^2$ in integers belonging to such a field. It is perhaps worth noting that the values for the six indeterminates obtained by the method might have to be multiplied by 2 (or possibly some factor of 2) in order that they should become integers.

Other new results are a theorem concerning maximal integral domains in a semi-simple algebra, a determination of the bases of all maximal domains of integral quaternions in a quadratic field, a proof that every normal division algebra of order n^2 is of rank n, and a generalization of Cayley's non-associative algebra of order 8. Further additions are a determination of all division algebras of rank 2 and those of order $\leqq 4$, certain theorems concerning algebraic numbers, a discussion of "quasi-fields," including a proof that every finite quasi-field is a field, and theorems concerning the extension of fields by the adjunction of indeterminates and concerning algebras in quasi-fields.

The final chapter on the theory of ideals in rational algebras, which was contributed by A. Speiser, seems interesting and well deserving of inclusion in spite of the fact that the introduction of ideals fails to restore unique factorization in contrast to the situation in algebraic fields. It is necessary here to distinguish between "right-sided," "left-sided," and "two-sided" ideals. The discussion is devoted in considerable part to the set of residues with respect to a two-sided ideal modulus. As these constitute an algebra over a finite field, a number of the fundamental theorems proved in preceding chapters apply to them. In the case of a two-sided prime ideal it is shown that the residues form a system that is isomorphic with respect to addition and multiplication with the totality of all matrices of a particular order with elements in a particular (finite) Galois field. An important theorem on algebraic numbers is generalized by proving that the prime divisors of the discriminant and only these are divisible by the squares of (two-sided) prime ideals.

The book is remarkably free from obscurities or inaccuracies. A few places of this sort in the English edition that the reviewer had noted have in nearly every instance been removed in the revision. An exception is possibly the proof of the theorem (section 71 of the new edition) that, if an algebra contains an idempotent element, it contains at least one such that is primitive. Here it appears to be assumed that the relation, $euAue < eAe$, follows as a consequence from $uAu < A$, a step that does not seem obvious to the reviewer. Only a slight modification of the proof however is necessary to remove this (apparent) obscurity.

In the proof of Theorem 8 in section 81 it may not be evident to a reader what is the relation between $[B]$ and B. In view of the proof on the next page that $A-N=(B-N_1)\times M$, where $N_1=N\wedge B$ is invariant in B, it follows that $[B]=B-N_1$.

The book is very attractive in its make-up, and the translation and proof-reading have been very carefully done. All concerned are to be congratulated on the result.

H. H. Mitchell

EISENHART'S COLLOQUIUM LECTURES

Non-Riemannian Geometry. By Luther Pfahler Eisenhart. New York, American Mathematical Society, 1927. viii+184 pp.

Generalization having become a familiar process to mathematicians, it is not strange that Riemannian geometry was hardly sixty-four years old before the invention of the non-Riemannian. The former had its origin in Riemann's Habilitationsschrift (1854); the latter, in an article by Weyl (Mathematische Zeitschrift, vol. 2, 1918). Just a decade after the creation of non-Riemannian geometry Professor Eisenhart has given in his colloquium lectures an account of the essential features of the new theory with special reference to the lines along which it has been developed in this country, notably at Princeton.

The fundamental idea of non-Riemannian geometry is the association with a given manifold of a connection which forms a basis for comparing vectors at different points. In the first chapter of Eisenhart's book the asymmetric type of connection is studied, and parallelism of vectors is discussed. The author is careful to distinguish between vectors arising by parallel displacement and parallel vectors: given a vector v at a point x and a curve C passing through x, there is at any point x' of C a unique vector v' which arises by parallel displacement of v along C, whereas all vectors $\phi \cdot v'$, where ϕ is an arbitrary factor and v' is the vector just defined, are called parallel to v with respect to C. The definition of parallel displacement, which is given only for a curve, might well have been stated in the following more general form: the vector v undergoes parallel displacement as x describes any integral variety of the differential equations

$$v^i{}_{,j}\, dx^j = 0,$$

where $v^i{}_{,j}$ denotes the covariant derivative of v. If the vector field is assigned a priori so that $|v^i{}_{,j}| \neq 0$, the integral variety referred to in the suggested modification of the definition is a point, and the vector actually undergoes no displacement. If the connection is euclidean and the coordinates cartesian, the integral variety is the whole of space for any vector field with constant components. From the notion of parallelism are obtained the equations of the paths of the space, a path being defined as a curve whose tangents are parallel with respect to the curve. The paths are thus the straightest lines of the space.

In §8 incompletely integrable systems of total differential equations are discussed. The author proves a theorem which is fundamental in the subsequent treatment (Chapters II and III) of the equivalence problems. The later repetition of this theorem adapted to the particular problem in hand becomes rather monotonous. It would perhaps be sufficient to set up the integrability conditions in each case and to make a concise reference to the original statement of the theorem.

The remainder of Chapter I is devoted to the parallel displacement of

a vector around an infinitesimal circuit, orthogonality of vectors, the generalization of the coefficients of rotation of an orthogonal ennuple, and the changes of connection which preserve parallelism. Of these topics, the last mentioned is perhaps the most important, since it leads to the projective theory of Chapter III.

Most of the results obtained for the asymmetric case to date are such direct generalizations of the symmetric or of the Riemannian case that their acquisition is not a great achievement. It is well, however, to develop the theory as far as possible without making the assumption of symmetry.

Chapter II deals with symmetric connections. Affine normal coordinates are defined in the usual way by means of the equations of the paths, but aside from an application to the definition of normal tensors and the process of extension, no subsequent use is made of them. If one were to judge from the way in which investigators fail to employ normal coordinates as a tool, one would conclude that they are beautiful but useless. In fact, little knowledge has been acquired so far by their use which cannot be as easily obtained with geodesic coordinates. There is an error in formula (22.8). For the correction, see Annals of Mathematics, (2), vol. 28, p. 556.

The theorem of Fermi to the effect that the Christoffel symbols are zero along any given curve in a properly chosen coordinate system is extended to symmetric connections. The proof is unnecessarily long. The following much simpler proof is indicated by Cartan (Mémorial des Sciences Mathématiques, No. 9, p. 18). Without loss of generality we may assume that $x^2=x^3=\cdots=x^n=0$ along the given curve. Assume for the new coordinates x' a transformation which is quadratic in $x^2, x^3, \cdots, x^n$ and which has for coefficients undetermined functions of x^1. Substitute these values of x' together with $x^2=x^3=\cdots=x^n=0$ and $\Gamma'=0$ in equations (28.1) in which the primes have been shifted:

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^k} + \Gamma'^i_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^j}\frac{\partial x'^\beta}{\partial x^k} = \Gamma^\alpha_{jk}\frac{\partial x'^i}{\partial x^\alpha}. \tag{28.1}$$

The coefficients of the zeroth and first orders are then determined by a system of ordinary differential equations to which Cauchy's existence theorem is applicable, and the coefficients of the quadratic terms are determined by finite relations. This same method can be applied to show that the coefficients of projective or conformal connection can be made zero along a curve. It is quite natural to ask, can we go still further and make the Γ's zero over a surface? The answer is in the negative. For taking the surface as $x^3=x^4=\cdots=x^n=0$, we must find a transformation to satisfy, among other conditions, the relations

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^1} = \Gamma^l_{j1}\frac{\partial x'^i}{\partial x^l}, \qquad \frac{\partial^2 x'^i}{\partial x^j \partial x^2} = \Gamma^l_{j2}\frac{\partial x'^i}{\partial x^l},$$

in which $x^3, x^4, \cdots, x^n$ have been put equal to zero. Differentiating the first of these with respect to x^2 and the second with respect to x^1, and comparing, we find that B^i_{j12} must vanish for $x^3=x^4=\cdots=x^n=0$, B representing the curvature tensor. Hence the transformation does not always exist.

The problem of Fermi and related problems, such as that of local coordinate systems, can be regarded as modified forms of the equivalence problem in which the equivalence is not to hold throughout an n-dimensional region, but only in a sub-space defined by certain relations among the independent variables. The equivalence theorem proper, which gives a necessary and sufficient condition that equations (28.1) have a solution $x'=f(x)$ satisfying them identically in all the independent variables, is discussed in detail in §28. The proof given is for symmetric connections, but the extension to the Riemannian and asymmetric cases is clearly indicated.

Chapter III concerns the projective geometry of symmetric connections. In the reviewer's opinion the theory is not presented in its most symmetrical or attractive form because the author does not free himself from the affine connection. The whole subject can be developed as the theory of a projective invariant, the projective connection. When it is so developed, the conditions obtained are in obviously projective form, involving only the Π's and their derivatives. Formulas (34.3), for example, appear on the surface to express affine conditions because they involve the affine connection, but actually they are projective. The condition obtained in the projective fashion would be the expression defined by (35.17) equated to zero. Even equations (34.8), when needed, can be immediately deduced from (35.12) by an appeal to the notion of normal affine connection (p. 105).

The final portion of the chapter is given to the discussion of collineations in affine and projective spaces. By a collineation is meant a transformation which makes the components of the connection exactly the same functions of the new variables as of the old. The analogue for a Riemannian space is a motion or an automorphism of the quadratic form. As the author remarks, the problem of determining whether a given connection admits collineations is a particular case of the equivalence problem, and can be treated as such. The author chooses, however, to use the method of infinitesimal transformations. Except for the flat case and the rather obvious theorem of page 128, the results are not definitive. They are conditional upon the existence of solutions of certain differential equations. The use of Lie's method might, therefore, seem to have no advantages over a consideration of the system of total differential equations which express the law of transformation of the connection. On the contrary, the advantages of the method employed are two: first, the equations of condition are rendered linear; second, variety of method is secured by the introduction of continuous groups. For $n=2$ a classification of all spaces admitting continuous groups of collineations might be possible and interesting.

Chapter IV develops the geometry first of hypersurfaces and then of the general sub-spaces. The generalized equations of Gauss and Codazzi are developed for both cases. The most noteworthy result of this chapter is perhaps that cited by the author in his preface: "For a sub-space of a Riemannian space there is in general an induced metric and consequently

an induced law of parallelism. There is not a unique induced affine connection in a sub-space of an affinely connected space."

The book ends with a bibliography of the books and articles to which reference is made in the text. Eisenhart's previous books have been conspicuous, among other things, for their carefully prepared and useful indexes. The omission of such an index from the present work is to be regretted. With the growth of mathematical literature, it is becoming more and more inconvenient to be forced to leaf through a whole book to locate a desired bit of information.

Those wishing an introduction to the subject will find *Non-Riemannian Geometry* a useful book. A familiarity with the methods of tensor analysis is, of course, presupposed on the part of the reader. The subject matter is carefully presented and the manipulations are easy to follow. The theorems are clearly stated and are proved in a straightforward manner, a number of the proofs being original with the author. The chief defect in the book—and this seems serious in a set of colloquium lectures—is the absence of any attempt to comment upon the significance of the results or to point out possible lines for future development.

J. M. Thomas

THREE BOOKS ON DIFFERENTIAL EQUATIONS

An Elementary Treatise on Differential Equations and their Applications. New edition, revised and enlarged. By H. T. H. Piaggio. London, G. Bell and Sons, 1928. 18+256+27 pp.

Gewöhnliche Differentialgleichungen. By J. Horn. Zweite, völlig umgearbeitete Auflage. (Göschens Lehrbücherei.) Berlin, Walter de Gruyter, 1927. 8+197 pp.

Ordinary Differential Equations. By E. L. Ince. London and New York, Longmans, Green, 1927. 8+558 pp.

Professor Piaggio's *Differential Equations* was first published in May, 1920, and was reprinted four times during the next six years. "The object of this book is to give an account of the central parts of the subject in as simple a form as possible, suitable for those with no previous knowledge of it, and yet at the same time to point out the different directions in which it may be developed." The only previous knowledge assumed is that of the differential and integral calculus. The style is admirably adapted to a text for beginners and the large number of examples with answers furnishes adequate drill material.

The usual standard forms of ordinary and partial differential equations occupy the greater part of the book: one chapter is devoted to numerical approximations and another to existence theorems. In the revised edition more examples have been included and a new chapter "Miscellaneous Methods" has been added, dealing with a number of disconnected topics, which may be regarded as in the nature of supplementary reading. The author calls particular attention to some difficulties in the theory of singular solutions, for which he is indebted to some unpublished work by Mr. H. B. Mitchell, formerly Professor at Columbia University.

Professor Horn's book is an abbreviation of his earlier work *Gewöhnliche Differentialgleichungen beliebiger Ordnung*, which was published as volume 50 of the Sammlung Schubert and is now out of print. It may be described as an introduction to the theory of ordinary differential equations, although one of the nine chapters is devoted to standard methods of integration and another to numerical and graphical approximations. The author assumes only an elementary knowledge of the theory of functions of a complex variable and has been guided in his choice of subject matter by a desire to lead the student directly to those topics which have most useful applications in physics and astronomy.

Professor Ince's treatise represents a notable addition to mathematical literature in English. The aim of the author has been to "introduce the student into the field of ordinary differential equations and thereafter to guide him to this or that standpoint from which he may see the outlines of unexplored territory." To quote further from the preface: "For some time I have felt the need for a treatise on differential equations whose scope would embrace not merely that body of theory which may now be regarded as classical, but which should cover, in some aspects at least, the main developments which have taken place in the last quarter of a century. During this period no comprehensive treatise on the subject has been published in England, and very little work in this particular field has been carried out; while, on the other hand, both on the Continent and in America investigations of deep interest and fundamental importance have been recorded."

The book is divided sharply into two parts of about equal length: Part I deals with differential equations in the real domain and Part II in the complex domain. Beginning with elementary methods of integration, the author passes on to the existence and nature of solutions and then to an account of continuous transformation-groups. The remainder of Part I is devoted to linear differential equations in real variables. The first three chapters of Part II discuss the existence theorems and singularities of non-linear equations and these are followed by seven chapters on linear equations in complex variables.

To describe in detail the topics discussed would be to give a résumé of the important articles on differential equations which have appeared in the journals during the last twenty-five years. The book forms an excellent account of the progress during that period. Necessarily some parts of it are rather condensed and the average reader will need to have his pencil and paper at hand in order to follow the argument intelligently. However, the references to original articles and standard texts are sufficiently complete to enable one to look up readily further discussion of every major point.

As an exposition of the present state of the theory of ordinary differential equations the book deserves the highest praise.

W. R. LONGLEY

COURNOT ON MATHEMATICAL ECONOMICS

Researches into the Mathematical Principles of the Theory of Wealth. By Augustin Cournot, 1838. Translated by Nathaniel T. Bacon, with an Essay on *Cournot and Mathematical Economics* and a *Bibliography of Mathematical Economics*, by Irving Fisher. New York, The Macmillan Company, 1927. xxiv+213 pp.

Marshall, in his celebrated treatise,* gives a remarkable estimate of two economists of the early part of the nineteenth century. Everywhere he pays tribute to Ricardo's skill in deductive reasoning and his power of abstraction in making hypotheses. "Ricardo himself had no mathematical training. But his instincts were unique; and very few trained mathematicians could tread as safely as he over the most perilous courses of reasoning. Even the acute logical mind of Mill was unequal to the task" (p. 836). And of Cournot he writes: " · · · Cournot's genius must give a new mental activity to everyone who passes through his hands · · · " (p. xi) and "In Cournot it [the French school] has had a constructive thinker of the highest genius" (p. 766). Ricardo's work, to a mathematician, suggests the additional clarity and completeness which could be obtained by means of mathematical formulation; and Cournot, in the book under review, actually carries out a systematic mathematical discussion of certain important questions in economics "rather to present a few new views than to arrange truths already sufficiently known" (p. 164). The reviewer is able to add his testimony that from Ricardo he received the inspiration to a mathematical study of economics, and in this book of Cournot,† which he still regards as the best elementary treatise on the theory of economics, found such a study already well developed.

This is not to say that the book is perfect. In particular, the last chapter, on *Variation in the Social Income, Resulting from the Communication of Markets*, seems not to be adequate in its analysis, for changes in the *form* of production, resulting from such communication, are not carefully considered. And there are occasional slips, usually not of serious character, which in this edition are noted or corrected. Nevertheless the reviewer maintains his opinion that the reader of inquiring mind will get more out of this book than from any other yet published on the subject.

Moreover Cournot is almost alone in holding to a clear realization of the difference between measurable and non-measurable quantities, pointing out in Chapter 1 that many of the controversies in the subject arise from ambiguities of this sort. In the next to the last chapter this clearness is manifest, for the author is able to build up a serviceable theory of the "Social Income," on the basis of money value.

On the other hand, one recent book on the mathematical principles of

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* *Principles of Economics*, London, 1920.

† The first French edition, in the Harvard College Library.

economics, typical of many others, builds its theory on the following basis.*

"Write $U(x, y, \cdots)$ for an algebraic function of measurable quantities $x, y \cdots$; let it be so related to an entity we will call $S(x, y, \cdots)$, where S is not a calculable function but the non-measurable satisfaction derived from quantities $x, y, \cdots$, that the following postulates are satisfied.

"*Postulates.* (1) When $x, y, \cdots$ vary without affecting the value of $U(x, y, \cdots)$, more x balancing less y, etc., $S(x, y, \cdots)$ remains unchanged.

"(2) When $x, y, \cdots$ vary so as to increase $U(x, y, \cdots)$, $S(x, y, \cdots)$ increases, and if U decreases, S decreases.

"(3) When there are successive variations of $x, y, \cdots$, the first increasing U from U_1 to U_2, the second from U_2 to U_3, so that the second increase is greater than the first $(U_3 - U_2 > U_2 - U_1)$, then the second increase in S is greater than the first; the postulate still to be true when less is written for greater."

Apparently this other author is unaware that he is begging the question. If loci of indifference are expressed by Pfaffian differential equations it does not follow that there is any function of which these are the level loci, for such equations are not necessarily completely integrable. The question is not of names, but of existence. These supposedly general treatments are much more special than their authors imagine.†

To carry the point further, we suggest that most nouns that denote values may be regarded as misnomers. They have possibly a significance "im kleinen"—a local significance,—derived from the local integration of the equation, but no significance in extenso. Not only in ethics and aesthetics would an adjective be "true-er" than a noun; it may be presumptuous even to insist, in the words of a recent controversy on a point in the teaching of the calculus, "that there is only one truth." With that remark, and the advice to economists to follow Cournot, rather than Jevons and his school, in this respect, these problems may well be left to the more experienced consideration of philosophers.

Besides the general problems discussed in the first two and last two chapters, to which we have already alluded, Cournot's book deals in considerable detail with the subjects of foreign exchange, law of demand, monopoly, competition, competition among a large number of producers, producers who furnish parts of a compound product, and the communication of markets. The effects of taxation are investigated with respect to the various forms of production.

Throughout, the problems are analyzed as far as possible in terms of functions of a single variable. Generalizations in this respect are important, but not so difficult for a mathematician to make as for the reader to follow, if he is of the type for which this book was intended. Such generalizations, among other contributions, have been made by Walras and Pareto. Still more significant extensions would arise from the abandonment of the hypothesis, which Cournot and almost all other economists make

* Bowley, *Mathematical Groundwork of Economics*, Oxford, 1924, p. 1.

† The trouble usually comes from reasoning about n dimensions with the intuition merely of a two-dimensional figure.

either explicitly or tacitly, that demand is a function of price, adequately represented by the notation $D=f(p)$. Cournot is also not alone in considering profit as the central quantity in economics, whose maximum determines the other quantities; and given the industrial habits of his time, he is hardly to be blamed.

The translation seems to be carefully done, and the editor has added numerous notes designed to assist the less mathematical reader. No attempt has been made to bring the bibliography down to the date of this edition; in the thirty years since the treatise first appeared in English the additions to the bibliography included in that edition would presumably have surpassed the desirable limits for the book. There would have been service rendered, however, if the editor had used the resources at his disposal to clear up the ambiguous character of some of the references already given. This is not said to diminish the gratitude which the public owes to both editor and translator in making again accessible, after ninety years, the eternal freshness of this brief classic.

G. C. Evans

THREE BOOKS ON NON-EUCLIDEAN GEOMETRY

Vorlesungen über Nicht-Euklidische Geometrie. By Felix Klein. Prepared anew for publication by W. Rosemann. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 26. Berlin, Julius Springer, 1928. xii+326 pp. Price 18 marks unbound, 19.50 marks in linen.

Nicht-Euklidische Geometrie. Hyperbolische Geometrie der Ebene. By Richard Baldus. Sammlung Göschen, No. 970. Berlin and Leipzig, Walter de Gruyter, 1927. 152 pp. Price 1.50 marks.

La Géométrie Non-Euclidienne. By P. Barbarin. Third edition, followed by notes by A. Buhl. Collection "Scientia," No. 15. Paris, Gauthier-Villars, 1928. 176 pp.

In a note to his Erlanger Programm, Klein justified his avoidance of the word "non-euclidean" in that address, saying: "There are connected with the designation 'non-euclidean geometry' a number of unmathematical conceptions which occasion as much enthusiasm on the one side as horror on the other, but which have nothing whatever to do with our purely mathematical considerations." This may be a reason why, aside from a few glances forward, that same word is absent from over one half of his admirable treatise, *Vorlesungen über Nicht-Euklidische Geometrie.* In fact, the reader with a purely euclidean past will be led so tactfully and persuasively from his usual geometry to the more general one that he may long wonder wherein lies the book's right to its title. He will be an admirer of the greater beauty and symmetry of non-euclidean space without realizing how far he has left euclidean space behind him.

Part one of the book is an introduction to projective geometry. We pass rapidly from elementary affine coordinates in euclidean space to homogeneous and general projective coordinates. Thus, the arithmetical basis

of cartesian geometry being assumed, no time need be given to axioms. Analysis situs enters early, when the projective plane is shown to be one-sided. The chapter on fundamental concepts treats, further, of homogeneous linear substitutions and projective transformations (the Erlanger Programm is to be our guide), n-dimensional manifolds, duality, cross ratios, imaginary elements. There follows a careful treatment of configurations of second degree (both class and order), a complete classification from the point of view of reality being given. A number of transitions from one non-degenerate form to another through a degenerate one are traced—such, for instance, as will later clarify the position of euclidean geometry between its two rivals. The last chapter on projective geometry deals with those collineations which leave invariant a configuration (degenerate or otherwise) of second degree. The fate of points, the families of invariant surfaces, the interpretation of real collineations as motions,—rotations, translations, screw-motions—are studied clearly and fully, for various types of fundamental configurations.

The second part deals with measurement, on a projective basis. Again taking thought that the reader begin with the familiar, Klein discusses the euclidean formulas for lengths and angles, and shows their projective relation to the circular points (or fundamental circle, if there are three dimensions). Then the metric of the bundle of lines or planes (e.g., in euclidean space) gives the first glimpse of measurement in elliptic and spherical geometries. It is noted, as a matter of interest, that elliptic geometry prevails on the euclidean plane at infinity. Only in the fifth chapter is a projective coordinate system set up in its own right, by the familiar repetition of the harmonic construction. A notable point in Klein's treatment is his insistence that coordinates can be set up by constructions entirely within a finite region. The way is then clear for metrical definitions on the basis of the absolute (first non-degenerate, then degenerate) by means of the logarithm of a cross ratio. Now a selection has to be made, among all the possible fundamental configurations, in order that angular measurement be elliptic; and there result, of course, the three ordinary types. The development of their distinctive properties is full and clear. Among the sections of unusual interest may be mentioned the equation, valid in all three geometries, between the dihedral and the trihedral angles of a tetrahedron and the analogous equation in n dimensions; the use of quaternions to represent motions in elliptic space; the excellent study of the Clifford surface and its euclidean geometry. Chapter IX, the last one of this part, discusses those forms of space which admit of elliptic, hyperbolic, euclidean metric. This discussion of analysis situs will probably be among the most influential sections in clarifying geometrical ideas. In the last paragraph we read: "From the assumption that the space about us displays a euclidean or a hyperbolic structure, it by no means follows that this space has infinite extent."

Chapter X is concerned with history, with differential geometry, and with a variety of conformal representations of non-euclidean planes on the euclidean. Finally, a few pages are given to applications of non-euclidean geometry—the use of hyperbolic motions in the study of linear

substitutions of a complex variable and in that of automorphic functions, the use of non-euclidean geometry in topological study, and of projective metric in special relativity. These chapters are brief—probably no fault, for the book has pictured the subject so alluringly that the references here given should lead many students on to wider reading than one treatise could furnish.

There is scarcely an improvement which the reviewer could suggest, even in such a small matter as proof-reading. One sentence, on page 26, with regard to those collineations of a line which result from two real projections, seemed at first reading self-contradictory: "Die so erhaltenen Kollineationen besitzen stets einen reellen Fixpunkt, der aber · · · im allgemeinen nicht vorhanden zu sein braucht." Clarity results if we change to "in den allgemeinen Kollineationen." On page 116, line 9, we should read "nullteiligen und ringartigen Flächen," not "nullteiligen und ovalen Flächen"—the only error of statement noted.

It is surprising that the two words "complex" and "imaginary" are used indiscriminately, whether real numbers are included in the domain intended or excluded from it. A few random cases are the following:

Page 73, line 4, "imaginary" collineations may be real.

Page 93, line 21, "complex" collineations may be real.

Page 93, line 24, "imaginary" point-pairs may not be real.

Page 100, footnote, "complex" values may not be real.

Page 185, line 15, "imaginary" region includes real elements.

Page 236, line 13, "imaginary" region includes real elements.

Page 236, line 18, "imaginary" lines may not be real.

Are not imaginary numbers those complex numbers which are not real?

A similar oversight occurs when Clifford parallels are discussed. On page 234 we read "A Clifford parallel to a line a is never in a plane with a, but always skew to it." In order that this be true, it is necessary that parallels be real. Yet, on page 237, imaginary lines, generators of the fundamental null surface, are made members of a family of parallels, and we have therefore intersecting Clifford parallels. Perhaps it would be best not to name each of the two displacements whose product is a general motion a "Schiebung längs einer *Parallelenschar*."

It may be due to a desire for brevity that, although curvature is carefully explained (p. 280), torsion is soon afterwards mentioned without definition (p. 305), and that a number of important theorems are not proved —for instance, that of the uniqueness of the common self-conjugate triangle of two conics. These are very slight departures from a fulfillment of the evident desire to write a book comprehensible immediately after a first course in calculus.

The value of the treatise, great in any case because of its elegance and comprehensiveness, is enhanced by a profusion of historical notes, by a generous use of cross references, looking both forward and backward, and by such a large collection of clear illustrations as no other work on non-euclidean geometry approaches. A few suggestions for future research are given. Thus: "a closed three-dimensional hyperbolic space-form of finite volume does not seem to have been found as yet"; "surfaces [of constant

curvature] are fixed by difficult differential equations, which have as yet been solved only in particularly simple special cases."

The work has been prepared for the press by Rosemann. As the editor states in the preface, he gave it years of unselfish labor, first in consultation with Klein, then alone. That there resulted such a valuable contribution to mathematical literature is cause for deep gratitude to him.

Although the book of Baldus on non-euclidean geometry, in the Sammlung Göschen, is much smaller, he has, by restricting himself to hyperbolic plane geometry, been able to give an excellent presentation.

The theory common to euclidean and hyperbolic geometries is developed as far as possible, largely on the same basis as in Hilbert's *Grundlagen der Geometrie*. Often, indeed, the reader is referred to that book for proofs of familiar theorems. In this development, elliptic geometry is excluded by the axiom: "Among three points of a line one and only one lies between the other two." Next it is shown that the axioms preceding that of parallelism are valid if we use the unit circle as the fundamental conic, and interpret congruence by means of automorphic collineations of that circle and the preservation of the usual cross ratio. There follows the hyperbolic axiom, in the following frugal form: "There exist one line g and one point P outside of g, such that (at least) two lines through P fail to meet the line g." Angles obtain measure, in this interpretation, without appeal to the cross ratio made with imaginary lines. For this purpose, an angle whose vertex is at the center is given its euclidean measure, while the hyperbolic magnitude of another is defined as that resulting after a "motion" has brought the vertex to the center. Indeed, the device of placing an important point of a configuration at this center, and then applying euclidean theorems in order to obtain hyperbolic ones, is used ingeniously many times. When length has been defined as half the logarithm of the cross ratio, the material is on hand for a good treatment of trigonometry and elementary analytic geometry, of properties of lines and the various conics of area. It is interesting to note that, although the concurrence of the medians of a triangle holds in hyperbolic geometry, the proof is via euclidean geometry and that (according to Baldus) no proof independent of the parallel axiom has been discovered.

Baldus, like Klein, has a satisfactory historical account. (In fact, the reviewer recalls no other branch of mathematics in which the reader is so sure to be entertained by the wisdom and errors of the past.) Much of the booklet's value lies in the philosophic quality of the author's thought. Particularly good are the sections on the significance of "absolute" geometry and on the formalization of geometry, and his reference to the doubts which might arise from the use of a euclidean circle as the basis for hyperbolic geometry, as well as the answer to those doubts.

Figures are abundant, very clear and carefully drawn. There is a short reference list—eleven books—at the beginning. All the works named are available in German. Perhaps, if the author had realized how welcome are the worthy, inexpensive members of the Sammlung Göschen in non-German parts of the world, he would have included such important treatises as those of Coolidge and MacLeod.

The third work on non-euclidean geometry which has come to hand in these days is the third edition of Barbarin's *La Géométrie Non-Euclidienne*. The first edition, in 1902, was a slight book, following the historical line of development. Later editions, in 1907 and 1928, have been rewritten and expanded as other related disciplines developed and as Barbarin's own researches were made and published in scattered journals. Now the volume, more than double its original size, contains, also, important supplementary notes by Buhl. There results a book written, in turn, for three publics: for those who, with scarcely higher mathematics than trigonometry, would learn the essential elementary facts of these geometries; for those who are interested in the more difficult study, though with simple tools, of questions of construction and the like; and for those who are anxious to learn, from a very competent writer, of the relations between non-euclidean geometry and some recent work in physics, in analysis, and in differential geometry.

The historical pages of this book are made more interesting by a number of portraits, more valuable by the inclusion of several theorems and proofs. Since the elementary point of view, that of the searchers after a proof of the parallel postulate, is maintained, there cannot be the same elegant symmetry as projective geometry grants. On the other hand, spherical geometry (which Barbarin prefers to elliptic) and hyperbolic geometry are presented to us somewhat symmetrically. Of two postulates—namely, the sixth (that two lines cannot enclose a space) and the fifth (in Euclid's form)—each is fulfilled by one and only one of the two theories. It is worth while to emphasize that Euclid's parallel postulate is true in Riemann's geometry. Unusual emphasis is placed on de Tilly, particularly on his proof that we must have one of three identities between the coordinates of five points in space,—identities which involve the distances of all pairs of points, and lead respectively to the three geometries. Those of us who prefer the projective approach to the subject will perhaps not rank de Tilly's work as highly as Barbarin does; nevertheless it is well that stress is laid on his researches. Another seemingly excessive enthusiasm is that for Euclid's *Elements*: "Leur emploi universel des arguments les plus complets et les plus rigoureux;" "les postulats d'Euclide sont absolument rigoureux"—it is hard to say what is meant by rigor of axioms.

The other chapters which remain from the first edition summarize the main facts on lines and the various types of circles, develop trigonometric formulas of triangles and trirectangular quadrilaterals, discuss areas and volumes. It is curious that the author does not notice that, in Riemannian geometry, equidistant curves (his "hypercycles") are circles. Unfortunately, proofs are so often absent that the reader may not be convinced.

Barbarin points out that Euclid's reasoning is always by means of figures of which the construction is known. He has followed this clew, and has interested himself in the discovery of methods of construction in non-euclidean geometry. Thus, the new chapters describe a mechanism for tracing horocycles; the construction of angles of rational trigonometric functions, of line segments whose trigonometric (or hyperbolic) functions

are rational, or roots of index 2^n, of rational numbers; inscription of regular polygons; the quadrature of the circle.

A chapter on the impossibility of proving the fifth postulate takes time to refute some very crude objections to the geometries of Lobachevsky and Riemann, discusses the surfaces on which their plane geometries are valid,—and, incidentally, contains a welcome photograph of Beltrami's paper pseudosphere.

With the supplementary notes of Buhl, who wrote the pamphlet on "Formules Stokiennes" of the Mémorial, we seem to begin a new book, one on those formulas. (Neither here nor in the Mémorial does he trouble to define the adjective.) The second note, starting with the derivation of Maxwell's equations from a Stokesian one, passes soon to the problem of the linear transformations leaving a homogeneous quadratic function invariant—that is, to that Cayleyan problem which was at the basis of Klein's treatise. On this basis, then, Buhl develops non-euclidean geometry anew. Twenty pages are next devoted to the much more general differential geometry of Riemann, once more with a start from Stokes. For its brevity, this is a remarkably good account; but it would be clearer if the author did not fear the words covariant and contravariant, and avoid all explanation of the distinction between subscript and superscript. The fourth note, entitled *The Geometry of Light*, deals condescendingly with restricted relativity.

Probably the simultaneous publication of several works on non-euclidean geometry is due, in part, to the influence of recent physical theories. The three books here reviewed are well fitted to lead to keen interest in that subject for its own sake.

E. S. Allen

SHORTER NOTICES

Précis d'Analyse Mathématique à l'usage des candidats au certificat de calcul différentiel et intégral. By E. Lainé. Vol. I, viii+231 pp.; Vol. II (in collaboration with G. Bouligand), 315 pp. Paris, Librairie Vuibert, 1927.

Among the large number of the French textbooks with the similar purposes, this is perhaps one of the most interesting. A definite and, in our opinion, a fortunate break with the old tradition is made by introducing the vector notation in the treatment of the fundamentals of differential geometry. The corresponding part (iv) of the book as well as the part (v) devoted to the partial differential equations (and written by G. Bouligand) are the best in the book. The usage of the vector notation makes intuitive and simple many results which otherwise are obtainable but with a great labor. The inductive method is also used in the treatment of the partial differential equations (the only ones being treated are those of the first order and some of the second order, of the Monge-Ampère type). The reader is compelled here to make his own conclusions on the basis of intuitive geometric discussion.

As to the remaining parts (Introduction: Review of some facts of algebra and analytic geometry; Part i: Functions of a real variable; Part ii: Analytic functions; Part iii: Ordinary differential equations), they reveal the tendency of putting too much material into too limited space (279 pp.). The resulting treatment is often rather brief and formal, and sometimes a little confusing. As for examples we may mention: infinite double integrals (Vol. I, 69–70); line and surface integrals (I, 76–77); extrema of functions of two variables (I, 82–83); differentiation under integral sign (I, 107). There are found some lapses and quite a few misprints. The text is supplemented by a large number of exercises; some of them, interesting as they are, appear rather difficult from the point of view of the means available in the text.

J. D. Tamarkin

The Early Mathematical Sciences in North and South America. By Florian Cajori. Boston, Richard G. Badger, 1928. 156 pp.

Excluding introductory matter and a six-page index, this book consists of eleven chapters distributed over 139 pages as follows: Mathematics, 18 pp.; Practical Astronomy and Surveying, 37 pp.; Surveys and Maps, 24 pp.; Meridian Measurements of the Earth, 8 pp.; Transit of Venus, 10 pp.; Comets, 5 pp.; Almanacs, 5 pp.; Orreries, 3 pp.; The Earliest Permanent Observatory in America, 8 pp.; Physics, 16 pp.; Societies, Academies and Journals, Conclusion, 5 pp. As may be inferred from these chapter headings, the work appears to be a series of notes or sketches on topics more or less closely related to mathematics, physics and astronomy rather than a systematic study of the development of these sciences

as sciences in North and South America during the sixteenth, seventeenth and eighteenth centuries. This inference becomes more apparent upon noticing the sub-headings in Chapter I, Mathematics: The Maya symbol for zero; The Peruvian and North American knot records; The calderón and cifrão in the writing of numbers; evolution of the dollar mark; magic squares; mathematical publications in America.

The fact that such paragraph titles as "Brattle using Halley's quadrant at Harvard" (p. 42), "Brattle and Kearsley using ring-dials" (p. 44) and "Laval at the gulf coast" (p. 43) occur along with "Early determinations of positions in and near Mexico" (p. 36) and "Marcgrave's observations in Brazil" (p. 41) suggests that the author has not given the form of this work the careful revision his other publications would lead us to expect.

Typographical errors are not unusually numerous but those which the reviewer has noticed lead him to wonder if the proof-reading has not been done chiefly in the publisher's office. For example, on p. 12 "Axtecs" appears in place of "Aztecs"; a reference to Hevelius on p. 46 is referred to in the index (p. 153) as "Hevelins"; under B in the index (p. 151) occurs "Brahe. See Tycho Brahe" and under T (p. 156) is found "Tycho, Brache" instead of "Tycho Brahe"; and although Professor J. L. Coolidge of Harvard must be well known personally to the author his name is cited as "J. J. Coolidge" in three different places (pp. 26, 28 and 152).

The mechanical makeup of the book is scarcely what might be expected from the Gorham Press of Boston. On various pages (e.g., pp. 74, 77, 93, 111, 124) certain lines stand out prominently and, in some places, the contrast is almost sufficient to give the appearance of a line of boldface type. Whether this variation comes from setting the text on one linotype machine and the corrections on another or from other causes, it is so easily corrected by a competent pressman that its appearance suggests careless or hurried presswork. Also, the reviewer has an impression that the illustrations which appear on plate paper might have been improved, both in appearance and in placing on the pages, without adding unduly to the cost of publication.

The reviewer has found the book quite interesting and, so far as he is able to judge, its statements of fact have been carefully verified.

U. G. Mitchell

NOTES

The American Mathematical Society is a beneficiary under the will of the late Miss Marion Reilly, of Bryn Mawr College, as residuary legatee of a trust fund.

The opening number of volume 31 of the Transactions of this Society (January, 1929) contains the following papers: *The boundary problem associated with a differential equation in which the coefficient of the parameter changes sign*, by R. E. Langer; *Triple and multiple systems, their geometric configurations and groups*, by A. Emch; *On the expansion of analytic functions of the complex variable in generalized Taylor's series*, by D. V. Widder; *Note on the expansion of analytic functions in series of polynomials and in series of other analytic functions*, by J. L. Walsh; *A general problem of minimizing an integral with discontinuous integrand*, by C. F. Roos; *An introduction to the theory of ideals in linear associative algebras*, by C. C. MacDuffee; *On the Padé approximants associated with the continued fraction and series of Stieltjes*, by H. S. Wall; *On curvilinear congruences*, by C. E. Weatherburn; *The multinomial solid and the chi test*, by B. H. Camp; *The singular points of analytic space-curves*, by A. Ranum; *Universal quadratic forms*, by L. E. Dickson; *Boundary value problems for potentials of a single layer*, by E. R. C. Miles; *On a generalization of the associative law*, by A. Suschkewitsch.

The opening number of volume 51 of the American Journal of Mathematics contains: *Discontinuous boundary value problems of the first kind for Poisson's equation*, by G. C. Evans; *On irreducible cross-cuts of plane simply connected regions*, by W. A. Wilson; *Number of abelian subgroups in every prime power group*, by G. A. Miller; *Finite groups in which conjugate operations are commutative*, by C. Hopkins; *Covariant conditions for multiple roots of a binary form*, by L. T. Moore and J. I. Tracey; *On certain finitely solvable equations between arithmetical functions*, by E. T. Bell; *On the number of representations of integers by certain ternary quadratic forms*, by J. V. Uspensky; *Representation of integers in the form* $x^2+2y^2+3z^2+6w^2$, by L. W. Griffiths; *Normal ternary continued fraction expansions for cubic irrationalities*, by P. H. Daus; *Triads of plane curves*, by A. R. Jerbert; *The symbolic development of the disturbing function*, by K. P. Williams; *An extension of Maschke's symbolism*, by N. L. Anderson; *On the approximate solution of Fredholm's homogeneous integral equation*, by R. B. Adams; *Functions of écart fini*, by H. E. Bray.

The November, 1928, number of the Annals of Mathematics (series 2, volume 29, No. 4) contains: *Concerning the complementary domains of continua*, by G. T. Whyburn; *Fermat's Note* XLV, by C. M. Walsh; *A new derivation of the Lorentz transformation*, by P. Y. Chou; *On m-dimensional cross-ratios*, by L. Weisner; *Twisted curves classified as to their osculating and quasi-osculating spheres*, by A. Ranum; *On the generalization of the algebra of the theory of numbers. The invariancy of infinite series*, by O. E.

Glenn; *Involutions that belong to a linear class*, by E. T. Browne; *Euclidean invariants of plane algebraic curves*, by L. Weisner; *Non-abelian groups whose groups of isomorphisms are abelian*, by C. Hopkins; *Oscillation theorems for the differential boundary value problems of the fourth order*, by S. A. Janczewsky; *A note on Stieltjes integrals*, by R. N. Haskell; *On motions which satisfy Kepler's first and second laws*, by L. R. Ford.

Professor G. H. Hardy, of Oxford University, who has been lecturing at Princeton University during the first semester of the present academic year on *Chapters in the theory of functions*, lectured at Lehigh University on January 11 on *Hilbert's Mathematical logic*, at the Ohio State University on January 18, 1929, on *The theory of primes*, and at the University of Chicago on January, 21 on *The analytic theory of numbers*. From January 27 to March 15, he was in residence at the California Institute of Technology, giving lectures on *The analytic theory of numbers*. On March 1 and 5 he lectured at the University of California at Los Angeles. Between March 7 and March 12 he delivered the Hitchcock lectures at the University of California at Berkeley. As announced elsewhere, he will speak at the invitation of the program committee at the meeting of this Society in New York City on March 29 on *Modern work in the theory of ordinary trigonometric series*.

Professor Hermann Weyl, of the Zurich Technical School, who has acted as research professor of mathematical physics at Princeton University during the academic year, has been appointed as Visiting Lecturer by the Council of the American Mathematical Society for the year 1928–29. (See this Bulletin, vol. 34, p. 22.) Professor Weyl has already delivered lectures in the Universities of Iowa and Michigan (in December), and at Columbia University (in March); and he will make an extended tour during the month of May, travelling slowly from Princeton westward to California, where he will teach in the Summer Session at the University of California at Berkeley. A detailed schedule of all of his lectures will appear later. As announced elsewhere, he will speak at the invitation of the program committee at the meeting of this Society in New York City on March 30, on *Fournier series and almost periodic functions from the standpoint of the theory of groups*.

Professor Arnold Emch, of the University of Illinois, who has been on leave of absence in Europe during the present academic year, lectured before the Mathematical Colloquium of the University of Zürich on January 28 on the topic *Cremona transformations and algebraic curves and surfaces*.

Professor James Pierpont's address on *Mathematical rigor, past and present* (this Bulletin, vol. 34, pp. 23–53) has been reprinted in the Revista Matemática Hispano-Americana, (2), vol. 3 (1928), Nos. 7–8, under the title *El rigor matemático. Pasado y presente*. The translation is by Alvarez Aguirre.

At the annual meeting of the London Mathematical Society, the following officers were elected: president, Professor E. T. Whittaker; vice-presidents, Mr. R. H. Fowler, Professor E. H. Neville, and Mr. E. C.

Titchmarsh; treasurer, Dr. A. E. Western; librarian, Professor Harold Hilton; secretaries, Professor G. N. Watson and Mr. F. P. White; new members of the Council, Professor Oswald Veblen and Mr. T. L. Wren.

Among the officers elected at the New York meeting of the American Association for the Advancement of Science are the following: president, Professor R. A. Millikan; vice-president of Section A (mathematics), Professor E. T. Bell; Secretary of Section A, Professor C. N. Moore; vice-president of Section K (economic and social sciences), Professor H. L. Rietz; secretary of Section K, Professor C. F. Roos.

Dr. Max Mason, director of the division of natural science of the Rockefeller Institute, and Professor M. I. Pupin, of Columbia University, have been appointed members of the National Research Council's committee to cooperate with the officials of the Chicago World's Fair Centennial Celebration. Mr. F. B. Jewett, vice-president of the American Telephone and Telegraph Company, is chairman of this committee.

Dr. Irving Langmuir, of the General Electric Company, has been elected president of the American Chemical Society.

The Paris Academy of Sciences announces the award of the following prizes for 1928: the Poncelet prize to Gaston Julia, for his mathematical work; the Francœur prize to Szolem Mandelbrojt, for his work in mathematical analysis; the Montyon prize in mechanics to Filippo Burzio, of Turin, for his work in ballistics; the Henri de Parville prize to F. C. Haus, of the Belgian Service Technique de l'Aéronautique, for his *Etude dynamique sur la vrille*; the Valz prize to Georges van Biesbroeck, of the University of Chicago, for his astronomical work; the Janssen medal to William Wright, of the Lick Observatory, for his astronomical work; the Plumey prize to Albert Thuloup, for his memoir entitled *Essai sur la fatigue des tuyaux mince à fibre moyenne plane ou gauche*; the Montyon prize in statistics to Georges Darmois, for his work entitled *Statistique Mathématique*; the Grand Prix for mathematical sciences to Georges Giraud, for his work in partial differential equations; the Vaillant prize to Maurice Fréchet, for his work in abstract sets; the Roux prize to François Divisia, for his work entitled *Economique Rationnelle*; a prize from the Gegner foundation to Maurice Vèzes, for his *Traité de Chimie Physique*; a prize from the Hirn foundation to Maurice Gevrey, for his work in partial differential equations; a prize from the Henri Becquerel foundation to Paul Lévy, for his work in the analysis of functionals.

The Royal Society of London has awarded the following medals: a royal medal to Professor A. S. Eddington, for his contributions to astrophysics; the Sylvester medal to Professor W. H. Young, for his contributions to the theory of functions of a real variable; the Rumford medal to Professor Friedrich Paschen, for his contributions to the knowledge of spectra; the Hughes medal to Maurice de Broglie, for his work on X-ray spectra.

The Buchan prize of the Royal Meteorological Society has been awarded to Dr. Harold Jeffreys, for papers contributed to the Quarterly Journal of

that Society during the years 1924–1927, on cyclones, fluid motions produced by differences of temperature and humidity, dynamics of winds, and related subjects.

The American Mathematical Society has awarded its Bôcher Memorial Prize to Professor J. W. Alexander, of Princeton University, for his memoir entitled *Combinatorial analysis situs*, in volume 28 of the Transactions of this Society. (See the present issue of this Bulletin, p. 162.)

The National Academy of Sciences has awarded its Comstock prize, for the most important research in electricity, magnetism, and radiant energy during the past five years, to C. J. Davisson, of the Bell Telephone Laboratories.

Professor A. H. Compton, of the University of Chicago, has been awarded the gold medal of the Radiological Society of America for his work in X-rays.

The New York Academy of Sciences has awarded its A. Cressy Morrison prize to B. P. Gerasimovic, of the Harvard College Observatory, and D. H. Menzel, of the Lick Observatory, for a thesis on *Subatomic energy and stellar radiation*.

Professor Serge Bernstein has been elected correspondent of the Paris Academy of Sciences, in the section of geometry.

Professor G. D. Birkhoff has been elected correspondent of the Paris Academy of Sciences.

Professor G. Koenigs, of the Sorbonne, has been elected an associate of the Royal Academy of Belgium.

Professor A. Fraenkel, of the University of Kiel, has been appointed head of the mathematical institute of the University of Jerusalem.

Associate Professor Georg Joos has been promoted to a full professorship of theoretical physics at the University of Jena.

Professor Adolf Smekal, of the University of Vienna, has been appointed professor of theoretical physics at the University of Halle.

Dr. F. A. Willers has been appointed professor of mathematics at the School of Mines of Freiberg i. Sa.

The following have been admitted as private docents: Dr. A. Berger, for insurance mathematics, at the University of Vienna; Dr. G. Thomsen, for mathematics, at the University of Hamburg.

Mr. Arthur Berry and Mr. H. W. Richmond have retired from university lecturerships at King's College, Cambridge.

Mr. A. S. Besicovitch has been appointed Cayley lecturer in mathmatics at Cambridge University.

Mr. H. S. Ruse has been appointed lecturer in mathematics at the University of Edinburgh.

Assistant Professor Herman Zanstra, of the University of Washington, has been appointed assistant professor of mathematics at the Imperial College, London.

The Institute of Mathematics and Applied Mathematics of the University of Paris, dedicated to the memory of Henri Poincaré, was formally opened on November 1, 1928; Premier Poincaré presided at the ceremonies. Funds for the erection of the building were provided by the Baron Edmond de Rothschild and the General Education Board, the latter subscribing also an endowment for a chair of applied mathematics. See pages 198–200 of this issue.

A memorial building in honor of the late Dean H. B. Fine, to be known as the Henry Burchard Fine Mathematical Hall, will be erected at Princeton University for the use of the departments of mathematics and mathematical physics. It is the gift of T. D. Jones, of the Princeton class of 1876, and his niece, Miss Gwenthalyn Jones.

The Carnegie Foundation has made a grant of $25,000 to Princeton University for the study of light conditions under which atoms exist.

Dr. P. A. M. Dirac, of St. John's College, Cambridge, will lecture on quantum mechanics at the University of Wisconsin during April and May.

Assistant Professor R. W. Babcock, of the University of Wisconsin, has been appointed as professor and head of the department at De Pauw University, to take effect in September, 1929.

Mr. W. M. Bond has been appointed professor of mathematics at Waynesburg College, Waynesburg, Pa.

Mr. Paul K. Rees, of Texas Technological College, has been appointed assistant professor of mathematics at the University of Mississippi; Mr. Thomas A. Bickerstaff has been appointed to a teaching fellowship at the College.

Professor A. A. Titsworth has retired after forty-two years of service in the mathematical department of Rutgers University.

Adjunct Professor G. T. Whyburn, of the University of Texas, has been appointed assistant professor of mathematics at the Johns Hopkins University, to take effect in September, 1929.

Professor G. H. Bryan, retired, of the University of North Wales, Bangor, died October 13, 1928, at the age of sixty-four.

Dr. J. W. L. Glaisher, F. R. S., senior fellow of Trinity College, Cambridge, died December 7, 1928, at the age of eighty.

Professor T. C. Chamberlin, of the department of geology of the University of Chicago, known for his planetesimal hypothesis, died November 15, 1928, at the age of eighty-five.

The Reverend J. N. Donahue, president of Columbia University, Portland, Ore., died recently.

Dr. Leonard Waldo, astronomer and consulting engineer, died January 25, 1929, at the age of seventy-five.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BAER (R.). Zur Einordnung der Theorie der Mischgruppen in die Gruppentheorie. Berlin, de Gruyter, 1928. 18 pp.

BIANCHI (L.). A Luighi Bianchi nel trigesimo della sua morte, la Scuole Normale Superiore. Pisa, Pacini-Mariotti, 1928.

BUHL (A). Aperçus modernes sur la théorie des groupes continus et finis. (Mémorial des Sciences Mathématiques, No. 33.) Paris, Gauthier-Villars, 1928.

CIANI (E.). Il metodo delle coordinate proiettive omogenee nello studio degli enti algebrici. 2a edizione. Torino, S.T.E.N., 1928. 267 pp.

FORD (W. B.). A first course in the differential and integral calculus. New York, Holt, 1928. 6+372 pp.

FRAENKEL (A.). Einleitung in die Mengenlehre. 3te umgearbeitete und stark erweiterte Auflage. (Die Grundlehren der Mathematischen Wissenschaften, Band 9.) Berlin, Springer, 1928. 14+424 pp.

FRÉCHET (M.). Les espaces abstraits et leur théorie considérée comme introduction à l'analyse générale. Paris, Gauthier-Villars, 1928. 9+296 pp.

FRÉCHET (M.) et ROULLET (—.). Nomographie. Paris, Colin, 1928. 79 pp.

FRICKE (R.). Lehrbuch der Algebra verfasst mit Benutzung von Heinrich Webers gleichnamigen Buche. Band 3: Algebraische Zahlen. Braunschweig, Vieweg, 1928. 8+506 pp.

FÜRSTWEGER (FRANZ) und FÜRSTWEGER (FRIEDRICH). Einführung in Theorie und Praxis der Herstellung von Perspektiven ohne zeichnerische Konstruktion. München, Selbstverlag, 1928. 21 pp.+3 tables.

FÜRSTWEGER (FRIEDRICH). See FÜRSTWEGER (FRANZ).

GAMBIER (B.). Applicabilité des surfaces étudiée au point de vue fini. (Mémorial des Sciences Mathématiques, No. 31.) Paris, Gauthier-Villars, 1928.

GUNN (G. A.). The slide rule. London, Spon, 1928. 40 pp.

HILL (T. H. W.). Elementary analytic geometry. London, Mills and Boon, 1928.

HÖLDER (O.). Die Arithmetik in strenger Begründung. 2te Auflage. Berlin, Springer, 1929. 73 pp.

KAYE (G. R.). The Bakhshāli manuscript. A study in mediaeval mathematics. (Archaeological Survey of India, new series, vol. 43, parts 1–2.) Calcutta, Government of India Central Publication Branch, 1927. 4+156 pp.+48 plates.

LIETZMANN (W.). Lustiges und Merkwürdiges von Zahlen und Formen. 3te durchgesehene und erweiterte Auflage. Breslau, Hirt, 1928. 6+308 pp.

MAENNCHEN (P.). Methodik des mathematischen Unterrichts. Frankfurt a. M., Verlag Diesterweg, 1928. 10+297 pp.

Millot (S.). La balance à calcul. Paris, Gauthier-Villars, 1928. 20 pp.

Miser (W. L.). See Palmer (C. I.).

Moreau (G.). See Russell (B.).

National Research Council. Bulletin No. 62: Algebraic numbers, II. By H. S. Vandiver and G. E. Wahlin. Washington, National Research Council, 1928. 111 pp.

Neumann (F.). Franz Neumanns gesammelte Werke. Herausgegeben von seinen Schülern. Band 1. Leipzig, Teubner, 1928. 10+428 pp.

Palmer (C. I.) and Miser (W. L.). College algebra. New York, McGraw-Hill, 1928. 14+377 pp.

Peters (J.). Sechsstellige Tafel der trigonometrischen Funktionen. Berlin, Dümmlers Verlag, 1929. 293 pp.

Prasad (G.). An introduction to the theory of elliptic functions and higher transcendentals. London, Longmans, 1928.

Riquier (C.). La méthode des fonctions majorantes et les systèmes d'équations aux dérivées partielles. (Mémorial des Sciences Mathématiques, No. 32.) Paris, Gauthier-Villars, 1928.

Roeser (E.). Komplementare Körper der beiden nichteuklidischen Geometrieen. Berlin, de Gruyter, 1928. 10 pp.

Roullet (—.). See Fréchet (M.).

Rowe (C. H.). See Salmon (G.).

Russell (B.). Introduction à la philosophie mathématique. Traduit de l'anglais par M. G. Moreau. Paris, Payot, 1928. 245 pp.

Salet (P.). Omar Khayyam, savant et philosophe. Paris, Librairie Orientale et Américaine, 1927.

Salmon (G.). A treatise on the analytic geometry of three dimensions. 7th edition. Volume 1. Edited by C. H. Rowe. London, Longmans, 1928.

Schuh (F.). Supplement 1927 op het tweede deel van de vraagstukken over differentiaal- en integraalrekening en over analytische en beschrijvende meetkunde. Groningen, Noordhoff, 1927.

Timerding (H. E.). Zeichnèrische Geometrie. Leipzig, Akademische Verlagsgesellschaft, 1928. 10+419 pp.

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Townsend (E. J.). Functions of real variables. New York, Holt, 1928. 10+405 pp.

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Warren (L. A. H.). See Wilson (N. R.).

Weber (H.). See Fricke (R.).

Willers (A.). Methoden der praktischen Analysis. Berlin, de Gruyter, 1928. 344 pp.

Wilson (N. R.) and Warren (L. A. H.). College algebra. New York, Oxford University Press, 1928. 8+451 pp.

PART II. APPLIED MATHEMATICS

Auerbach (F.) und Hort (W.). Handbuch der physikalischen und technischen Mechanik. 1ter Band, 2te Lieferung.

Beaufils (H.). See Harang (F.).

Boswall (R. O.). The theory of film lubrication. London, Longmans, 1928. 11+280 pp.

Bragg (W.). An introduction to crystal analysis. London, Bell, 1928. 7+168 pp.

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Butler (J. A. V.). The fundamentals of chemical thermodynamics. Part 1. London, Macmillan, 1928. 11+207 pp.

Cajori (F.). A history of physics in its elementary branches including the evolution of physical laboratories. Revised and enlarged edition. New York, Macmillan, 1929. 14+424 pp.

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Duhem (P.). L'œuvre scientifique de Pierre Duhem. Paris, Blanchard, 1928. 554 pp.

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Fermi (E.). Introduzione alla fisica atomica. Bologna, Zanichelli, 1928. 330 pp.

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Françon (M.). See Bridgman (P. W.).

Fuchs (F. J.). See Gerlach (W.).

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Gudden (B.). Lichtelektrische Erscheinungen. Berlin, Springer, 1928. 9+325 pp.

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Habibur Rahman Khan. The discovery and formulation of a new theory of the generation and propagation of wireless waves in space. Simla, Thucker Spink, 1928.

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Harang (F.) et Beaufils (H.). Notions élémentaires de géométrie descriptive appliquée au dessin. 5e édition revue et corrigée. Paris, Dunod, 1928. 8+192 pp.

Hartmann (L.). See Ohm (G. S.).

Hatschek (E.). The viscosity of liquids. London, Bell, 1928. 12+239 pp.

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Kistiakowsky (G. B.). Photochemical processes. (American Chemical Society Scientific Monographs.) New York, Chemical Catalogue Company, 1928. 280 pp.

Kneser (A.). Das Prinzip der kleinsten Wirkung von Leibnitz bis zur Gegenwart. Leipzig, Teubner, 1928. 70 pp.

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Pichler (O.). Die Biegung kreissymmetrischer Platten von veränderlicher Dicke. Berlin, Springer, 1928. 60 pp.

Pippard (A. J. S.). Strain energy methods of stress analysis. London, Longmans, 1928. 10+146 pp.

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Salkowski (E.). Grundzüge der darstellende Geometrie. Leipzig, Akademische Verlagsgesellschaft, 1928. 8+100 pp.

Sampson (R. A.). Science and reality. London, Benn, 1928. 80 pp.

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Tweedy (S. K.). See Partington (J. R.).

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Vidmar (J. E.). Vorlesungen über die wissenschaftlichen Grundlagen der Elektrotechnik. Berlin, Springer, 1928. 10+451 pp.

de Villamil (R.). Rational mechanics. Chapters in modern dynamics and energetics. London, Spon, 1928. 224 pp.

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Wellings (A. W). Volumetric analysis. A practical course based on modern theoretical principles. London, Methuen, 1928.

Whitehead (S.). Dielectric phenomena. 2: Electrical discharges in liquids. Edited with a preface by E. B. Wedmore, London. Benn, 1928. 137 pp.

THE FEBRUARY MEETING IN NEW YORK

The two hundred sixty-seventh regular meeting of the Society was held at Columbia University, on Saturday, February 23, 1929, extending through the usual morning and afternoon sessions. The attendance included the following fifty-nine members.

R. L. Anderson, R. G. Archibald, H. E. Arnold, Benton, G. A. Campbell, J. A. Clark, Cramlet, Demos, Doermann, J. E. Donahue, Dresden, Engstrom, Fite, D. A. Flanders, R. M. Foster, Gehman, Gill, Glenn, Gourin, Gronwall, Hazeltine, L. S. Hill, Himwich, E. H. Johnson, M. I. Johnson, Kasner, Kenny, Kline, Koopman, Mark Kormes, Kramer, Marden, Mullins, F. H. Murray, Newman, Parsons, Pfeiffer, Pierpont, Post, Rawles, Reddick, M. S. Rees, R. G. D. Richardson, Ritt, Robertson, Seely, Serghiesco, Simons, Smail, P. A. Smith, J. M. Thomas, J. L. Walsh, Weinstein, Weisner, Whittemore, H. B. Williams, W. J. Willis, W. A. Wilson, Zippin.

There was no meeting of the Council or of the Trustees of the Society.

Associate Secretary Dresden presided at the morning session, and Professor Whittemore in the afternoon.

At the request of the Program Committee, Professor J. L. Walsh, of Harvard University, delivered, at the beginning of the afternoon session, an address entitled *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions*. This address will appear in full in an early issue of this Bulletin.

Titles and abstracts of the other papers read at this meeting follow below. The papers of Beckenbach, Calugaréano, Douglas and Emch were read by title. Mr. Beckenbach was introduced by Professor Bray, and Dr. Calugaréano by Professor Kasner.

1. Professor J. E. Donahue: *Geometric proof of Kasner's pentagon theorem.*

In a recent paper (American Mathematical Monthly, vol. 35 (1928), pp. 352 ff.), Kasner proves (using coordinates) the following theorem: *For*

any plane pentagon P the inscribed pentagon of its diagonal pentagon and the diagonal pentagon of its inscribed pentagon are identical. In the present paper there is given a synthetic proof employing Pascal's line and Brianchon's point.

2. Miss Edna E. Kramer: *The Laguerre group and allied topics.*

It is well known that the Laguerre group can be represented analytically by the linear fractional transformations of the variable $w=u+jv$, where $j^2=0$ and u and v are functions of the Hessian coordinates of a directed line. Using this representation, the present paper deals with the Laguerre group in a manner analogous to that in which the inversion group is often treated. Some new properties of Laguerre inversion are discussed, and a large family of anallagmatic curves is found. The invariants and path-curves of the groups of integral linear transformations are discussed. For $W=w+b$ these curves are cycles (directed circles); for $W=aw$ each curve (non-oriented) is a tractrix. For the direct transformations of the Laguerre group it is found that there may be 0, 2, ∞^1, or $2\infty^1$ fixed rays in the finite plane, and for the indirect transformations, 1, 2, ∞^1 or $2\infty^1$. The transformations resulting from combinations of the integral linear transformations with reciprocation are classified kinematically. In the second part of the paper a theory of polygenic functions of w is developed, analogous to that of Kasner for polygenic functions of z, and points of similarity and divergence between the two theories are noted.

3. Professor Edward Kasner: *The biharmonic functions of Poincaré related to functions of two complex variables.*

Poincaré applies the term biharmonic to those functions of four real variables $F(x, y, x', y')$ which can be regarded as the real part of an analytic function of two complex variables $z=x+iy$, $z'=x'+iy'$. If a conformal substitution $x'=\phi(x, y)$, $y'=\psi(x, y)$ is performed, F obviously becomes a harmonic function of x, y. In the present paper two easy converses of this theorem are proved. (I) Biharmonic functions are the only functions of four real variables which are converted into harmonic functions by *every* conformal substitution. (II) The only substitutions which convert *all* biharmonic functions into harmonic functions are the conformal substitutions. Connection is made with the author's geometric representation of functions of two complex variables, first by points in four dimensions, then by bipoints or point-pairs in the plane. See abstracts in this Bulletin, vol. 15 (1908–09), pp. 67, 159.

4. Mr. Eli Gourin: *On the irreducible polynomials in several variables which become reducible when the variables are replaced by powers of themselves.*

Given an irreducible polynomial $Q(x_1, \cdots, x_p)$, consisting of more than two terms, the sets of positive integers $t_1, \cdots, t_p$ for which $Q(x_1^{t_1}, \cdots, x_p^{t_p})$ is reducible can be grouped into a finite number of classes of sets. In each class, the sets are of the form $a_1T_1, \cdots, a_pT_p$, where the T's are elements of a certain set, having the required property, and the a's are arbitrary

positive integers. Those of the T's which are distinct from unity are each equal to one and the same prime number p. If M denotes the largest among the m_i's, where m_i is the degree of Q in x_i, and if P is the largest prime number which does not exceed M^2, then $p \leqq P$.

5. Professor J. F. Ritt: *Algebraic combinations of exponentials.*

This paper investigates functions defined by an equation

$$(1) \qquad \sum_{i=0}^{n} (a_{0i} + a_{1i}e^{\alpha_1 x} + \cdots + a_{mi}e^{\alpha_m x})y^i = 0,$$

where the α's and a's are any complex constants. The theory of these functions contains in itself the theory of algebraic functions. It is shown that, for large x's, every branch of a function defined by (1) can be represented by Dirichlet series with complex exponents. It is then proved that if (1) defines more than a single analytic function, the first member of (1) can be factored into expressions similar to itself but of lower degree in y. In particular, a uniform function defined by (1) must satisfy an equation of the first degree, and hence is the quotient of two exponential polynomials.

6. Professor J. F. Ritt: *Representation of analytic functions as infinite products.*

Let $f(z)$ be analytic, and equal to unity, at $z=0$. It is proved in this paper that $f(z)$ admits, for the neighborhood of the origin, a representation as an absolutely convergent product $(1+a_1z)(1+a_2z^2)\cdots(1+a_nz^n)\cdots$, with constant a's. The representation is unique.

7. Professor W. A. Wilson: *On linear upper semi-continuous collections of continua.*

Let M be a bounded upper semi-continuous collection of continua $\{X\}$ such that no two have common points and $X=f(t)$ is upper semi-continuous in the interval $a \leqq t \leqq b$; a collection of this kind may be called *linear* and M the *image* of ab. If there is no upper semi-continuous aggregate function $Y=g(t)$, where each Y is a continuum defined in ab such that $g(t) \subset f(t)$ at every point and $g(t) \neq f(t)$ at some point, $f(t)$ is called a minimal upper semi-continuous function. It was shown in a previous communication that, if $f(t)$ is a minimal upper semi-continuous function in ab, M is a bounded plane continuum, and no $f(t)$ separates $f(a)$ from $f(b)$, then M is irreducible between $f(a)$ and $f(b)$. The present paper discusses the case that the set of points $T'=\{t'\}$ for which $f(t')$ separates $f(a)$ from $f(b)$ is not void. It is shown that the conclusion of the quoted theorem remains valid if the set T' is totally disconnected, and that it is sometimes valid and sometimes not if T' contains an interval. Finally, there exist cases where $f(t)$ is a minimal upper semi-continuous function defined in an interval ab and the image of ab fills a portion of the plane.

8. Dr. O. E. Glenn: *A theory of integral invariants.*

The general question studied in this paper is as follows. A curve C: $f(x, y)=0$, is transformed by $x_1=\sigma(x, y)$, $y_1=\tau(x, y)$ into C_1: $g(x_1, y_1)$.

Suppose any function ϕ of x, y, y' to be given. It is required to find the line of integration C such that $\int_\alpha^\beta \phi(x_1, y_1, y_1')dx = M\int_\gamma^\delta \phi(x, y, y')dx$, the line of integration being C_1 in the first integral and C in the second. The limits (α, β) are to be arbitrary, and (γ, δ) are limits transformed from (α, β) by $x_1 = \sigma$, $y_1 = \tau$. We also consider the problem where ϕ is to be determined, C being assigned.

9. Dr. G. Calugaréano: *Polygenic functions considered as integrals of differential equations.*

In this paper, which will appear in the Transactions of this Society, the author applies polygenic (that is, non-analytic) functions to the solution of certain ordinary differential equations of nth order. The first derivative is defined as in Kasner's papers (Science, vol. 66 (1927), p. 981; Proceedings of the National Academy, vol. 13 (1928, p. 75). The second and higher derivatives are the special case which Kasner describes as rectilinear (Transactions of this Society, vol. 30 (1928), pp. 803–818). By means of the polygenic solutions, analytic solutions may also be obtained. The discussion is purely analytic, as in the author's Paris thesis (Nov., 1928).

10. Professor Arnold Emch: *On the mapping of the involutorial G_4 in a plane upon a Steiner surface.*

This paper appears in full in the present issue of this Bulletin.

11. Dr. Jesse Douglas (National Research Fellow): *Solution of the problem of Plateau.*

In a paper presented to the Society, Dec. 28, 1926, the author reduced the problem of Plateau to the integral equation (I): $*\int_\Gamma K(t, \tau)$ ctn $(\frac{1}{2}[\phi(\tau) - \phi(t)])d\tau = 0$ (or its equivalent by the substitution ctn $(\frac{1}{2}\phi) = \psi$: $*\int_\Gamma K(t, \tau)d\tau/[\phi(\tau) - \phi(t)] = 0$). For the notation see the abstract in this Bulletin, vol. 33, p. 143. In a subsequent paper presented April 6, 1928 (see this Bulletin, vol. 34, p. 405) the author showed that (I) expressed the vanishing of the functional derivatives of the functional

$$A(\phi) = -\int_\Gamma\int_\Gamma K(t, \tau) \log \sin \tfrac{1}{2}|(\phi(\tau) - \phi(t)|\, dtd\tau.$$

It thus becomes a question of proving the existence of a minimizing function ϕ for $A(\phi)$. This is accomplished in the present paper. It is shown that $A(\phi)$ is a lower semi-continuous function on the set of all functions ϕ which, by means of $\theta = \phi(t)$, establish a one-one continuous correspondence between the contour Γ and the unit circle C; only these functions are admissible. In order to have a closed set $[\phi]$ it is necessary to include functions ϕ whose cartesian t, θ graph presents (1) horizontal or (2) vertical line segments as well as (3) properly monotonically increasing continuous functions ϕ. The set $[\phi]$ is compact as well as closed, and we can therefore apply the results of Fréchet's thesis (Palermo Rendiconti, vol. 22 (1906)) to affirm the existence of a minimizing ϕ. It is shown that, for a function ϕ of type (1), $A(\phi) = +\infty$, and for one of type (2), $A'(\phi, t)$ cannot be identically zero. Hence the existing minimizing ϕ is of the proper type (3), and the problem is solved.

12. Mr. E. F. Beckenbach: *An inequality for definite Hermitian determinants.*

This paper appears in full in the present issue of this Bulletin.

13. Dr. T. H. Gronwall: *Straight line geodesics in Einstein's parallelism geometry.* Preliminary communication.

This paper gives a set of partial differential equations forming the necessary and sufficient conditions that, in Einstein's parallelism geometry, all geodesics be straight lines and vice versa.

14. Dr. T. H. Rawles: *On the inverse problem in the calculus of variations.*

The problem considered in this paper is that of finding the integrand function which is minimized by a given two-parameter family of extremals. By approaching the problem from the standpoint of the invariant integral a method is obtained which is simpler than that of Darboux in that it is not necessary to solve a partial differential equation.

15. Professor James Pierpont: *Foucault's pendulum in elliptic space.*

The equations of motion are deduced from Hamilton's principle. We find $2\dot{\psi}(\dot{\omega}+kp)+\psi\ddot{\omega}=k^2p'\psi \sin\omega\cos\omega-k^2pp'(r/l)\sin\omega$, where ω is the azimuth of the plane of vibration, ψ the small angle that the pendulum makes with the vertical, k the angular velocity of the earth, ϕ the latitude of the station, $p=\sin\phi$, $p'=\cos\phi$, $l=\sin L$, $r=\sin\rho$, L being the length of the pendulum, ρ being the distance of the point of suspension from the center of the earth, both expressed in elliptic measure. The space constant is taken as unity. The above equation is entirely analogous to the corresponding equation in classical mechanics. Under similar conditions we may say, therefore, that in first approximation the angular velocity of the plane of vibration is $\dot{\omega}=-k\sin\phi$. The author wishes to note that the analysis employed may be readily extended to hyperbolic space.

16. Dr. T. H. Gronwall: *On the differential equation of the vibrating membrane.*

This paper considers the question of the analytic continuation of a solution of the partial differential equation $\nabla^2u+k^2u=0$ (with boundary condition $u=0$ or $\partial u/\partial n=0$) across a straight line part of the boundary. The result is applied to the determination of the frequencies and characteristic functions for a rectangular membrane.

17. Mr. Leo Zippin: *A characterization of the simple closed surface and of the plane.*

It is proved that a continuous curve C, such that (i) any (1, 1) bicontinuous transformation of a simple closed curve of C into itself or another simple closed curve of C can be extended to C, and (ii) there is at least one simple closed curve of C which disconnects it, is a simple closed surface if C is bounded and homeomorphic with the plane if C is unbounded.

R. G. D. Richardson,
Secretary.

MODERN HYDRODYNAMICAL THEORY, WITH SPECIAL REFERENCE TO AERONAUTICS*

BY F. D. MURNAGHAN

A. General Notions and Theorems

1. *Particle and Local Differentiation.* In hydrodynamical theory matter is regarded as distributed continuously, but its various points are supposed to be capable of identification; we shall refer to them as particles. In order to identify a particle, a material reference frame is introduced, and we shall suppose for simplicity that this is a rectangular Cartesian frame. At any convenient initial instant $t=t_0$, the coordinates of a particle may be denoted by (a, b, c); and at any later instant the coordinates of the same particle may be denoted by (x, y, z). We make the assumption that (x, y, z) are differentiable functions of a, b, c, and t. This assumption

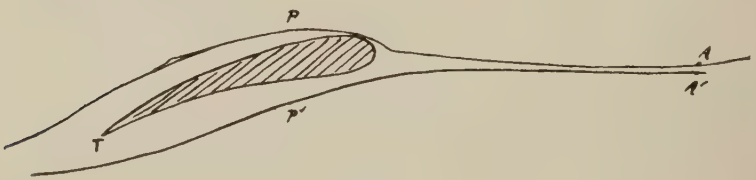

Fig. 1

must be carefully noted since the principal object of interest to us later on will be the flow of a fluid around an obstacle and the fluid will divide at the obstacle. Thus two particles initially adjacent (as A and A' in Fig. 1) will not be adjacent when separated by the obstacle as at P and P'. Our subse-

* This paper is a somewhat formal and detailed presentation of the contents of an address delivered at the invitation of the program committee at the meeting of this Society on October 29, 1927. The adjective modern in the title of the address refers to the first quarter of the present century, but in order to make the paper easily intelligible to mathematical readers who have not specialized in hydrodynamics, I give a statement of the more important results which had been previously obtained.

quent remarks will apply, then, separately to the two portions of the fluid which separate at the obstacle but not to the fluid as a whole. A particle may be identified either by means of (a, b, c, t) which are known as the particle, or Lagrangian, variables or by means of (x, y, z, t) which are known as the local, or Eulerian, variables; we shall denote time differentiation when the particle variables are used by d/dt and when the local variables are used by $\partial/\partial t$. The velocity components $(dx/dt, dy/dt, dz/dt)$ being denoted by (u, v, w), we have the relation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{1}$$

In particular the acceleration components $(du/dt, dv/dt, dw/dt)$ are given, in terms of the local variables, by formulas of the type

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}. \tag{2}$$

The paths of the various particles are found, if (u, v, w) are known in terms of the local variables, by integrating the *three* equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt. \tag{3}$$

At any given instant $t = t_1$ the *stream-lines* are given by integrating the *two* equations

$$\frac{\delta x}{u_1} = \frac{\delta y}{v_1} = \frac{\delta z}{w_1}, \tag{4}$$

where $u_1 = u(x, y, z, t_1)$, and so on. The ∞^3 paths must then be distinguished from the ∞^2 stream-lines. They coincide geometrically when the stream-lines are permanent, that is, when $\partial u/\partial t = 0$, $\partial v/\partial t = 0$, $\partial w/\partial t = 0$ identically in t. In this case the motion is said to be *steady*, and in the applications with which we shall be concerned steady motion is assumed.

2. *The Principle of Conservation of Mass.* This postulates the existence of a positive density function ρ, which we assume to be differentiable, and such that the volume integral $\int\rho\ \delta(x, y, z)$ does not vary with t, the volume of integration consisting of the same particles. In other words $\int\rho\ \delta(x, y, z)$ is an integral invariant of the equations (3). If the initial value of ρ be denoted by ρ_0, the differential form of statement of the principle of conservation is, in terms of the particle variables,

$$\rho\frac{d(x,y,z)}{d(a,b,c)} = \rho_0,$$

and in terms of the local variables,

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0. \tag{5}$$

This equation is known as the equation of continuity and it may be put in the equivalent form

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = 0.$$

For an incompressible fluid $d\rho/dt=0$ and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{6}$$

3. *The Circulation Integral.* With any closed curve C of particles we may associate, at any instant t, the number

$$\Gamma = \oint (u\delta x + v\delta y + w\delta z), \tag{7}$$

where δ indicates the differential with respect to the parameter of the curve. The integral Γ is known as the circulation around the curve; and an application of Stokes' theorem enables us to write it in the equivalent form

$$\Gamma = \int \xi\,\delta(y,z) + \eta\,\delta(z,x) + \zeta\,\delta(x,y), \tag{8}$$

where

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

are the vorticity components and the surface of integration is bounded by C. A simple calculation shows that

$$\frac{d\Gamma}{dt} = \oint \left\{ \left(\frac{\partial}{\partial y}\frac{dw}{dt} - \frac{\partial}{\partial z}\frac{dv}{dt} \right) \delta(y,z) \right. \tag{9}$$

$$\left. + \left(\frac{\partial}{\partial z}\frac{du}{dt} - \frac{\partial}{\partial x}\frac{dw}{dt} \right) \delta(z,x) + \left(\frac{\partial}{\partial x}\frac{dv}{dt} - \frac{\partial}{\partial y}\frac{du}{dt} \right) \delta(x,y) \right\}.$$

Hence the necessary and sufficient conditions that $d\Gamma/dt = 0$ for every closed curve C are

$$\frac{\partial}{\partial y}\frac{dw}{dt} - \frac{\partial}{\partial z}\frac{dv}{dt} = 0,$$

and so forth. In other words, Γ is a (relative) integral invariant of the equations (3) provided that the acceleration components are derivable from a potential function Q, that is, provided that

$$\frac{du}{dt} = -\frac{\partial Q}{\partial x}, \quad \frac{dv}{dt} = -\frac{\partial Q}{\partial y}, \quad \frac{dw}{dt} = -\frac{\partial Q}{\partial z}, \tag{10}$$

and conversely. In our applications this will be the situation. A particular instance of importance occurs when $\Gamma = 0$ throughout a region of the fluid; then the vorticity components (ξ, η, ζ) are zero and the motion is said to be *irrotational* in that region. The velocity components are derivable from a single function ϕ as follows:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}; \tag{11}$$

and ϕ is called the velocity potential.

4. *The Fundamental Equations and the Concept of a Perfect Fluid.* If we focus our attention on a particular point (x, y, z) of the fluid, the orientation of a particular element of area δS through (x, y, z) may be specified by means of the direction cosines (l, m, n) of one of the two directed normals to δS and we may call the side of δS towards which the normal is drawn its positive side. Then the fluid on the negative side of δS exerts on the fluid on the positive side a force whose components may be denoted by $(P_x\,\delta S,\ P_y\,\delta S,\ P_z\,\delta S)$, where (P_x, P_y, P_z) are given in terms of (l, m, n) and nine coefficients $(p_{xx}, p_{xy}, \cdots, p_{zz})$ by the equations

$$\begin{aligned} P_x &= lp_{xx} + mp_{yx} + np_{zx}, \\ P_y &= lp_{xy} + mp_{yy} + np_{zy}, \\ P_z &= lp_{xz} + mp_{yz} + np_{zz}. \end{aligned} \tag{12}$$

The nine coefficients $(p_{xx}, p_{xy}, \cdots, p_{zz})$ constitute what is called the stress or pressure tensor. If the fluid is subjected to a system of forces (X, Y, Z) per unit mass, a simple application of the fact that the total force acting on any portion of the fluid is equal to the time rate of change of its linear momentum yields the equations

$$\rho\frac{du}{dt} = \rho X - \left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right); \tag{13}$$

$$\rho\frac{dv}{dt} = \qquad ; \quad \rho\frac{dw}{dt} = \qquad .$$

In deriving these we have used the fact that if there are surface forces, whose components are (F, G, H) per unit area, acting on the fluid then (see (1)) we have

$$F + lp_{xx} + mp_{yx} + np_{zz} = 0\ ;\ G + \quad = 0\ ;\ H + \quad = 0\ ; \tag{14}$$

where (l, m, n) are the direction cosines of the outward drawn normal. The theorem that the moment of the forces acting on the fluid about any axis is equal to the time rate of change of its angular momentum about this axis yields the fact that the pressure tensor is symmetric, that is,

$$(15) \qquad p_{zy} = p_{yz}, \qquad p_{xz} = p_{zx}, \qquad p_{yx} = p_{xy}.$$

A perfect fluid is by definition one such that the stress across any element is normal to it; it is this quality of inability to withstand tangential stresses that distinguishes fluids from solids. The equations (12) combined with $P_x : P_y : P_z = l : m : n$ for arbitrary values of the ratios $l : m : n$ show that the pressure tensor for a perfect fluid is the *scalar* tensor $(p, 0, 0; 0, p, 0; 0, 0, p)$, where p is the common value of p_{xx}, p_{yy}, p_{zz}. The number p is called the *pressure* of the perfect fluid and it is well to remember that the concept of pressure, as here defined, does not exist for a non-perfect or *viscous* fluid. For a perfect fluid the equations of motion are

$$(16) \qquad \rho\frac{du}{dt} = \rho X - \frac{\partial p}{\partial x}, \quad \rho\frac{dv}{dt} = \rho Y - \frac{\partial p}{\partial y}, \quad \rho\frac{dw}{dt} = \rho Z - \frac{\partial p}{\partial z},$$

to which must be added the boundary conditions

$$(17) \qquad F + lp = 0, \quad G + mp = 0, \quad H + np = 0,$$

where p denotes, in these latter equations, the pressure at the boundary of the fluid. It is clear from (16) that an acceleration potential Q exists for a perfect fluid if (1) (X, Y, Z) are derivable from a potential function V, that is, if

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z},$$

and (2) if

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial P}{\partial x}, \quad \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial P}{\partial y}, \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial P}{\partial z},$$

where P is any function of x, y, z and t. Then $Q = V + P$. We shall be concerned mainly with the case where the fluid is *homogeneous*, that is, $\partial\rho/\partial x = 0$, $\partial\rho/\partial y = 0$, $\partial\rho/\partial z = 0$, and in this case $P = p/\rho$ so that

$$(18) \qquad Q = V + \frac{p}{\rho}.$$

If our first condition is satisfied but not our second we have from (9)

$$\frac{d\Gamma}{dt} = \int\left[\frac{\partial\left(p,\frac{1}{\rho}\right)}{\partial(y,z)}\delta(y,z) + \frac{\partial\left(p,\frac{1}{\rho}\right)}{\partial(z,x)}\delta(z,x)\right.$$

(19)

$$\left. + \frac{\partial\left(p,\frac{1}{\rho}\right)}{\partial(x,y)}\delta(x,y)\right]$$

$$= \int \delta\left(p,\frac{1}{\rho}\right) = \oint_C p\delta\left(\frac{1}{\rho}\right).$$

5. *Bernoulli's Theorem connecting Pressure and Velocity.* When an acceleration potential Q exists the three equations $du/dt = -\partial Q/\partial x$, etc., may be written in the form

$$\frac{\partial u}{\partial t} + \eta w - \zeta v = -\frac{\partial}{\partial x}\left(Q + \frac{1}{2}q^2\right),$$

and so forth, where (ξ, η, ζ) are the vorticity components and $q^2 = u^2+v^2+w^2$ is the square of the velocity. These yield at once the equation

$$\frac{d}{dt}\left(Q + \frac{1}{2}q^2\right) = \frac{\partial Q}{\partial t},$$

so that when Q is free of t, $Q+q^2/2$ is a first integral of the equations (3). In other words $Q+q^2/2$ has a constant value along any one path but it will, in general, vary from path to path. If, however, the motion is *steady* ($\partial u/\partial t = 0$, etc.), and *irrotational* ($\xi = 0$, etc.), we have $\partial(Q+q^2/2)/\partial x = 0$, etc., so that $Q+q^2/2$, being free of t, is a number independent of x, y, z and t. In the application with which we shall be concerned we shall have $V = 0$, $P = p/\rho$, so that $Q = V+P = p/\rho$, with ρ a numerical constant. Hence we may write

(20) $$p + \tfrac{1}{2}\rho q^2 = p_0\,; \quad \text{or} \quad p = p_0 - \tfrac{1}{2}\rho q^2,$$

where p_0 is a constant throughout the fluid. Since p_0 is

what p becomes wherever $q=0$, it is called the *static* pressure. The additional term $-\rho q^2/2$ is called the *dynamic* pressure.

6. *The Forces and Turning Moments Experienced by Solids in the Fluid.* The equations (16) may be written in the form

$$(21)\quad \rho X - \frac{\partial p}{\partial x} = \rho\frac{du}{dt} = \rho\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw),$$

and so forth, on using (5). Upon integrating this equation throughout the region of the fluid between the immersed solid, or solids, S and any convenient outer boundary surface S', and using the fact that at any point of the surface of a solid the velocity of the fluid is tangential to the surface, we find that the force components (F_x, F_y, F_z) exerted by the fluid on the immersed solids are given by equations of the form

$$(22)\quad F_x = \int \rho X d\tau - \int \frac{\partial}{\partial t}(\rho u) d\tau - \int \rho u(lu + mv + nw) dS' - \int p l dS',$$

where (l, m, n) are the direction cosines of the normal to dS', drawn away from the fluid. In the cases to which we shall apply this result we shall assume the applied mass forces negligible ($X=0$, etc.), the motion steady ($\partial u/\partial t = 0$, etc.) and irrotational (so that (20) is applicable) and the fluid is compressible and homogeneous; we find then

$$(23)\quad F_x = \frac{\rho}{2}\int \{l(u^2 + v^2 + w^2) - 2u(lu + mv + nw)\} dS';$$

$$F_y = \qquad ; \quad F_z = \qquad .$$

In a similar way, the moments (M_x, M_y, M_z) about the coordinate axes of the forces exerted by the fluid on the immersed obstacle are given by equations of the form

$$(24)\quad M_x = \frac{\rho}{2}\int \{(yn - zm)(u^2 + v^2 + w^2) - 2(yw - zv)(lu + mv + nw)\} dS',$$

and so forth. It is with these forces and moments that we shall be mainly concerned and the significance of the formulas given lies in the fact that the desired values are obtained by integration over an *arbitrary* surface S' enclosing the obstacles. It will be usually convenient, when the fluid is supposed infinite in extent, to regard S' as receding to infinity and we see that a knowledge of the nature of the flow at infinity suffices to determine the forces exerted on the solids by the fluid and also the moments of these forces.

B. Two-Dimensional Irrotational Motion

7. *The Complex Velocity and Potential.* In many applications we may assume, at any rate to a first approximation, that the velocity of each particle is parallel to a given plane and independent of its distance from it. Taking the given plane as our (x, y) plane we have $w=0$ and (y, v) independent of z. We shall find it convenient to introduce the complex variable $x+iy$ and as we shall not have occasion to use the symbol z in this section in its previous sense we shall denote $x+iy$ by z. Assuming the fluid to be *incompressible* we have, from (6), the relation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{25}$$

and if we further assume the motion to be irrotational we have

$$\zeta \equiv \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0. \tag{26}$$

The equations (25) and (26) are equivalent to the statement that $u-iv$ is a function of z. We shall denote this function by $w(z)$, since there is no possibility of confusion with the third velocity component which is now everywhere zero; $w(z)$ is called the *complex velocity*. Upon introducing the function $f(z)$ defined, save for an additive constant, by the equation

$$\frac{df}{dz} = w(z), \tag{27}$$

and denoting the real and imaginary parts of f by ϕ and ψ, respectively, we have, from (27),

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{28}$$

Hence ϕ is the velocity potential and the level curves $\psi = \text{constant}$ are the stream-lines. The function f is called the complex potential. The equations (28) tell us that the resolved part v_s of the velocity vector in any direction s is given by

$$v_s = \frac{\partial \phi}{\partial s} = \frac{\partial \psi}{\partial\left(s + \frac{\pi}{2}\right)}. \tag{29}$$

Once the complex potential f is known the velocity components (u, v) follow upon differentiation and from these the pressure at any point may be calculated by means of (20), the motion being assumed to be steady.

8. *The Flow around a Circular Cylinder.* We shall suppose that the cylinder, of radius a and with generators perpendicular to the (x, y) plane, has a uniform velocity of translation V making an angle $-\alpha$ with the positive x axis. The flow relative to the cylinder will, therefore, have a velocity V making an angle $\pi - \alpha$ with the positive x axis at points remote from the cylindrical obstacle. In other words the complex velocity w will have the value $Ve^{i(\pi+\alpha)} = -Ve^{i\alpha}$ at $z = \infty$ so that, at $z = \infty$, f will have the form $-Ve^{i\alpha}z$ + terms whose derivatives vanish at $z = \infty$. Since the circle in which the cylinder intersects the (x, y) plane must be a portion of a stream line, ψ must be constant along it and this constant may be taken to be zero since f is undefined to the extent of an additive constant; hence the correspondence set up between points in the z plane and points of the

$f=\phi+i\psi$ plane is such that points on the circle in the z plane correspond to points on the axis of reals in the f plane. If the center of the circle in the z plane be denoted by z_0 and if $z_0 \pm ae^{i\delta}$ are two diametrically opposite points on the circle the relation

$$\frac{f-f_0}{f+f_0}=C\left(\frac{z-z_0-ae^{i\delta}}{z-z_0+ae^{i\delta}}\right)^2$$

will set up a correspondence of the kind desired, the ends of the segment of the real axis in the f plane being denoted by $\pm f_0$ and C being a constant. In order that f may have the proper form, $f=-Ve^{i\alpha}z+\cdots$, at ∞, it is necessary that $C=1$ and $\delta=\pi-\alpha$, $f_0=2aV$. The terms in f whose derivatives vanish at $z=\infty$ may include, since f need not be uniform owing to the fact that the portion of the z plane outside the circle is not simply connected, a term of the form $A \log (z-z_0)$; for if A is a pure imaginary, the imaginary part of $A \log (z-z_0)$ is constant as we go along the circle in the z plane. If we write $A=-i\Gamma/(2\pi)$, the quantity Γ will be the circulation around any closed curve in the (x, y) plane which enlaces the cylinder once and we have

$$(30)\quad f=-V\left[(z-z_0)e^{i\alpha}+\frac{a^2e^{-i\alpha}}{z-z_0}\right]-\frac{i\Gamma}{2\pi}\log(z-z_0).$$

We shall see in the next paragraph how Γ is determined for obstacles with a sharp trailing edge. It was precisely this determination of Γ which constituted the first significant result belonging to what we have termed the modern period.

On differentiating f it is seen that there are two points where w is zero. If $\Gamma \leqq 4\pi aV$, these two points are on the circle and are symmetrically situated with respect to the diameter of the circle which is perpendicular to the direction of flight. The two points are known as *rest points* and they are distinguished from one another by the terms *front* and *rear*. If the argument of the rear rest-point be denoted by $\pi+\beta$ we have

$$\Gamma = 4\pi a V \sin(\alpha + \beta). \tag{31}$$

9. *Flow around an Airfoil.* If we have an obstacle in the z plane which can be mapped on a circle in an auxiliary ζ plane by means of a relationship of the type

$$\zeta = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \tag{32}$$

which makes the points at ∞ in the two planes correspond to one another, it is clear from (30) that the desired complex potential is given by

$$f = -V\left[(\zeta - \zeta_0)e^{i\alpha} + \frac{a^2 e^{-i\alpha}}{\zeta - \zeta_0}\right] - \frac{i\Gamma}{2\pi}\log(\zeta - \zeta_0), \tag{33}$$

where ζ is defined by (32), a is the radius, and ζ_0 the center of the circle in the auxiliary plane. We shall be interested mainly in obstacles whose contour is of the type illustrated in Fig. 1; these have a cusp or sharp double point at the trailing edge T and so $dz/d\zeta$ will be zero there. Now the complex velocity

$$w = \frac{df}{dz} = \frac{df}{d\zeta} : \frac{dz}{d\zeta},$$

and this will be infinite at the trailing edge unless $df/d\zeta$ is zero there. This condition suffices to determine the unknown circulation Γ. Physically it amounts to saying that for a stream-lined airfoil the flow is smooth at the trailing edge. Geometrically stated, the rear rest-point on the circle in the auxiliary ζ plane must be mapped on the trailing edge of the obstacle in the z plane.

10. *The Blasius Formulas for Lift and Turning-Moment.* If ds' denote the element of arc of any curve C' enclosing the contour of the obstacle, the components of the force *per unit length* exerted on the obstacle by the fluid are, from (23),

$$F_x = \frac{\rho}{2} \oint_{C'} \{l(u^2 + v^2) - 2u(lu + mv)\} ds',$$

$$F_y = \frac{\rho}{2} \oint_{C'} \{m(u^2 + v^2) - 2v(lu + mv)\} ds',$$

where (l, m) are the (x, y) direction cosines of the outward drawn normal to C'. Since $l = dy'/ds'$ and $m = -dx'/ds'$, we readily find that

$$\overline{F} \equiv F_x - iF_y = \frac{i\rho}{2} \oint_{C'} w^2 dz. \tag{34}$$

This important formula, due to Blasius, enables us to write down at once F_x and F_y when the development of w^2 near $z = \infty$ is known.

Similarly the turning moment M, about an axis through the origin perpendicular to the (x, y) plane, of the forces exerted by the fluid on a unit length of the cylinder, being equal to

$$\frac{\rho}{2} \oint_{C'} \{(xm - yl)(u^2 + v^2) - 2(xv - yu)(lu + mv)\} ds',$$

is seen to be the real part of

$$-\frac{\rho}{2} \oint_{C'} zw^2 dz,$$

that is,

$$M = \text{R.P.} \left(-\frac{\rho}{2} \oint_{C'} zw^2 dz\right). \tag{35}$$

In order to apply these results we turn to (32) and (33). Thus, from

$$\zeta = z\left(1 + \frac{a_1}{z^2} + \cdots\right),$$

we find

$$\frac{1}{\zeta} = \frac{1}{z} - \frac{a_1}{z^3} + \cdots ; \quad \frac{1}{\zeta^2} = \frac{1}{z^2} - \frac{2a_1}{z^4} + \cdots ,$$

whence

$$\frac{df}{d\zeta} = - Ve^{i\alpha} - \frac{i\Gamma}{2\pi z} + \left(Va^2 e^{-i\alpha} - \frac{i\Gamma\zeta_0}{2\pi} \right) \frac{1}{z^2} + \cdots .$$

Multipliying this by

$$\frac{d\zeta}{dz} = 1 - \frac{a_1}{z^2} + \cdots ,$$

we find

$$w = - Ve^{i\alpha} - \frac{i\Gamma}{2\pi z} + \left(Va^2 e^{-i\alpha} - \frac{i\Gamma\zeta_0}{2\pi} + a_1 Ve^{i\alpha} \right) \frac{1}{z^2} + \cdots ,$$

$$w^2 = V^2 e^{2i\alpha} + \frac{iV\Gamma e^{i\alpha}}{\pi z} - \left(\frac{\Gamma^2}{4\pi^2} + 2V^2 a^2 - \frac{i\Gamma\zeta_0 Ve^{i\alpha}}{\pi} + 2a_1 V^2 e^{2i\alpha} \right) \frac{1}{z^2} + \cdots .$$

From this expression it follows, by (34), that

$$\begin{aligned} \overline{F} &\equiv F_x - iF_y = - i\rho V\Gamma e^{i\alpha} , \\ F &\equiv F_x + iF_y = i\rho V\Gamma e^{-i\alpha} . \end{aligned} \tag{36}$$

In other words *the force per unit length exerted by the fluid on the cylindrical obstacle has the magnitude* $\pm\rho V\Gamma$ *and is perpendicular to the direction of flight*, the sense of rotation from the direction of flight to the direction of the force being that of the circulation Γ. See Fig. 2. This force per unit length is called the *lift* per unit length and is denoted by L, so that we have the Kutta-Joukowski formula

$$L = \rho V\Gamma . \tag{37}$$

If the potential function f were assumed uniform, Γ would be zero and there would be no force experienced by the obstacle; a result known as d'Alembert's paradox. The component of the force in the direction opposite to the direction of flight is called the *drag* and the drag per unit length is de-

noted by D. We see, then, that the present theory gives $D=0$, which is not in accord with experience. It will appear in the next section that when account is taken of the fact that the length of the obstacle is not infinite this discrepancy can be removed.

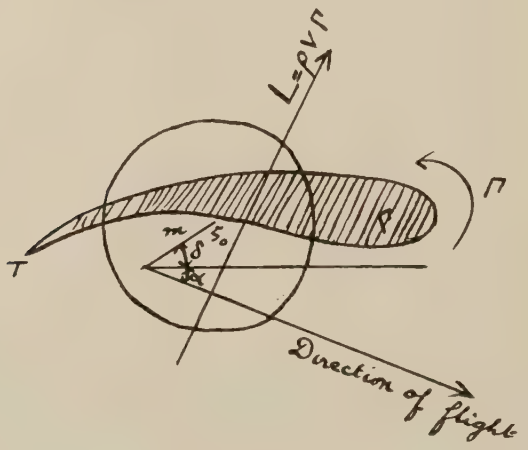

FIG. 2

If we introduce the notation $\zeta_0=me^{i\delta}$, $a_1=-b^2e^{2i\gamma}$, where m, δ, b and γ are all real, it follows that

$$M = \rho V\Gamma m\cos(\alpha+\delta) + 2\pi\rho b^2V^2\sin 2(\alpha+\gamma),$$

and hence the turning moment about the center $\zeta_0=0$ of the circle in the auxiliary ζ plane, the two planes being supposed superposed, is

$$M_0 = 2\pi\rho b^2V^2\sin 2(\alpha+\gamma). \tag{38}$$

11. *The Circular Arc and Straight Line Profile.* The relationship $((\zeta-l)/(\zeta+l))^2=A(z-c/2)/(z+c/2)$, where A is any constant, maps a doubly covered circular arc through the points $\pm c/2$ in the z plane on a circle through the points $\pm l$ in the ζ plane. In order that the mapping may be of the type (32) we must have $A=1$, $l=c/4$ and the mapping is

$$z=\zeta+\frac{c^2}{16\zeta}, \qquad \zeta=z-\frac{c^2}{16z}+\cdots, \tag{39}$$

it being necessary to take that one of the two determinations of ζ which tends to ∞ with z. If we denote by 2ϵ the angle of the circular arc, we have

$$(40)\quad \zeta_0 = il \operatorname{ctn} \epsilon = \frac{ic}{4} \operatorname{ctn} \epsilon, \qquad a = l \csc \epsilon = \frac{c}{4} \csc \epsilon,$$

$$\beta = \frac{\pi}{2} - \epsilon, \qquad b^2 = \frac{c^2}{16}, \qquad \gamma = 0,$$

so that, by (31)

$$(41)\qquad \Gamma = \pi c V \csc \epsilon \cos (\alpha - \epsilon),$$

$$(42)\quad L = \pi \rho V^2 c \csc \epsilon \cos (\alpha - \epsilon), \qquad M_0 = (\pi \rho c^2 V^2 \sin 2\alpha)/8.$$

The distance h of the lifting force $L = \rho V \Gamma$ from the center of the auxiliary circle is

$$h = M_0/L = c \sin 2\alpha \sin \epsilon / [8 \cos(\alpha - \epsilon)].$$

The particular case where $\epsilon = \pi/2$ is very important. Here we are dealing with a plane wing flying at an *angle of attack* α and we find

$$(43)\quad \Gamma = \pi c V \sin \alpha, \qquad L = \pi c \rho V^2 \sin \alpha, \qquad h = c \cos \alpha/4.$$

The number c is the length of the chord of the wing and the expression for h indicates that the point where the line of action of L meets the chord, that is, the center of pressure, does not vary with α but remains constantly one-quarter of the length of the chord from the leading edge; for ζ_0 is now the middle point of the chord. The number $c_L = L/(\frac{1}{2}\rho V^2 c)$ is called the *lift coefficient* c_L and we have the result that for a plane wing

$$(44)\qquad c_L = 2\pi \sin \alpha.$$

For small angles of attack we may put

$$(45)\qquad c_L = 2\pi\alpha.$$

so that there is a linear relation connecting c_L and α. This prediction is verified by experience for angles of attack up to about 15°; but the factor of proportionality 2π is found to be about 40% too large. The reason for this discrepancy is that the wing is not infinite in length and a corrected expression will be given in the next section.

12. *Joukowsky Profiles and their Extensions.* The relationship between z and ζ which is expressed by (39) maps the exterior of a circle through the points $\pm c/4$ in the ζ plane upon the exterior of a doubly covered circular arc, with ends at the points $\pm c/2$, in the z plane. A profile suitable for airfoils is obtained if the same mapping relationship is applied to a circle through the point $-c/4$ but *not* through the point $+c/4$ in the ζ plane. If the circle does not differ very much from the circle through the points $\pm c/4$ which maps into the doubly covered circular arc, the contour will have a rounded leading edge and a sharp trailing edge. A linear fractional substitution on a complex variable being resolvable into a translation, rotation, reflexion in a line and an inversion may be called a generalized inversion and it is evident that the mapping (39) may be written as

$$z' = \zeta'^2, \quad z' = \frac{z - \frac{c}{2}}{z + \frac{c}{2}}, \quad \zeta' = \frac{\zeta - \frac{c}{4}}{\zeta + \frac{c}{4}}. \tag{46}$$

Now a generalized inversion sends circles into circles (straight lines being regarded as circles through the point at infinity). The generalized inversion in the ζ plane, written down in (46), sends any circle through $-c/4$ in this plane into a straight line and this is sent by the square-mapping $z' = \zeta'^2$ into a parabola in the z' plane. The profile obtained is then the generalized inverse of a parabola. Such profiles are known as Joukowsky profiles. Generalizations are obtained by writing, instead of (46),

$$(47) \qquad z' = \zeta'^n, \quad z' = \frac{z - \frac{c}{n}}{z + \frac{c}{n}}, \quad \zeta' = \frac{\zeta - \frac{c}{2n}}{\zeta + \frac{c}{2n}},$$

which is of the form (32) near $z = \infty$. This sends the exterior of a circle through the points $\pm c/(2n)$ in the ζ plane into the exterior of a region bounded by two circular arcs through the points $\pm c/n$ in the z plane, the angle between the two arcs being $(2-n)\pi$. By taking a circle in the ζ plane through $-c/(2n)$ but not exactly through $+c/(2n)$ we obtain a profile with a rounded leading edge and a sharp trailing edge; it has the crescent formed by the two circular arcs as a "skeleton" just as the Joukowsky profile has the doubly covered circular arc as skeleton. The essential difference is that the trailing edge is now a double point of the contour and not a cusp as in the case of a Joukowsky profile. The new profiles are obtained by applying a generalized inversion to the curves obtained by applying the nth power mapping $z' = \zeta'^n$ to a straight line (passing nearly through the origin) in the ζ' plane.

C. The Induced Drag of a Finite Airfoil

13. *The Formulas for the Lift and Drag.* It has been seen in the previous section that when the length of the airfoil is assumed infinite it experiences, owing to the circulation around it, a force at right angles to the direction of flight; in other words there is a lift but no drag. In order to give an explanation of this discrepancy between theory and experiment Prandtl advanced the idea that since an actual airfoil is not infinite in length the circulation around it may vary from point to point along its length. Under this hypothesis the flow loses at once its two-dimensional character but it is assumed that the results obtained for the infinite wing may be used as first approximations; the fact that the circulation is not constant along the span of the airfoil introduces a correction to this approximation. To get an

estimate of the nature and amount of this correction we shall consider the case of a plane wing whose chord c is small in comparison with its length or *span* s and we shall suppose the airfoil sufficiently represented by its trailing edge (it is then called a lifting line). If we introduce a coordinate system with its origin at the middle of the lifting line (see Fig. 3), we may draw the x axis in the direction of flight, the z axis vertically downwards and the y axis along the lifting line. The circulation around the wing in any plane perpendicular to it is assumed to be a function $\Gamma(y)$

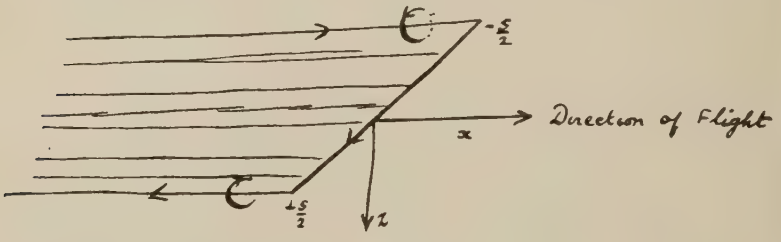

FIG. 3

of y. If then we consider the ring-shaped area bounded by two closed curves enlacing the wing, one in the plane y and the other in the plane $y+\Delta y$, there will be a flux of vorticity through this area, outwards from the wing, of amount $-(d\Gamma/dy)\Delta y$. (See (8).) We assume that this flux is concentrated at the trailing edge and is in the direction of the negative x axis, that is, is in the direction opposite to the direction of flight. Hence there will be a sheet of vortices trailing off from the trailing edge of the wing such that the strength of the vortices leaving an element Δy is $-(d\Gamma/dy)\Delta y$. Now if we have a single rectilinear vortex filament of strength μ in an indefinitely extended fluid which is at rest at infinity, the velocity distribution must be symmetrical around it and hence at a distance r from it the velocity must be of magnitude $\mu/(2\pi r)$. We *assume* that for a *semi*-infinite filament extending, as in the present case, along the negative x axis, the magnitude of the velocity at a point along the y axis distant r from the filament will be $\mu/(4\pi r)$. The direction

of the velocity will be that of the positive or negative z axis. This argument leads to the conclusion that the presence of the semi-infinite strip of trailing vortices induces at any point y of the lifting line a velocity $w(y)$ in the direction of the z axis, that is downwards, of amount

$$w(y) = \frac{1}{4\pi} \int_{-s/2}^{s/2} \frac{(d\Gamma/dy)_{y=y'}}{y - y'} dy'. \tag{48}$$

This assumption of an induced *downwash* is the cardinal one of Prandtl's theory for the finite wing. For an infinite plane wing flying at a small angle of attack α with uniform velocity V an easy evaluation of the indeterminate form df/dz shows that the relative velocity of the wind at the trailing edge has the magnitude $V \cos \alpha$ and the direction of the negative chord of the wing and for small angles of attack α this can be regarded as of magnitude V in the direction opposite to the direction of flight. The net effect of compounding with this relative velocity V the downwash w is to rotate the relative velocity through the angle $\tan^{-1}(w/V)$ and since w is merely a small corrective term this can be equated to w/V. This rotation of the relative wind, due to the sheet of trailing vortices, may be regarded as equivalent to a rotation of the direction of flight through the same angle, that is to a diminution of the angle of attack (which is the angle from the direction of flight to the chord of the wing). The diminished angle α_e is called the *effective* angle of attack (as opposed to the *geometric* angle of attack α_g) and we have the equation

$$\alpha_e = \alpha_g - \frac{w}{V}. \tag{49}$$

On making the assumption that we can apply the results developed for the infinite wing, once we have replaced the geometric angle of attack by the effective angle of attack, we find that the lift per unit length L' at any point y on the span of the wing is given by

$$2\pi c \cdot \frac{1}{2}\rho V^2 \cdot \left(\alpha_g - \frac{w}{V}\right),$$

and on using $L' = \rho V \Gamma$ and equation (48), we have the equation

$$\Gamma(y) = \pi c V \alpha_g - \frac{c}{4} \int_{-s/2}^{s/2} \frac{(d\Gamma/dy)_{y=y'}}{y - y'} dy' \tag{50}$$

for the unknown function $\Gamma(y)$. For a given airfoil both c and α_g will be given functions of y; if α_g is not constant the airfoil is said to be twisted. It is apparent from (49) that even when α_g is constant α_e may vary owing to the possibility of a variable downwash $w(y)$. This variability of α_e is known as aerodynamic twisting of the airfoil.

Since the lift experienced by an infinite wing is perpendicular to the direction of flight and since the effect of the downwash is to decrease, in effect, the angle of attack by w/V, the direction of the force at any point of the wing will be rotated through an angle w/V from the direction of lift. It is this rotation of the direction of the force which introduces a component in the direction opposite to that of flight. Once $\Gamma(y)$ is determined from (50), the total lift L and drag D experienced by the wing are given by the expressions

$$L = \rho V \int_{-s/2}^{s/2} \Gamma dy, \qquad D = \rho \int_{-s/2}^{s/2} w \Gamma dy, \tag{51}$$

(where we have identified the sine of w/V with w/V). The drag D given by (51) is known as *induced* drag, it being a matter of experience that it is too small by a certain amount depending on the shape of the airfoil and known as the profile drag.

14. *The Solution of the Integral Equation and the Determination of the Airfoil of Minimum Induced Drag having a Given Lift.* In order to obtain a numerical solution of (50) we make the substitution $y = -(s/2)\cos\theta$, so that θ runs from 0 to π along the span of the wing and imagine $\Gamma(y) = F(\theta)$ developed

in a Fourier sine series (in order that there may be no discontinuity in $\Gamma(y)$ at the wing tips, $y = \pm s/2$, we must have $\Gamma(\pm y/2) = 0$). Writing this series as $A_1 \sin\theta + A_2 \sin 2\theta + \cdots$, we see from the fact that $\Gamma(y)$ is symmetric about the middle of the wing that $A_2 = A_4 = A_6 = \cdots = 0$. In actual calculations it will be sufficient to consider the four terms $A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta + A_7 \sin 7\theta$ in the development of $\Gamma(y)$. A simple substitution in (50) and a knowledge of the fact that

$$\int_0^\pi \frac{\cos n\phi\, d\phi}{\cos\phi - \cos\theta} = \pi \frac{\sin n\theta}{\sin\theta}$$

gives the equation

$$(52) \qquad \sum_1^\infty A_n \sin n\theta \left\{ \sin\theta + n\frac{\pi c}{2s} \right\} = \pi c V \alpha_g \sin\theta$$

for the determination of the coefficients A_n. On substituting the values $\theta = \pi/8$, $\theta = \pi/4$, $\theta = 3\pi/8$, $\theta = \pi/2$, in this and dropping the terms after A_7 we have four linear equations for the determination of the four coefficients A_1, A_3, A_5, and A_7.

The coefficients A_n being supposed known, the expressions for the lift L and the drag D in terms of them follow readily from the formulas (51). We find

$$(53) \qquad L = \rho V s A_1/4, \quad D = \frac{\pi\rho}{8} \sum_1^\infty n A_n^2,$$

so that the lift depends only on the first term in the Fourier development of $\Gamma(y) \equiv F(\theta)$. The downwash w is given by

$$(54) \qquad w(y) = \frac{1}{2s} \sum_1^\infty \frac{n A_n \sin n\theta}{\sin\theta},$$

and the drag D by

$$(55) \qquad D = \frac{\pi\rho}{8} \sum_1^\infty n A_n^2.$$

It follows that D is a minimum, for a given value of L, when

the remaining coefficients $A_3, A_5, \cdots$ after A_1 in the Fourier development of $F(\theta)$, are zero. In this case the downwash has the constant value $A_1/(2s)$ and there is no aerodynamic twisting of the wing. Since

$$\Gamma(y) = A_1 \sin\theta = A_1\left(1 - \frac{4y^2}{s^2}\right)^{1/2},$$

the graph of $\Gamma(y)$ against y is an ellipse and the wing is said to be elliptically loaded. Since for an infinite plane wing the lift per unit length is, for a given angle of attack, proportional to the chord c (see (43)), we may surmise that an elliptically loaded wing would be approximated to by one whose profile is approximately elliptical. A criterion of good design of an airfoil is that, for a given L, D should be as small as possible and so for a well designed airfoil we may assume the loading approximately elliptical.

15. *Comparison with Experiment.* For a finite wing having a constant chord c the ratio of the span s to the chord c is known as the aspect ratio. For a plane wing whose chord is not constant we first define the mean chord c as the quotient of the area S of the wing by its span s; then the aspect ratio A is the ratio of the span of the wing to its mean chord. Hence $A = s^2/S$. The lift coefficient c_L of the wing is defined by the equation

$$c_L = L/(\tfrac{1}{2}\rho V^2 S), \tag{56}$$

and there is a similar equation defining the drag coefficient. For an elliptically loaded wing we have, by (53) and (56),

$$c_L = \frac{\pi A}{2V_s} A_1, \tag{57}$$

while the downwash w has the constant value

$$w = \frac{A_1}{2s} = \frac{V}{\pi A} c_L. \tag{58}$$

Hence the diminution in the angle of attack, induced by the

trailing vortices, is $c_L/(\pi A)$. The total lift L of the wing is

$$2\pi\alpha_e \cdot \tfrac{1}{2}\rho V^2 \cdot \int_{-s/2}^{s/2} c\,dy = 2\pi\alpha_e \cdot \tfrac{1}{2}\rho V^2 \cdot S,$$

so that $c_L = 2\pi\alpha_e$ and we have

$$\alpha_g = \alpha_e + \frac{c_L}{\pi A} = \alpha_e\left(1 + \frac{2}{A}\right). \tag{59}$$

The aspect ratio of a wing is usually about 6, so that the lift formula for the infinite wing (using α_g) might be expected to give a result which is too large in the ratio 4:3. This is confirmed by experience.

The drag coefficient c_D has, for an elliptically loaded wing, the value $\pi\rho A_1^2/8$ (by (55)) so that, by (57),

$$c_D = c_L^2/(\pi A). \tag{60}$$

Hence a curve plotting D against L should be parabolic and this is verified experimentally; the actual curve obtained empirically indicates a relationship of the type

$$c_D = c_{D_0} + c_L^2/(\pi A), \tag{61}$$

and c_{D_0} is called the coefficient of profile drag. The formulas (59) and (61) are of importance because they enable us to calculate the effect of different aspect ratios; thus for two airfoils having the same lift coefficient

$$\alpha_g' - \alpha_g = \frac{c_L}{\pi}\left(\frac{1}{A'} - \frac{1}{A}\right), \quad c_D' - c_D = \frac{c_L^2}{\pi}\left(\frac{1}{A'} - \frac{1}{A}\right). \tag{62}$$

D. Concluding Remarks

In the present address, I have tried to present in an elementary way some of those fundamental principles of modern hydrodynamical theory, which have proved useful in aeronautics, in the hope of interesting some of our hearers and readers in these questions. It would be easy to point out many unsolved problems and indeed a casual reading of the previous sections will show that even in the most elementary

problems assumptions have continually to be made the legitimacy of which is at least not proved. We append a bibliography which has been deliberately restricted to those works which we would recommend to a student wishing to orient himself in the subject and to reach research problems as soon as possible.

TEXTBOOKS

I. H. Lamb, *Hydrodynamics*. 5th ed. Cambridge University Press. The standard text on hydrodynamics.

II. H. Glauert, *The Elements of Aerofoil and Airscrew Theory*. Cambridge University Press. A fine and elementary introduction to the modern problems and methods.

III. P. Appell, *Traité de Mécanique rationnelle*, vol. 3. Gauthier Villars.

IV. L. Bairstow, *Applied Aerodynamics*. Longmans-Green.

V. R. von Mises, *Fluglehre*. Julius Springer.

These two texts are less mathematical and more physical than the other texts and are particularly valuable as complementing them.

VI. Fuchs-Hopf, *Aerodynamik*. R. C. Schmidt. This is volume 2 of a *Handbuch der Flugzeugkund*, and is very complete.

VII. A. Betz, *Tragflügel und Hydraulische Maschinen*. Handbuch der Physik, vol. 7, pp. 215–288.

VIII. Th. v. Karman and T. Levi-Civita, *Hydro- und Aerodynamik*. J. Springer.

Many papers of interest will be found in the volumes of the Zeitschrift für Angewandte Mathematik and in the publications of the National Advisory Commitee for Aeronautics. Of these latter the following may be especially recommended:

IX. N.A.C.A. Report No. 116 (Prandtl). Government Printing Office, Washington, D.C.

X. N.A.C.A. Report No. 213 (Ames). Government Printing Office, Washington, D.C.

The Johns Hopkins University

THE EQUATION OF THE V_{n-1}^{n-1} IN S_n*

BY C. A. RUPP

Segre, dealing with synthetic and enumerative geometry, has twice mentioned† that the manifold formed by all the lines of S_n which meet n given S_{n-2}'s is a V_{n-1}^{n-1}. In a recent article, Wong‡ mentions that the equation of the general V_{n-1}^{n-1} is unknown. It is the purpose of this paper to exhibit the equation.

The method is the following: through a general point of one of a set of n given S_{n-2}'s is passed a line which is required to meet each of the remaining $(n-1)$ S_{n-2}'s. The eliminant of the system of equations thus set up is the desired equation, as will shortly appear.

Let the equations of the n given S_{n-2}'s be

$$x_i = 0 = \sum_{j=0}^{j=n} a_{ij}x_j, \qquad (i = 1,2, \cdots, n), \qquad (a_{ii} = 0).$$

Berzolari§ has shown that the above display does not particularize the set of S_{n-2}'s, but is a mere question of a suitable choice of the reference system. We shall need the Grassmann-Plücker coordinates of the S_{n-2}'s, that is, the two-rowed determinants from the matrices of the coefficients in their equations. In each set of coordinates we here find $n(n-1)/2$ elements are zero; the remaining n are some of the numbers a_{ij}, prefaced with a proper sign.

Let y be the coordinates of a point on one of the S_{n-2}'s, say the first one. We then have that

$$\text{(A)} \qquad y_1 = 0, \qquad \sum_{j=0}^{j=n} a_{1j}y_j = 0.$$

* Presented to the Society, December 27, 1928.

† Segre, *Mehrdimensionale Räume*, Encyklopädie, vol. III C7, pp. 815 and 832.

‡ Wong, B. C., this Bulletin, vol. 34 (1928), pp. 553–554.

§ Berzolari, Rendiconti del Circolo Matematico di Palermo, vol. 29 (1905), p. 229.

Let x be the coordinates of a point on a line through y, and distinct from y. The coordinates of this line may be denoted by

$$p_{ij} = x_i y_j - x_j y_i, \qquad (i,j = 0,1,\cdots, n),\ (i \neq j).$$

The condition that this line meet an S_{n-2} is that the summation of the product of corresponding coordinates vanish. When we stipulate that the line shall meet all the S_{n-2}'s of the set, we adjoin to equations (A) the $n-1$ equations (B), where

$$\text{(B)} \qquad \sum_{j=0}^{j=n} a_{ij} p_{ij} = 0, \qquad (i = 2,3,\cdots, n).$$

When we eliminate y from the two sets of equations, the resulting equation in x represents the equation of the V_{n-1}^{n}, for the line from x to y has been made to meet each of the n S_{n-2}'s and is hence a generator of the V_{n-1}^{n-1}. One may readily verify that the final form of the eliminant does not depend upon the particular member of the set of S_{n-2}'s whereon one chooses to place the point y.

If the interested reader will consider the following S_3's in S_5,

$$x_0 = x_1 = 0, \quad x_2 = x_3 = 0, \quad x_4 = x_5 = 0, \quad x_0 = x_2 = x_4, \quad x_1 = x_3 = x_5,$$

locate the point y on one of the S_3's, compute the coordinates of the line from x to y as well as the coordinates of the remaining four S_3's, write down the four incidence relations that cause the line to meet the four S_3's, eliminate y from the set, and reduce, he will have the pleasure of finding the equation of a V_4^4 in S_5; the result is

$$0 = x_0x_1x_2x_5 + x_0x_3x_4x_5 + x_1x_2x_3x_4 - x_0x_1x_3x_4 - x_0x_2x_3x_5 - x_1x_2x_4x_5.$$

The Pennsylvania State College

A PARTIAL ISOMORPHISM BETWEEN THE FUNCTIONS OF LUCAS AND WEIERSTRASS*

BY E. T. BELL

1. *Introduction.* In a former paper† it was pointed out that certain identities in the theory of multiplication, real or complex, of elliptic functions, are of precisely the same form as others between Lucas' U_n, V_n. The latter were defined only for integer values of n by Lucas as follows,

$$U_n \equiv \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n \equiv \alpha^n + \beta^n,$$

where α, β are real or complex numbers such that $\alpha+\beta=p$, $\alpha\beta=q$, where p, q are rational numbers. Hence U_n, V_n are rational numbers and are solutions of

$$W_{n+2} = pW_{n+1} - qW_n,$$

with the initial conditions

$$(U_0, U_1) = (0,1), \ (V_0, V_1) = (2, p).$$

It follows, as shown by Lucas,‡ that U_n, V_n are simply expressible as circular functions of a real or imaginary argument, according as α/β is imaginary or real. Thus every formula in elliptic functions will contain as a degenerate case one concerning the U_n, V_n functions, but this is not the isomorphism sought.

We first generalize Lucas' definition in one respect and specialize it in another, replacing n by the complex variable z and restricting q to be 1. We shall write the functions thus defined as

$$(1) \qquad U(z) \equiv \frac{\alpha^z - \alpha^{-z}}{\alpha - \alpha^{-1}}, \quad V(z) \equiv \alpha^z + \alpha^{-z},$$

* Presented to the Society, December 31, 1928.

† This Bulletin, vol. 29 (1923), pp. 401–406.

‡ American Journal of Mathematics, vol. 1, (1878), p. 189.

where α, p are constants, real or complex, such that $\alpha+\alpha^{-1}=p$. The principal determinations of exponentials and logarithms are to be understood. From $U(z)$, $V(z)$ we shall construct certain functions exhibiting a partial isomorphism with the $\sigma(z)$, $\zeta(z)$, $\wp(z)$ of Weierstrass. A complete isomorphism is obviously impossible, $U(z)$, $V(z)$ being only singly periodic.

To make the sense in which we are using isomorphism precise, we recall a well known theorem for the sigma function.* Let $f'(z)$ denote the derivative of $f(z)$ with respect to z, and suppose that $f(z)$ satisfies the following conditions.

(A) $f(z)$ is a transcendental integral function of z.

(B) $f'(0) = 1$.

(C) $$f(z+a)f(z-a)f(b+c)f(b-c) + f(z+b)f(z-b)f(c+a)f(c-a) + f(z+c)f(z-c)f(a+b)f(a-b) = 0,$$

where z, a, b, c are independent complex variables. Then necessarily $f(z)\equiv\sigma(z)$. If now $f(z)$ satisfies (A) and (C), but not (B), we shall call $f(z)$ a *partial isomorph* of $\sigma(z)$. As shown by Kronecker,† the "three-term relation" (C) is formally equivalent to Jacobi's formula for the multiplication of four elliptic theta functions. Hence, if preferred, Jacobi's relation may replace (C) in what follows. The notation sh x, ch x is used for the hyperbolic sine and cosine of x.

2. *The Functions* λ, μ, ν. With α, p as in (1), define the constants k, g by

$$k \equiv \log\alpha, \qquad 2g \equiv k\,\mathrm{hs}\,k, \tag{2}$$

where $\mathrm{hs}\,x\equiv 1/\mathrm{sh}\,x$. Then $p=2\,\mathrm{ch}\,k$, and

$$W(z+2) - 2\,\mathrm{ch}\,kW(z+1) + W(z) = 0$$

has the solutions $W(z)\equiv U(z)$, $V(z)$.

* Halphen, *Traité des Fonctions Elliptiques*, vol. 1, p. 187.

† Journal für Mathematik, vol. 102, p. 260.

Write ch $x/\text{sh}\ x \equiv \text{cs}\ x$, and define the functions λ, μ, ν by

$$\lambda(z) \equiv \frac{1}{2g}U(z) = \frac{1}{k}\,\text{sh}\ kz, \tag{3}$$

$$\mu(z) \equiv \frac{g^2V^2(z)}{U^2(z)} - k^2 = k^2\,\text{hs}\ kz, \tag{4}$$

$$\nu(z) \equiv g\frac{V(z)}{U(z)} = k\,\text{cs}\ kz. \tag{5}$$

The hyperbolic forms can be verified by inspection from (2) and

$$U(z) = \frac{2g}{k}\,\text{sh}\ kz, \quad V(z) = 2\,\text{ch}\ kz. \tag{6}$$

3. *Correspondence between* λ, μ, ν *and* σ, $\wp$, ζ. We shall now show that $\lambda(z)$, $\mu(z)$, $\nu(z)$ correspond, up to a certain point, to $\sigma(z)$, $\wp(z)$, $\zeta(z)$ respectively. Note first that $\lambda'(0) = 1$, so that §1 (B) does not hold. The addition and subtraction theorems for $U(z)$ are

$$2U(z \pm w) = U(z)V(w) \pm V(z)U(w),$$

as is evident from (6), and therefore, if $K \neq 0$ and L are arbitrary constants, and if for the moment we write $W(z) \equiv KU(z)$, we have the identity

$$\left[\left\{\frac{V(z)}{2W(z)}\right\}^2 + L\right] - \left[\left\{\frac{V(w)}{2W(w)}\right\}^2 + L\right] = -\frac{W(z+w)W(z-w)}{W^2(z)W^2(w)}.$$

The correspondence will be effected by properly choosing K, L, as the last identity is of the same form as

$$\wp(z) - \wp(w) = -\frac{\sigma(z+w)\sigma(z-w)}{\sigma^2(z)\sigma^2(w)}, \tag{D}$$

from which the three-term relation for σ follows at once, in the usual manner, by means of the algebraic identity

$$(b - c)(d - a) + (c - a)(d - b) + (a - b)(d - c) \equiv 0.$$

The constants are determined by assuming $W(z)$ to be the correspondent of $\sigma(z)$, and recalling the definitions

$$\zeta(z) \equiv \sigma'(z)/\sigma(z), \qquad \wp(z) = -\zeta'(z). \tag{7}$$

A simple calculation gives the functions $\lambda(z)$, $\mu(z)$, $\nu(z)$ as defined in §2, (3)–(5). Hence, or directly from (3)–(5), we see that

(8) $\lambda(z)$ *satisfies* §1 (A), (C), *but not* (B), *so that* $\lambda(z)$ *is a partial isomorph of* $\sigma(z)$;

$$\nu(z) = \lambda'(z)/\lambda(z), \qquad \mu(z) = -\nu'(z), \tag{9}$$

corresponding to (7);

$$\lambda(-z) = -\lambda(z), \quad \nu(-z) = -\nu(z), \quad \mu(-z) = \mu(z), \tag{10}$$

corresponding to the like for $\sigma(z)$, $\zeta(z)$, $\wp(z)$ *respectively*;

$$\mu'^2(z) = 4\mu^3(z) + 4k^2\mu^2(z)\ ; \tag{11}$$

and hence, if $\theta(z) \equiv \mu(z) + k^2/3$,

$$\theta'^2(z) = 4\theta^3(z) - k_1\theta(z) - k_2, \tag{12}$$

corresponding to the Weierstrass normal form with the invariants $k_1 = 4k^4/3$, $k_2 = (-2k^2/3)^3$.

The consequences of this correspondence can be developed indefinitely, without further computations. For example, since the correspondent of (D) is

$$\mu(z) - \mu(w) = -\frac{\lambda(z + w)\lambda(z - w)}{\lambda^2(z)\lambda^2(w)}, \tag{13}$$

we infer the following correspondent for the addition theorem for the $\wp$ function in one of its usual forms,

$$\mu(z \pm w) = \mu(z) - \frac{1}{2}\frac{d}{dz}\left[\frac{\mu'(z) \mp \mu'(w)}{\mu(z) - \mu(w)}\right],$$

and the like with θ in place of μ. Again, since the limit of $\lambda(t)/t$ as t approaches 0 is 1, we divide (13) throughout by

$z-w$, take the limit as w approaches z, and find $\mu'(z) = -\lambda(2z)/\lambda^4(z)$, the correspondent of $\wp'(z) = -\sigma(2z)/\sigma^4(z)$.

If we wish to interpret this partial isomorphism in terms of Lucas' U_n, V_n, where n is an integer, we may so do in the special case (cf. §1) in which $q=1$. The derivatives $\lambda'(z)$, $\mu'(z)$, $\cdots$ are then replaced by their equivalents in terms of $U(z)$, $V(z)$ with z finally replaced by n.

California Institute of Technology

AN INEQUALITY FOR DEFINITE HERMITIAN DETERMINANTS*

BY E. F. BECKENBACH

The proof of M. Ragnar Frisch's theorem,† *The absolute value of a symmetric, definite determinant of real elements is at most equal to the product of the absolute values of the elements of the principal diagonal,* may be generalized to establish the following theorem of which the above is clearly a special case: *The absolute value of a definite Hermitian determinant is at most equal to the product of the absolute values of the elements of the principal diagonal.*

An Hermitian determinant

$$H \equiv \begin{vmatrix} h_{11} \cdots h_{1n} \\ \cdots\cdots\cdots \\ h_{n1} \cdots h_{nn} \end{vmatrix} \tag{1}$$

is a determinant such that

$$h_{rs} = \bar{h}_{sr}, \qquad (r,s = 1,2, \cdots, n),$$

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* Presented to the Society, February 23, 1929.

† *Sur le théorème des déterminants de M. Hadamard,* Comptes Rendus, vol. 185 (1927), p. 124. This is not a new theorem; see Bachmann, *Die Arithmetik der Quadratischen Formen,* 1923, pp. 250–251. It is more the method of proof than the result that makes Frisch's paper of interest. Though Bachmann does not give the generalized proof of the present paper, his proof holds equally for it.

where $\bar{h}_{sr}$ designates the conjugate imaginary of h_{sr}. A definite Hermitian determinant is an Hermitian determinant such that the corresponding Hermitian form

$$h \equiv \sum_{r,s=1}^{n} h_{rs} x_r \bar{x}_s, \qquad (h_{rs} = \bar{h}_{sr}), \tag{2}$$

is definite. The reader may readily verify the fact that an Hermitian form can never take on a complex value. An Hermitian form which is positive (negative) or zero for all values of the variables is said to be positive (negative) definite; in either case, the form is said to be definite.

We can suppose the definite form (2) corresponding to the definite determinant (1) to be positive definite, for otherwise we could consider the form

$$-h \equiv \sum_{r,s=1}^{n} (-h_{rs}) x_r \bar{x}_s, \qquad (h_{rs} = \bar{h}_{sr}),$$

which would be positive definite, and whose corresponding determinant would be equal in absolute value to H. And we can suppose the form to be non-singular, for if the determinant were equal to zero the theorem would be trivial.

The determinant

$$F(\lambda) \equiv \begin{vmatrix} h_{11} - \lambda & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} - \lambda & \cdots & h_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} - \lambda \end{vmatrix}$$

may be expanded into a polynomial in λ of degree n, where the constant term, H, is unequal to zero. Hence we can construct*

$$F(\lambda) \equiv \frac{[P(\lambda)]^2 - \lambda}{Q(\lambda)}, \tag{3}$$

where $P(\lambda)$ and $Q(\lambda)$ are polynomials in λ and $P(\lambda)$ is of

* See Bôcher, *Introduction to Higher Algebra*, pp. 297–299.

degree less than n. As is well known,* the roots of the characteristic equation,

$$F(\lambda) = 0,$$

are all real and greater than zero. Note, then, in the above construction, that the coefficients in $P(\lambda)$ are all real.

Let

$$M \equiv \begin{pmatrix} h_{11} \cdots h_{1n} \\ \cdots\cdots\cdots \\ h_{n1} \cdots h_{nn} \end{pmatrix}$$

be the matrix corresponding to H. By a fundamental theorem,†

$$F(M) \equiv 0.$$

Hence, (3) gives

$$[P(M)]^2 \equiv M. \tag{4}$$

It is easy to show that $P(M)$, being a polynomial with real coefficients in an Hermitian matrix, is itself an Hermitian matrix of the same order. Now let

$$P(M) \equiv \begin{pmatrix} b_{11} \cdots b_{1n} \\ \cdots\cdots\cdots \\ b_{n1} \cdots b_{nn} \end{pmatrix}, \qquad (b_{rs} = \bar{b}_{sr}).$$

Then from (4),

$$h_{ii} = \sum_{j=1}^{n} b_{ij}\bar{b}_{ij}, \qquad (i = 1,2, \cdots, n).$$

Whether the form is positive or negative definite, we still have

$$\text{mod } h_{ii} = \sum_{j=1}^{n} b_{ij}\bar{b}_{ij}, \qquad (i = 1,2, \cdots, n), \tag{5}$$

where by mod h_{ii} we mean the modulus, or absolute value, of h_{ii}.

* See Kowalewski, *Einführung in die Determinantentheorie*, Leipzig, 1909, pp. 280–284.

† See Bôcher, loc. cit., p. 296.

I am indebted to Professor H. E. Bray for the following proof of Hadamard's Theorem. Given the matrix of real or complex elements

$$\alpha \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{pmatrix}, \qquad \alpha_i \equiv (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in}),$$

consider the matrix

$$\beta \equiv \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{pmatrix},$$

where

$$\begin{aligned} \beta_1 &= \alpha_1, \\ &\cdots\cdots\cdots \\ \beta_i &= \sum_{j=1}^{i-1} k_j^i \beta_j + \alpha_i. \end{aligned}$$

We note that

$$|\beta| = |\alpha|.$$

Choose the k_l^i so that, for given i and all $j<i$,

$$(\beta_i, \bar{\beta}_j) \equiv \beta_{i1}\bar{\beta}_{j1} + \beta_{i2}\bar{\beta}_{j2} + \cdots + \beta_{in}\bar{\beta}_{jn} = 0.$$

That is,

$$\left(\left[\sum_{l=1}^{i-1} k_l^i \beta_l + \alpha_i \right], \bar{\beta}_j \right) = k_j^i (\beta_j, \bar{\beta}_j) + (\alpha_i, \bar{\beta}_j) = 0,$$

or

$$k_j^i = -\frac{(\alpha_i, \bar{\beta}_j)}{(\beta_j, \bar{\beta}_j)}.$$

Hence

$$\sum_{j=1}^{i-1} \bar{k}_j^i (\alpha_i, \bar{\beta}_j) = -\sum_{j=1}^{i-1} k_j^i \bar{k}_j^i (\beta_j, \bar{\beta}_j),$$

which is real and negative. Now

$$0 \leqq (\beta_i, \bar{\beta}_i) = \left(\left[\sum_{j=1}^{i-1} k_j^i \beta_j + \alpha_i \right], \left[\sum_{j=1}^{i-1} \bar{k}_j^i \bar{\beta}_j + \bar{\alpha}_i \right] \right),$$

whence

$$0 \leqq (\beta_i \bar{\beta}_i) = \sum_{j=1}^{i-1} k_j^i \bar{k}_j^i (\beta_j, \bar{\beta}_j) + \sum_{j=1}^{i-1} \bar{k}_j^i (\alpha_i, \bar{\beta}_j) + \sum_{j=1}^{i-1} k_j^i (\bar{\alpha}_i, \beta_j) + (\alpha_i, \bar{\alpha}_i)$$

$$= (\alpha_i, \bar{\alpha}_i) - \sum_{j=1}^{i-1} k_j^i \bar{k}_j^i (\beta_j, \bar{\beta}_j) \leqq (\alpha_i, \bar{\alpha}_i).$$

But

$$|\alpha| \cdot |\bar{\alpha}| = |\beta| \cdot |\bar{\beta}| = \prod_{i=1}^{n} (\beta_i, \bar{\beta}_i).$$

Hence

$$|\alpha| \cdot |\bar{\alpha}| \leqq \prod_{i=1}^{n} (\alpha_i, \bar{\alpha}_i),$$

or

$$\operatorname{mod} |\alpha| \leqq \prod_{i=1}^{n} [(\alpha_i, \bar{\alpha}_i)]^{1/2}.$$

This inequality expresses Hadamard's Theorem: *The absolute value of any determinant is at most equal to the product of the norms (either horizontal or vertical) of the determinant.* The quantities

$$+ [(\alpha_i, \bar{\alpha}_i)]^{1/2}, \quad \text{or} \quad + \left(\sum_{j=1}^{n} \alpha_{ij} \bar{\alpha}_{ij} \right)^{1/2}, \qquad (i = 1, 2, \cdots, n),$$

are called the horizontal norms of the determinant $|\alpha|$, and a similar definition holds for the vertical norms.

Applying this theorem to the determinant $|P(M)|$, we get

$$\operatorname{mod} |P(M)|^2 \leqq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} b_{ij} \bar{b}_{ij} \right). \tag{6}$$

Equations (4), (5), and (6) give

$$\operatorname{mod} H \leqq \prod_{i=1}^{n} (\operatorname{mod} h_{ii}). \qquad Q.E.D.$$

THE RICE INSTITUTE

A THEOREM CONCERNING SIMPLY TRANSITIVE PRIMITIVE GROUPS*

BY W. A. MANNING

The theorem here presented has evolved by easy stages from a paragraph in Jordan's *Memoir on primitive groups*.† In the discussion of a particular class of simply transitive primitive groups, he showed that the degree of a doubly transitive constituent of the subgroup leaving one letter fixed cannot be greater than the sum of the degrees of all the other transitive constituents.

THEOREM. *Let H be the subgroup that fixes one letter of a simply transitive primitive group. If one of the constituents of H is a doubly transitive group of degree m, there is in H a transitive constituent whose degree is greater than m and divides $m(m-1)$.*

Let G, of degree n and of order nh, be the given simply transitive primitive group. Let H, the subgroup of G that leaves the letter x fixed, be denoted by $G(x)$.

Let it be assumed: (1) that $G(x)$ has exactly k similar‡ doubly transitive constituent groups: A on the letters $a_1, a_2, \cdots, a_m$; B on $b_1, b_2, \cdots, b_m$; $\cdots$; K on $k_1, k_2, \cdots, k_m$; (2) that $G(a_1)$ has $k-1$ doubly transitive constituents: B_1 on $b_1, a_2, \cdots, a_m$; C_1 on $c_1, b_2, \cdots, b_m$; $\cdots$; K_1 on $k_1, j_2, \cdots, j_m$; (3) that n is greater than $km+1$. These assumptions, when $k=1$, reduce to the hypothesis of our theorem. We wish to prove by induction that there is a transitive

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* Presented to the Society, April 6, 1928.

† C. Jordan, Bulletin de la Société Mathématique de France, vol. 1 (1873), p. 198, §64.

Manning, American Journal of Mathematics, vol. 39 (1917), p. 298; Transactions of this Society, vol. 20 (1919), p. 66; *Primitive Groups*, 1921, p. 83; Transactions of this Society, vol. 29 (1927), p. 821, §8.

‡ Manning, Transactions of this Society, vol. 29 (1927), p. 821, §8.

constituent of some degree μ in $G(x)$ such that μ divides $m(m-1)$ and is greater than m.

The letters of G are supposed to be so chosen that each substitution of $G(x)$ permutes the m subscripts of the letters of the similar groups $A, B, \cdots, K$ in exactly the same way.

The subgroup $G(x)(a_1)$ of $G(x)$ (that fixes a_1) is of order h/m. The subgroups $G(x)(a_1)$ and $G(a_1)(x)$ are identical, and therefore in $G(a_1)$ x is a letter of a transitive constituent of degree m. $G(a_1)(x)(=G(b_1)(x)=G(c_1)(x)=\cdots=G(k_1)(x))$, because of assumption (1) has k transitive constituents of degree $m-1$ each on the letters $a_2, a_3, \cdots, a_m$; $b_2, b_3, \cdots, b_m$; $\cdots$; $k_2, k_3, \cdots, k_m$, respectively. Since G is primitive, $\{G(x), G(a_1)\}=G$, and therefore $G(a_1)$ cannot have a transitive constituent of degree $m-1$ on the letters $k_2, k_3, \cdots, k_m$, nor a transitive constituent of degree m on the letters $x, k_2, \cdots, k_m$. In $G(a_1)$ the letters $k_2, k_3, \cdots, k_m$ all belong to the same transitive constituent of unknown degree $\mu(\geqq m)$. Then $G(a_1)(k_2)$ is of order h/μ and if x belongs to a transitive constituent of degree $\delta(\geqq 1)$ in $G(a_1)(k_2)$, $G(a_1)(k_2)(x)$ is of order $h/\mu\delta$. Then since $G(x)(a_1)(k_2)=G(x)(k_1)(k_2)$, $h/\mu\delta=h/[m(m-1)]$, and μ divides $m(m-1)$. Either our theorem is proved or $\mu=m$. If $\mu=m$, $G(a_1)$ has this transitive constituent (L_1) on the letters $l_1, k_2, \cdots, k_m$ (where $l_1\neq x$); L_1 is doubly transitive because it contains a transitive subgroup of degree $m-1$ on $k_2, k_3, \cdots, k_m$.

The substitution $(a_1a_i\cdots)\cdots$ of $G(x)$ transforms $G(a_1)$ into $G(a_i)$ and L_1 into L_i, a doubly transitive constituent of $G(a_i)$ on the m letters $l_i, k_1, k_2, \cdots, k_{i-1}, k_{i+1}, \cdots, k_m$ $(l_i\neq x)$. Nor is $l_i=l_1$, for if it were, $\{G(a_1), G(a_i)\}$ would have a transitive constituent of degree $km+1(<n)$ on the letters $a_1, a_2, \cdots, k_m, l_1$. Then $l_1, l_2, \cdots, l_m$ are the letters of a transitive constituent L of $G(x)$. Any substitution of $G(x)$ that replaces a_1 by a_i replaces l_1 by l_i, so that the substitutions of $G(x)$ permute the letters of A and the letters of L in exactly the same way. The groups A and L are similar.

Suppose now that $n=(k+1)m+1$. Then $G(a_1)$ has a transitive constituent X_1 on the m letters $x, l_2, \cdots, l_m$. $G(a_1)(x)$

$(=G(b_1)(x)=\cdots=G(l_1)(x))$ has $k+1$ transitive constituents on $a_2, a_3, \cdots, a_m$; $b_2, b_3, \cdots, b_m$; $\cdots$; and $l_2, l_3, \cdots, l_m$. No two of the groups $G(b_1), G(c_1), \cdots, G(l_1)$ can have the transitive constituent $a_2, a_3, \cdots$ in common. Because B_1 (on $b_1, a_2, \cdots, a_m$) is in $G(a_1)$, in one of these k groups $x, a_2, \cdots, a_m$ are the letters of a doubly transitive constituent. Let these subgroups be transformed by $(x)(a_1a_2\cdots)\cdots$ into $G(b_2), G(c_2), \cdots, G(l_2)$. One of the transformed groups has a doubly transitive constituent on the letters $x, a_1, a_3, \cdots, a_m$, and therefore contains a substitution $S=(xa_1)\cdots$ which permutes the letters $a_3, a_4, \cdots, a_m$ among themselves. Then S should transform the constituent A of $G(x)$ into the constituent X_1 of $G(a_1)$; but this, because $m\geqq 3$, S cannot do. Hence $n>(k+1)m+1$. We have shown too that $G(x)$ has $k+1$ doubly transitive constituents $A, B, \cdots, L$ whose letters are permuted by $G(x)$ in exactly the same way, and that $G(a_1)$ has k transitive constituents $B_1, C_1, \cdots, L_1$. Thus the three conditions of our assumption are reproduced with $k+1$ replacing k.

Since n is finite, this process can lead only to the conclusion that the degree of some transitive constituent of $G(x)$ is greater than m and divides $m(m-1)$.

STANFORD UNIVERSITY

A THEOREM OF KUMMER'S CONCERNING THE SECOND FACTOR OF THE CLASS NUMBER OF A CYCLOTOMIC FIELD*

BY H. S. VANDIVER

Kummer† in a letter to Kronecker dated December 28, 1849, gave without proof the result that if

$$E_n = \epsilon^t \; ; \; t = 1 + sr^{-2n} + s^2r^{-4n} + \cdots + s^{(l-3)/2}r^{-(l-3)n},$$

and

$$\epsilon = \left(\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)(1-\zeta^{-1})}\right)^{1/2}, \qquad \zeta = e^{2i\pi/l},$$

and

$$\prod_{i=1}^{(l-3)/2} E_i^{a_i} = \eta^l \tag{1}$$

where η is a unit in the cyclotomic field defined by ζ, then each $E_i^{a_i}$ is the lth power of a unit in the field, the a's being rational integers. Kummer refers to this result as follows:

"Sehr wichtiger Satz, welcher eine grosse Schwierigkeit hebt."

In the above statement the Kronecker-Hilbert‡ symbolic powers are employed; r is a primitive root of the odd prime l and s stands for the substitution (ζ/ζ^r). These symbolic powers have the following properties.§ If we denote by ω_k the integer in the field $k(\zeta)$ obtained from the integer $\omega = \omega_0$ by means of the substitution (ζ/ζ^{r^k}) then we write

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* Presented to the Society, New York, March 30, 1929.

† Abhandlungen zur Geschichte der Mathematik, vol. 29 (1910), pp. 84–85.

‡ Hilbert, *Die Theorie der Algebraischen Zahlkörper*, Berichte der Deutschen Mathematiker-Vereinigung, 1894, p. 271.

§ Landau, *Vorlesungen über Zahlentheorie*, vol. 3, pp. 253–4, proves analogous properties for a field relative to $k(\zeta)$.

$$\omega_0^{b_0}\omega_1^{b_1}\cdots\omega_{l-2}^{b_{l-2}} = \omega^{F(s)},$$

the b's being rational integers and

$$F(s) = b_0 + b_1 s + b_2 s^2 + \cdots + b_{l-2}s^{l-2}.$$

It then follows that

$$(\omega^{F(s)})^{G(s)} = \omega^{F(s)G(s)}.$$

Employing this relation we shall now prove Kummer's statement. A direct application of it gives

$$E^{s-r^{2i}} = \eta_i^{\,l},$$

and

$$E^{s^c-r^{2ci}} = \eta_{i,c}^{\,l},$$

where the η's are units in $k(\zeta)$. Applying the last equation to (1) we may write after employing the substitution s^c,

$$\prod_{i=1}^{(l-3)/2} E_i^{a_i r^{2ic}} = \xi^l,$$

where ξ is a unit in $k(\zeta)$. We obtain $(l-1)/2$ equations by taking $c=0, 1, 2, \cdots, (l-3)/2$. Raising each equation to the power r^{-2cd}, the c's varying as to the particular equation used, we then obtain

$$E_d^{a_d}\prod E_i^{a_i r^{2c(i-d)}} = \xi_c^{\,l},$$

where the product ranges over the values $i=1, 2, \cdots, (l-3)/2$ excepting d.

Multiplying the $(l-1)/2$ equations together we obtain

$$E_d{}^{a_d{}^{(l-1)/2}}\prod E_j^{a_j k} = \xi^l,$$

where

$$k = \frac{r^{l-1}-1}{r^{2(i-d)}-1}.$$

The numerator of this fraction is divisible by l and the denominator is prime to l since $(i-d)$ is not divisible by $(l-1)$. Hence we have

$$E_d{}^{a_d{}^{(l-1)/2}} = \xi_d{}^l.$$

Raising each side of this equation to the power l' where

$$l' \frac{l-1}{2} \equiv 1, \pmod{l},$$

we have Kummer's result for $d = 1, 2, \cdots, (l-3)/2$.

I think it highly probable that Kummer encountered this question in connection with a problem involving the second factor of the class number of $k(\zeta)$. I shall prove in another article that if the second factor of the class number is divisible by l, then (1) holds with not all the a's divisible by l. A somewhat similar result is proved by Hilbert.*

THE UNIVERSITY OF TEXAS

ON THE RANK EQUATION OF ANY NORMAL DIVISION ALGEBRA†

BY A. A. ALBERT‡

1. *Introduction.* The different types of normal division algebras which have been discovered up to the present depend upon equations with different groups. It has been thought that, as the rank equation of an algebra is invariant under a change of basal units, the groups of the rank equations of these various types of algebras might serve to show their non-equivalence. This notion is shown to be false here, as the group of the rank equation of any normal division algebra is the symmetric group. In proving this theorem a new theorem in the Hilbert theory of an irreducible polynomial whose coefficients are rational functions, with coefficients in any infinite field K, of several parameters is developed.

2. *General Theory.* We shall first give several presupposed

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* Loc. cit., pp. 435–437.

† Presented to the Society, December 27, 1928.

‡ National Research Fellow.

results as lemmas. Lemma 3 is an immediate consequence of Lemmas 1 and 2.

LEMMA 1. *The group of the general equation for the field K determined by its coefficients and any constants finite in number is the symmetric group.*

LEMMA 2. THE HILBERT IRREDUCIBILITY THEOREM. *Let K be any infinite field. Consider the equation*

$$(1) \quad f(x; \lambda_1, \cdots, \lambda_r) \equiv x^n + F_1(\lambda_1, \cdots, \lambda_r)x^{n-1} + \cdots + F_n(\lambda_1, \cdots, \lambda_r) = 0,$$

where $F_i(\lambda_1, \cdots, \lambda_r)$ are rational functions, with coefficients in K, of the independent parameters $\lambda_1, \cdots, \lambda_r$. Let the group of $f(x)$ with respect to $K(\lambda_1, \cdots, \lambda_r)$ be Γ. Then there exist an infinity of rational values of the parameters $\lambda_1, \cdots, \lambda_r$, such that the resulting numerical equation has the group Γ with respect to K.

LEMMA 3. *There exist an infinity of equations with leading coefficients unity and further coefficients in K such that the group of each equation with respect to K is the symmetric group.*

LEMMA 4. *Let A be the algebra of all n-rowed square matrices in K. Let*

$$(2) \quad \phi(\omega) \equiv \omega^n + \alpha_1\omega^{n-1} + \cdots + \alpha_n = 0$$

be any equation of degree n in K. Then there exists an element of A whose characteristic equation is $\phi(\omega) = 0$.

PROOF. The matrix $(\lambda_{ij} | \ i, j = 1, 2, \cdots, n)$, where

$$\lambda_{ij} = -\alpha_j, \qquad (j = 1, 2, \cdots, n),$$

$$\lambda_{i+1,i} = 1, \qquad (i = 1, 2, \cdots, n-1),$$

and all other elements are zero, has for its characteristic equation $\phi(\omega) = 0$.

As a consequence of these lemmas we have the following theorems.

THEOREM 1. *There exists an element of A, the algebra of all n-rowed square matrices with coefficients in K, whose mini-*

mum equation has degree n and the symmetric group with respect to K.

THEOREM 2. *Let* $f(x; \lambda_1, \cdots, \lambda_r) = 0$ *be an equation* (1) *with group* Γ. *Let* $\lambda_1', \cdots, \lambda_r'$ *be any set of scalars and let the resulting numerical equation*

$$(3) \quad f(x; \lambda_1', \cdots, \lambda_r') \equiv x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

have the group Γ_0 *with respect to* $K(\lambda_1', \cdots, \lambda_r')$. *Then* Γ_0 *is a sub-group of* Γ.

PROOF. Form all permutations $i_1, i_2, \cdots, i_n$ of the numbers $1, 2, \cdots, n$. Consider the product

$$P \equiv \Pi(u + x_{i_1}u_1 + \cdots + x_{i_n}u_n)$$

taken over all such permutations of the roots $x_1, \cdots, x_n$ of (1). This is a polynomial with coefficients in $K(\lambda_1, \cdots, \lambda_r)$ of the indeterminates $u, u_1, \cdots, u_n$. Let $G(u, u_1, \cdots, u_n; \lambda_1, \cdots, \lambda_r)$ be any factor of P with coefficients in $K(\lambda_1, \cdots, \lambda_r)$, irreducible in $K(\lambda_1, \cdots, \lambda_r)$. Then the group Γ of the equation $f(x; \lambda_1, \cdots, \lambda_r)$ is defined as the set of all permutations leaving G unaltered. If we replace the indeterminates $\lambda_1, \cdots, \lambda_r$ by $\lambda_1', \cdots, \lambda_r'$ as in the theorem and consider the equation $f(x; \lambda_1', \cdots, \lambda_r') = 0$, we shall obtain its group by finding all permutations leaving any factor g of $G(u, u_1, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$, having coefficients in $K(\lambda_1', \cdots, \lambda_r')$ and irreducible in $K(\lambda_1', \cdots, \lambda_r')$, unaltered. But the group of $f(x; \lambda_1', \cdots, \lambda_r')$ is unique, and hence any permutation leaving g unaltered leaves any of the other factors of $G(u, u_1, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$ unaltered and hence the product $G(u, u_1, \cdots; u_n, \lambda_1', \cdots, \lambda_r')$. Hence *all* substitutions of Γ_0 leave $G(u, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$ unaltered. The number of permutations that leave $G(u, u_1, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$ unaltered is not greater than the number leaving $G(u, \cdots, u_n; \lambda_1, \cdots, \lambda_r,$ unaltered since the degree of both polynomials in the indeterminates $u_1, \cdots, u_n$ is the same. Hence *the only* substitutions leaving $G(u, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$ unaltered are substitutions of Γ.

Hence the substitutions of Γ_0 are substitutions of Γ and Γ_0 is a subgroup of Γ. This leads to the following conclusion.

THEOREM 3. *Let D be a normal division algebra in n^2 units over a field K. Then the group of the rank equation of D is the symmetric group.*

PROOF. Let $x=\sum_{i=1}^{n^2}\xi_i u_i$ be an element of D whose coordinates $\xi_1, \cdots, \xi_{n^2}$ are independent variables in K and let $R(\omega; \xi_1, \cdots, \xi_{n^2})=0$ be the rank equation of D. Then $R=0$ is an equation with leading coefficient unity and further coefficients polynomials in $\xi_1, \cdots, \xi_{n^2}$ with coefficients in K, and the degree of R is n. Let K' be an extension of K such that the algebra D', with the same basal units as D, is equivalent, by a transformation of basal elements, to the algebra of all n-rowed square matrices with coefficients in K'. This can be done by the adjunction of a finite number of constants to K.* It is also easily shown that the rank equation of D' is $R(\omega; \xi_1', \cdots, \xi_{n^2}')=0$, where $\xi_1', \cdots, \xi_{n^2}'$ are independent variables in K'.

By Theorem 1, D' contains an element y whose minimum equation is of degree n and has the symmetric group with respect to K'. Let $y=\sum_{i=1}^{n^2}\bar{\xi}_i' u_i$. Then the minimum equation of y is $R(\omega; \bar{\xi}_1', \cdots, \bar{\xi}_{n^2}')=0$. Hence for some values of $\xi_1, \cdots, \xi_{n^2}$ in K' the group of $R(\omega; \xi_1, \cdots, \xi_{n^2})$ with respect to K' is the symmetric group. Again, by Theorem 2, the symmetric group is a subgroup of the group Γ of $R(\omega; \xi_1, \cdots, \xi_{n^2})$ with respect to $K(\xi_1, \cdots, \xi_{n^2})$. Hence Γ is the symmetric group. This proves the theorem. As a corollary we have the following result.

COROLLARY. *Every normal division algebra in n^2 units over K contains an infinity of elements whose minimum equations each have degree n and the symmetric group with respect to K.*

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* L. E. Dickson, *Algebren und ihre Zahlentheorie*, p. 137.

ON THE NUMBER OF APPARENT TRIPLE POINTS OF SURFACES IN SPACE OF FOUR DIMENSIONS

BY B. C. WONG

Two hypersurfaces in 4-space of orders μ and ν respectively intersect in a surface F of order $\mu\nu$. F has a certain number H of apparent triple points, that is, lines that can be drawn through a given point meeting F three times. If F degenerates into an F_1 of order m_1 and an F_2 of order m_2 where $m_1+m_2=\mu\nu$, then H is the sum of the numbers h_{30}, h_{21}, h_{12}, h_{03}, where h_{ij} is the number of lines that pass through a given point and meet F_1 i times and F_2 j times. It is the purpose of this paper to determine H and, if F is composite, to determine the distribution of the h_{ij} lines.

The formula for H can be readily obtained by calculating the order of the restricted system of equations resulting from imposing upon two binary equations of orders μ and ν respectively the conditions that they have three common roots.* But this method does not offer a ready means for the determination of the distribution of the h_{ij} lines if F is composite. The following method seems well adapted for the purpose.

Suppose, temporarily, that the two hypersurfaces giving F be composed of μ and ν hyperplanes α_k, β_l $[k=1, 2, \cdots, \mu;$ $l=1, 2, \cdots, \nu]$. Then F is made up of $\mu\nu$ planes $\alpha_k\beta_l$. We construct the rectangular array

$$
\begin{array}{ll}
 & \alpha_1\beta_1 \quad \alpha_2\beta_1 \quad \alpha_3\beta_1 \cdots \alpha_\mu\beta_1 \\
 & \alpha_1\beta_2 \quad \alpha_2\beta_2 \quad \alpha_3\beta_2 \cdots \alpha_\mu\beta_2 \\
\text{(A)} & \alpha_1\beta_3 \quad \alpha_2\beta_3 \quad \alpha_3\beta_3 \cdots \alpha_\mu\beta_3 \\
 & \cdots\cdots\cdots\cdots\cdots \\
 & \cdots\cdots\cdots\cdots\cdots \\
 & \alpha_1\beta_\nu \quad \alpha_2\beta_\nu \quad \alpha_3\beta_\nu \cdots \alpha_\mu\beta_\nu
\end{array}
$$

* Salmon, *Modern Higher Algebra*, 4th ed., Lesson 19.

and interpret it as the symbolic representation of F, proper or improper. In this interpretation we remove the assumption that the hypersurfaces are composed of hyperplanes and the $\alpha_k\beta_l$ are to be regarded as mere symbols.

Each of the constituents of the array (A), taken alone, represents a plane. A pair of constituents represents a quadric surface or two incident planes if the constituents are in the same row as $\alpha_1\beta_1$, $\alpha_2\beta_1$ or in the same column as $\alpha_1\beta_1$, $\alpha_1\beta_2$; two non-incident planes if they are in different rows and columns from $\alpha_1\beta_1$, $\alpha_2\beta_2$. Three constituents in the same row as $\alpha_1\beta_1$, $\alpha_2\beta_1$, $\alpha_3\eta_1$ or in the same column as $\alpha_1\beta_1$, $\alpha_1\beta_2$, $\alpha_1\beta_3$ represent a cubic surface lying wholly in an S_3: if the constituents are such that one of them lies in the same column with another and in the same row with the third as $\alpha_1\beta_2$, $\alpha_1\beta_1$, $\alpha_2\beta_1$, the cubic surface is a 4-space surface. Three non-incident planes are represented by three constituents all in different rows and columns from $\alpha_1\beta_1$, $\alpha_2\beta_2$, $\alpha_3\beta_3$. Since from a given point only one line can be drawn meeting three non-incident planes each once, the presence of such a triple of constituents, all lying in different rows and columns, in the array means the presence of an apparent triple point on F. The total number of possible triples of this sort in (A) is the total number of possible apparent triple points of F and the formula for this number is evidently

$$H = \mu\nu(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2)/6. \tag{1}$$

Now if F is composed of an F_1 of order m_1 and an F_2 of order m_2, the constituents of (A) are divided into two groups: one of m_1 constituents representing F_1 and the other of m_2 constituents representing F_2. Then h_{30} is the number of triples of constituents lying in different rows and columns of the first group; h_{03} the number of similar triples in the second group; h_{21} the number of triples each consisting of a pair of constituents in the first group and one constituent in the second; h_{12} the number of triples each consisting of one in the first and two in the second. Evidently H is the sum of all the h_{ij}, that is

$$(2)\quad (m_1 + m_2)(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) = 6(h_{30} + h_{21} + h_{12} + h_{03}).$$

From the very nature of the case we have also

$$(3)\quad \begin{aligned} m_1(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) &= a_0 h_{30} + a_1 h_{21} + b h_{11}, \\ m_2(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) &= a_0 h_{03} + a_1 h_{12} + b h_{11}, \end{aligned}$$

and

$$(4)\quad (m_1 - m_2)(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) = a_0(h_{30} - h_{03}) + a_1(h_{21} - h_{12}),$$

where a_0, a_1 are numerical constants, b is a function of μ and ν, and h_{11} is the order of the cone of lines through a given point meeting F_1 and F_2 each once, or the number of apparent intersections of the sections of F_1 and F_2 by an S_3. The values of a_0, a_1, being independent of μ and ν, can be determined without difficulty. If we put $m_2 = 0$, and consequently $m_1 = \mu\nu$, $h_{03} = h_{21} = h_{12} = 0$ in (4), we have $h_{30} = H$ and $a_0 = 6$. To determine a_1 let F_2 be of order μ, represented by a row of constituents in (A). Then $m_1 = \mu\nu - \mu$, $m_2 = \mu$, $h_{30} = \mu(\mu-1)(\mu-2) \cdot (\nu-1)(\nu-2)(\nu-3)/6$, $h_{21} = \mu(\mu-1)(\mu-2)(\nu-1)(\nu-2)/2$, $h_{03} = h_{12} = 0$. Substituting in (4), we find $a_1 = 2$. To determine b, it is only necessary to make $m_2 = 1$. Then $h_{30} = h_{12} = 0$ and $h_{11} = (\mu-1)(\nu-1)$. Substituting in the second of (3), we obtain $b = (\mu-2)(\nu-2)$. Then (3) and (4) become

$$(5)\quad \begin{aligned} m_1(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) &= 6h_{30} + 2h_{21} + (\mu - 2)(\nu - 2)h_{11}, \\ m_2(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) &= 6h_{03} + 2h_{12} + (\mu - 2)(\nu - 2)h_{11}, \end{aligned}$$

and

$$(6)\quad (m_1 - m_2)(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) = 6(h_{30} - h_{03}) + 2(h_{21} - h_{12}).$$

From (2) and (5) we obtain

$$(7)\qquad 2(h_{21} + h_{12}) = (\mu - 2)(\nu - 2)h_{11},$$

$$(8)\quad (m_1+m_2)(\mu-1)(\mu-2)(\nu-1)(\nu-2) = 6(h_{30}+h_{03})+3(\mu-2)(\nu-2)h_{11}.$$

From (5), (7), (8) one can calculate the h's if any two of them are known. From the divided array representing the degenerate F it is not difficult to obtain the values of two of the h's. Take a simple illustration. Let $\mu=\nu=3$ and $m_1=6$, $m_2=3$. If F_1 is symbolized by

$$\begin{array}{ccc} \alpha_1\beta_1 & \alpha_2\beta_1 & \alpha_3\beta_1 \\ \alpha_1\beta_2 & \alpha_2\beta_2 & \cdot \\ \alpha_1\beta_3 & \cdot & \cdot \end{array}$$

and F_2 by

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha_3\beta_2 \\ \cdot & \alpha_2\beta_3 & \alpha_3\beta_3 \end{array}$$

then we have, by inspection, $h_{30}=1$, $h_{03}=0$. Either from the formulas or by further inspection we see that $h_{11}=10$, $h_{21}=4$, $h_{12}=1$.

It is to be added that the method outlined above, applied to r-space, enables us to show that the number of apparent $(r-1)$-fold points of an $(r-2)$-dimensional variety which is the intersection of two hypersurfaces in S_r of orders μ and ν respectively is

$$H_{r-1}=(r-1)!\binom{\mu}{r-1}\binom{\nu}{r-1}.$$

The same process of reasoning yields the following formulas analogous to (5) and (6):

$$\frac{m_1(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!}=\sum_{i=0}^{t-1}a_ih_{r-i-1,i}+\sum_{j=1}^{t}b_jh_{jj},$$

$$\frac{m_2(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!}=\sum_{i=0}^{t-1}a_ih_{i,r-i-1}+\sum_{j=1}^{t}b_jh_{jj},$$

$$\frac{(m_1-m_2)(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!}=\sum_{i=0}^{t-1}a_i(h_{r-i-1,i}-h_{i,r-i-1})$$

where $t=(r-1)/2$ if r is odd and $(r-2)/2$ if r is even. There is no difficulty in calculating

$$a_i = (r-2)!(r-2i-1),$$

but some difficulty is encountered in calculating b_j which are functions of μ and ν. The following are some of their values:

$$b_1 = (\mu-2)!(\nu-2)!/D,$$
$$b_2 = 4(\mu-4)!(\nu-4)!/D,$$
$$b_3 = 72(\mu-6)!(\nu-6)!/D,$$
$$b_4 = 2880(\mu-8)!(\nu-8)!/D, \quad \text{etc.},$$

where $D=(\mu-r+1)!(\nu-r+1)!$

For $r=3$, $a_0=2$, $b_1=1$ as is well known.* For $r=4$, $a_0=6$, $a_1=2$, $b_1=b=(\mu-2)(\nu-2)$ as we have seen above.

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* Salmon, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 1, pp. 357, 358.

ON THE INDEPENDENCE OF THE FIRST AND SECOND MATRICES OF AN ALGEBRA*

BY C. C. MACDUFFEE

1. *Introduction.* It is well known† that every linear associative algebra with a principal unit (modulus) is isomorphic with the algebra of its first matrices, and also with the algebra of its transposed second matrices. If the algebra has no principal unit, it can be represented as a matric algebra of $(n+1)$th order matrices.

The condition that the algebra have a principal unit is not, however, necessary in order that the algebra be isomorphic with the algebra of its first or second matrices, as can readily be seen from examples. In this paper necessary and sufficient conditions for this isomorphism are obtained.

2. *The Correspondence of Poincaré.* Consider a linear associative algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ with n basal numbers $e_1, e_2, \cdots, e_n$, the constants of multiplication being c_{ijk}. Let us denote by R_i the matrix‡ (c_{isr}), and by S_i the matrix (c_{ris}), where r determines the row and s the column in which an element stands.

The conditions for associativity in $\mathfrak{A}$ may be written§

$$(1) \qquad \sum_k c_{ikr}c_{jsk} = \sum_k c_{ijk}c_{ksr}, \quad (i,j,r,s = 1,2, \cdots, n).$$

If we form the matrices in which the respective members of the above equation stand in the rth row and sth column, we have

$$R_iR_j = \sum_k c_{ijk}R_k.$$

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* Presented to the Society, Chicago, March 30, 1929.

† L. E. Dickson, *Algebras and their Arithmetics*, Chicago, 1923, p. 96.

‡ R_i and S_i are the first and transposed second matrices, respectively, of e_i. Dickson, loc. cit., p. 95.

§ Dickson, loc. cit., p. 92.

We may also write conditions (1) in the form

$$\sum_k c_{rik}c_{kjs} = \sum_k c_{ijk}c_{rks},$$

whence it follows that

$$S_iS_j = \sum_k c_{ijk}S_k.$$

Thus both R_i and S_i are isomorphic under multiplication with the basal numbers e_i.

Every number a of $\mathfrak{A}$ can be written

$$a = a_1e_1 + a_2e_2 + \cdots + a_ne_n,$$

where the a_i are in $\mathfrak{F}$. We define the *first matrix* $R(a)$ of a by the equation

$$R(a) = a_1R_1 + a_2R_2 + \cdots + a_nR_n,$$

and the *second matrix* $S(a)$ by

$$S(a) = a_1S_1 + a_2S_2 + \cdots + a_nS_n.$$

Thus the algebra $\mathfrak{A}$ is isomorphic with the algebra of matrices $R(a)$ if and only if $R_1, R_2, \cdots, R_n$ are linearly independent, and isomorphic with the algebra of matrices $S(a)$ if and only if $S_1, S_2, \cdots, S_n$ are linearly independent.

3. *Two Invariants.* If we apply to the basal numbers $e_1, e_2, \cdots, e_n$ the linear transformation

$$e_i = \sum_j a_{ij}e_j', \qquad a = |a_{rs}| \neq 0, \tag{2}$$

with coefficients in $\mathfrak{F}$, the constants of multiplication are subject to the induced transformation*

$$\sum_i c_{rsi}a_{ij} = \sum_{p,q} a_{rp}a_{sq}c'_{pqj}, \qquad (r,s,j = 1,2,\cdots,n). \tag{3}$$

This may be written

$$ac_{isr} = \sum_{p,q,t} A_{rt}a_{ip}c'_{pqt}a_{sq},$$

* For example, see MacDuffee, Transactions of this Society, vol. 31 (1929), p. 81.

where A_{rt} denotes the cofactor of a_{rt} in $A=(a_{rs})$. Then

$$R_i = \bar{A}^{-1}(\Sigma_p a_{ip} R_p')\bar{A}, \qquad (i = 1, 2, \cdots, n), \tag{4}$$

where $\bar{A}$ is the transpose of A.

If we denote by

$$\sum_j k_{ij} R_j = 0, \qquad (i = 1, 2, \cdots, \rho), \tag{5}$$

a maximal set of linearly independent linear relations among the R_j, we have

$$\bar{A}^{-1} \sum_{h,j} k_{ij} a_{jh} R_h' \bar{A} = 0,$$

so that

$$\sum_h \left(\sum_j k_{ij} a_{jh} \right) R_h' = 0, \qquad (i = 1, 2, \cdots, \rho).$$

Since

$$\left(\sum_j k_{rj} a_{js} \right) = (k_{rs})A,$$

and the matrix (k_{rs}) of ρ rows and n columns is of rank ρ, we see that there are at least ρ linearly independent linear relations among the matrices $R_1', R_2', \cdots, R_n'$. Since (2) has an inverse, there are just ρ such relations. Hence ρ is invariant under transformation of coordinates.

Similarly we find that the number σ of linearly independent linear relations among the matrices $S_1, S_2, \cdots, S_n$ is likewise invariant.

4. *A Condition for the Independence of the Matrices.* Suppose that exactly ρ independent relations (5) hold among the matrices R_j. Form a matrix $B \equiv (b_{rs})$ so that $b_{ij} = k_{ij}$ for $i = n-\rho+1, \cdots, n$, and take for the remaining b_{ij} any convenient numbers of $\mathfrak{F}$ so that B is non-singular. Apply a transformation (2) using $A = B^{-1}$. From (4) we have

$$R_i' = \bar{B}^{-1} \sum_j b_{ij} R_j \bar{B}, \qquad (i = 1, 2, \cdots, n).$$

Hence $R'_{n-\rho+1} = \cdots = R'_n = 0$ while $R'_1, R'_2, \cdots, R'_{n-\rho}$ are linearly independent. We drop primes.

We now have

$$c_{ijk} = 0, \qquad (i > n-\rho\,;\ j,k = 1,2,\cdots,n). \tag{6}$$

The associativity conditions (1) may be written

$$\sum_{k=1}^{n} c_{ijk}c_{ksr} = \sum_{k=1}^{n} c_{ikr}c_{jsk}.$$

We consider only those equations in which $j > n-\rho$, and pass to matrices, obtaining

$$\sum_{k=1}^{n-\rho} c_{ijk}R_k = 0.$$

Since $R_1, R_2, \cdots, R_{n-\rho}$ are linearly independent,

$$c_{ijk} = 0, \quad (j > n-\rho\,;\ k \leqq n-\rho\,;\ i = 1,2,\cdots,n). \tag{7}$$

Consider the linear set $\mathfrak{Z}$ composed of all numbers

$$z = z_{n-\rho+1}e_{n-\rho+1} + \cdots + z_n e_n.$$

We see readily from (6) and (7) that

$$\mathfrak{Z}\mathfrak{A} = 0, \qquad \mathfrak{A}\mathfrak{Z} \leqq \mathfrak{Z}, \tag{8}$$

so that $\mathfrak{Z}$ is an invariant zero subalgebra of $\mathfrak{A}$ which has the additional property that $\mathfrak{Z}\mathfrak{A} = 0$.

Conversely, let us suppose that $\mathfrak{A}$ has an invariant zero subalgebra $\mathfrak{Z}$ of order ρ such that $\mathfrak{Z}\mathfrak{A} = 0$. We take the basis numbers of $\mathfrak{Z}$ for $e_{n-\rho+1}, \cdots, e_n$ of a basis for $\mathfrak{A}$. Since $\mathfrak{Z}\mathfrak{A} = 0$, we have (7) and therefore $R_{n-\rho+1} = \cdots = R_n = 0$.

Similar results hold for the second matrix.

THEOREM 1. *A necessary and sufficient condition in order that there be exactly $\rho(\sigma)$ linearly independent linear relations among the matrices $R_1, R_2, \cdots, R_n$ $(S_1, S_2, \cdots, S_n)$ is that $\mathfrak{A}$ have an invariant zero subalgebra $\mathfrak{Z}(\mathfrak{W})$ of order $\rho(\sigma)$ such that $\mathfrak{Z}\mathfrak{A} = 0$, $(\mathfrak{A}\mathfrak{W} = 0)$, and no such subalgebra of order greater than $\rho(\sigma)$.*

It is known that in every algebra with a principal unit (including every semi-simple algebra) both the first and second matrices are independent.*

It can be seen directly that if $\mathfrak{A}$ is nilpotent or the direct sum of a nilpotent algebra and another algebra, the first matrices and also the second matrices are dependent. For we may choose normalized† basis numbers e_i for the nilpotent algebra such that

$$e_i e_n = e_n e_i = 0, \qquad (i = 1, 2, \cdots, n).$$

Hence e_n serves both as $\mathfrak{Z}$ and $\mathfrak{W}$ for Theorem 1.

That ρ is not necessarily equal to σ is shown by the following example:

$$e_1^2 = e_1, \qquad e_1 e_2 = 0, \qquad e_2 e_1 = e_2, \qquad e_2^2 = 0.$$

We have

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here R_1 and R_2 are linearly independent and give a representation of the algebra, while S_1 and S_2 are dependent. Thus $\rho = 0$, $\sigma = 1$, and $\mathfrak{W}$ is composed of multiples of e_2.

5. *The Characteristic Equations.* Let

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

be the general number of an algebra $\mathfrak{A}$, and define

$$\delta(\omega) = \left| R(x) - \omega I \right|, \qquad \delta'(\omega) \equiv \left| S(x) - \omega I \right|.$$

It is well known‡ that $x\delta(x) = 0$ and $x\delta'(x) = 0$, and if $\mathfrak{A}$ has a principal unit, also $\delta(x) = 0$ and $\delta'(x) = 0$.

It is possible, however, for $\delta(x) = 0$ or $\delta'(x) = 0$ even though $\mathfrak{A}$ has no principal unit.

THEOREM 2. *A sufficient condition in order that* $\delta(x) = 0$ $(\delta'(x) = 0)$ *is that* $R_1, R_2, \cdots, R_n$ $(S_1, S_2, \cdots, S_n)$ *be linearly independent.*

* Dickson, loc. cit., p. 95.

† Dickson, loc. cit., p. 176.

‡ Dickson, loc. cit., p. 100.

The proof is immediate, for by the Hamilton-Cayley theorem

$$\delta(R(x)) = 0, \qquad \delta'(S(x)) = 0.$$

Since $\mathfrak{A}$ is isomorphic with the algebra of matrices $R(x)$ (or $S(x)$), we have $\delta(x)=0$ (or $\delta'(x)=0$).

For the example of §4 we have

$$\delta(\omega) = \omega^2 - \omega x_1, \qquad \delta'(\omega) = \omega^2 - 2\omega x_1 + x_1^2.$$

Hence $\delta(x)=0$, while $\delta'(x)=x_1^2-x_1^2e_1-x_1x_2e_2$.

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ON THE NUMBER $(10^{23}-1)/9$

D. H. LEHMER

The purpose of this note is to save any further effort* in trying to factor the number $N=(10^{23}-1)/9=111,\ 11111,\ 11111,\ 11111,\ 11111$ which in a previous paper was found to be composite.† This assertion was based on a negative result giving $3^{N-1}\not\equiv 1 \pmod{N}$.

On the basis of this conclusion Kraitchik‡ attempted to factor N arriving at another negative result that N had no factors and therefore was a prime. This conflict of results led us to recompute the value of $3^{N-1} \pmod{N}$ which shows clearly a mistake in the original calculation arising from the choice of 3 for a base instead of another number prime to $10^{23}-1$. Such another base would have furnished the extra check which would have detected the error.

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* A recent letter from Mr. R. E. Powers informs us that he has been to the trouble of finding 150 quadratic residues of N.

† This Bulletin, vol. 33 (1927), p. 338.

‡ Mathesis, vol. 42 (1928), p. 386.

The recomputation revealed the following results:

$$3^{N-1} \equiv 1 \pmod{N},$$
$$3^{(N-1)/11} \equiv 14\ 45009\ 64787\ 71867\ 25049 = r_1 \pmod{N},$$
$$3^{(N-1)/4093} \equiv 98\ 37816\ 77563\ 73768\ 37434 = r_2 \pmod{N},$$

$((r_1 - 1), N) = ((r_2 - 1), N) = 1.$

By Theorem 3 of my paper cited above, it follows that the factors of N belong to the forms

$$\left.\begin{matrix} 23n + 1 \\ 121n + 1 \\ 4093n + 1 \end{matrix}\right\} 11390819n + 1.$$

If we seek to express N as the difference of squares $(a^2 - b^2)$, we have

$$a = 129750757490761n + 115222895547343.$$

If we restrict a modulo 12 and 25, the smallest admissible value to try is

$$a = 5435003952668544.$$

The total range for a is given by the inequalities

$$N^{1/2} < a < \frac{1}{2}\left(W + \frac{N}{W}\right),$$

where $W = 22781638$, that is,

$$a < 243861122499491.$$

The maximum value of a is less than the smallest possible value; therefore a does not exist and N is a prime.

The results of Kraitchik's investigations will occupy a whole chapter of his forthcoming book.* Those interested in the factorization of large numbers will await with interest the exposition of the method by which Kraitchik was able to identify this sixth largest prime known.

Brown University

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* *Recherches sur la Théorie des Nombres*, vol. 2.

ON THE ATTRACTION OF SPHERES IN ELLIPTIC SPACE*

BY JAMES PIERPONT

1. *Introduction.* C. Neumann, Killing, and Liebmann have treated the motion of a material particle about a center of attraction in elliptic (hyperbolic) space. The question arises do these results hold when the center of attraction is replaced by a spherical mass.

Let the sphere be placed at the origin of coordinates O, let the polar coordinates of an element of volume at P of the sphere be ρ, ϕ, θ, where ϕ, θ are latitude and longitude. The element of volume is then

$$dv = \sin^2 \rho \cos \phi d\theta d\rho d\phi,$$

where for simplicity we take the space constant $R=1$. We will suppose the elementary mass A attracted by the sphere is on the z axis. Let $OA = \alpha$, $AP = \epsilon$, in elliptic measure. The force of attraction we will take to be

$$F = \frac{cdv}{\sin^2 \epsilon}, \qquad c \text{ a constant.}$$

If ψ is the angle AP makes with the z axis, the work done by the force F for a small displacement of A of extent $\delta\alpha$ along the z axis is

$$\delta W = F \cos \psi dv \cdot \delta\alpha.$$

It will be convenient to set

$$a = \sin \alpha, \quad r = \sin \rho, \quad e = \sin \epsilon, \quad p = \sin \phi,$$
$$a' = \cos \alpha, \quad r' = \cos \rho, \quad e' = \cos \epsilon, \quad p' = \cos \phi.$$

We have then

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* Presented to the Society, New York, March 29, 1929.

$$\cos\psi = \frac{(r' - a'e')}{ae},$$

$$e = (\alpha p^2 + \beta p + \gamma)^{1/2},$$

where

$$\alpha = -a^2r^2, \qquad \beta = -2aa'rr', \qquad \gamma = 1 - a'^2r'^2.$$

Then

$$\delta W = 2\pi c(aG - a'H)\delta\alpha,$$

where

$$G = \int_0^\rho r^2r'Ld\rho, \qquad L = \int_{-\pi/2}^{\pi/2} p'd\phi/e^3,$$

$$H = \int_0^\rho r^3Md\rho, \qquad M = \int_{-\pi/2}^{\pi/2} pp'd\phi/e^3.$$

Now

$$L = \left[\frac{4\alpha p + 2\beta}{\Delta(\alpha p^2 + \beta p + \gamma)^{1/2}}\right]_{-1}^{1}, \qquad \Delta = 4\alpha\gamma - \beta^2 = -4a^2r^2,$$

$$\alpha + \beta + \gamma = 1 - (a'r' + ar)^2,$$

$$(\alpha + \beta + \gamma)^{1/2} = \sin(\alpha - \rho),$$

$$(\alpha - \beta + \gamma)^{1/2} = \sin(\alpha + \rho),$$

$$4\alpha + 2\beta = -4ar\cos(\alpha - \rho),$$

$$4\alpha - 2\beta = 4ar\cos(\alpha + \rho).$$

Hence

$$L = \frac{2}{ar}\,\frac{\sin 2\rho}{\cos 2\rho - \cos 2\alpha}, \qquad G = \frac{1}{a}\int_0^\rho \frac{\sin^2 2\rho d\rho}{\cos 2\rho - \cos 2\alpha}.$$

Turning to M, we have

$$M = -\left[\frac{4\gamma + 2\beta p}{\Delta(\alpha p^2 + \beta p + \gamma)^{1/2}}\right]_{-1}^{1},$$

$$4\gamma - 2\beta = 4(1 - a'r'\cos(\alpha + \rho)),$$

$$4\gamma + 2\beta = 4(1 - a'r'\cos(\alpha - \rho)).$$

Hence

$$M = \frac{M'}{a^2r^2} + \frac{a'r'}{a^2r^2}M'',$$

where

$$M' = \frac{-4\sin\rho\cos\alpha}{\cos 2\alpha - \cos 2\rho}, \quad M'' = \frac{-2\sin 2\rho}{\cos 2\rho - \cos 2\alpha}.$$

Hence

$$M = \frac{2a'}{a^2}\cdot\frac{r}{a^2 - r^2}.$$

Thus

$$G = \frac{2}{a}\int_{-1}^{1}\frac{r^2(1-r^2)^{1/2}}{a^2 - r^2}dr,$$

$$H = \frac{2a'}{a^2}\int_{-1}^{1}\frac{r^4 dr}{T},$$

where $T = (a^2 - r^2)(1 - r^2)^{1/2}$. Then

$$aG - a'H' = 2\int_0^r \frac{r^2(1-r^2)dr}{T} - \frac{2}{a^2}\int_0^r \frac{r^2 dr}{T}$$

$$= \frac{2}{a^2}\int_0^r \frac{r^2 dr}{(1-r^2)^{1/2}} = \frac{1}{2a^2}(2\rho - \sin 2\rho).$$

Now the volume of a sphere of radius ρ in elliptic measure is $V = \pi(2\rho - \sin 2\rho)$. As $m = cV$ is the mass of the sphere, this gives

$$\delta W = \frac{m}{a^2}\cdot\delta\alpha.$$

The attraction of the sphere on the point A is thus m/a^2, which is the same as if the mass of the sphere were concentrated at its center.

2. *Another Method.* Let us define the potential of an element of mass dm at a distance ϵ by $dU = c\ \mathrm{ctn}\ \epsilon\cdot dv$, c constant, and for the whole body supposed homogeneous, by

$$U = c\int \mathrm{ctn}\ \epsilon\cdot dv,$$

Then α being as in §1,

$$\frac{\partial U}{\partial \alpha} = c\int dv\cdot\frac{\partial}{\partial \alpha}\operatorname{ctn}\epsilon = c\int dv\cdot\frac{\partial \operatorname{ctn}\epsilon}{\partial \epsilon}\cdot\frac{\partial \epsilon}{\partial \alpha}.$$

Let A' be the point on the z axis at a distance $\alpha+\delta\alpha$ from O. Let B lie on PA' at a distance ϵ from P. Then in the small triangle ABA' we have

$$\cos\psi = \frac{\sin BA'}{\sin AA'}, \quad \text{or} \quad \cos\psi = \frac{\partial \epsilon}{\partial \alpha}.$$

Thus

$$-\frac{\partial U}{\partial \alpha} = c\int\frac{\cos\psi}{\sin^2\epsilon}\cdot dv = \int F\cos\psi\cdot dv = \int F_z\, dv,$$

where F_z is the component of F along the z axis.

This result gives us another method of finding the attraction of a homogeneous sphere on a point at A. We have

$$U = c\int \operatorname{ctn}\epsilon\, dv = c\int\frac{e'}{e}r^2p'd\theta d\rho d\phi$$
$$= 2\pi c(a'G_1 + aH_1),$$

where

$$G_1 = \int_0^{\rho} r^2r'L_1d\rho, \qquad L_1 = \int_{-1}^{1}\frac{dp}{Q},\ Q = (\alpha p^2+\beta p+\gamma)^{1/2},$$

$$H_1 = \int_0^{\rho} r^3M_1d\rho, \qquad M_1 = \int_{-1}^{1}\frac{pdp}{Q}.$$

Now

$$L_1 = \frac{1}{(-\alpha)^{1/2}}\left\{\sin^{-1}\frac{\beta-2\alpha}{(-\Delta)^{1/2}} - \sin^{-1}\frac{\beta+2\alpha}{(-\Delta)^{1/2}}\right\},$$

$$\beta - 2\alpha = -2ra(a'r' - ar) = -2ar\cos(\alpha+\rho),$$

$$\beta + 2\alpha = -2ar\cos(\alpha-\rho).$$

Hence

$$L_1 = \frac{1}{ar}\left\{\sin^{-1}(\cos(\alpha+\rho)) - \sin^{-1}(\cos(\alpha-\rho))\right\} = \frac{2\rho}{ar}.$$

Similarly

$$M_1 = \left[\frac{(\alpha p^2 + \beta p + \gamma)^{1/2}}{\alpha}\right]_{-1}^{1} - \frac{\beta L_1}{2\alpha}$$

$$= \frac{1}{\alpha}\left\{(\alpha+\beta+\gamma)^{1/2} - (\alpha-\beta+\gamma)^{1/2}\right\} - \frac{\beta L_1}{2\alpha}$$

$$= \frac{1}{a^2r^2}\left\{\sin(\alpha+\rho) - \sin(\alpha-\rho)\right\} - \frac{2a'r'\rho}{a^2r^2} = \frac{2a'}{a^2r^2}(r - r'\rho).$$

Hence

$$G_1 = \frac{1}{4a}(\sin 2\rho - 2\rho\cos 2\rho),$$

$$H_1 = \frac{2a'}{a^2}(\tfrac{1}{2}\rho - \tfrac{1}{4}\sin 2\rho - \tfrac{1}{8}\sin 2\rho + \tfrac{1}{4}\rho\cos 2\rho).$$

Thus

$$U = \frac{a'}{2a}(2\rho - \sin 2\rho) = \frac{a'm}{a} = m\,\mathrm{ctn}\,a\ ;$$

that is, the potential of a sphere at A is the same as if the whole mass were concentrated at its center.

3. *Attraction at a Point Within.* We consider now the potential W of a homogeneous spherical shell at a point within the cavity. This may be treated briefly. Proceeding as in §2, we have

$$W = 2\pi c(a'G_2 + aH_2),$$

where

$$G_2 = \int r^2r'L_2d\rho, \qquad H_2 = \int r^3M_2d\rho.$$

Here L_2, M_2 are the same as L_1, M_1 in §2 except that now we must take $\rho > \alpha$. Hence

$$L_2 = \frac{2\alpha}{ar} = -ar, \qquad M_2 = \frac{2r'}{ar^2} + a'r'.$$

These give

$$G_2 = -a\int_{\rho_1}^{\rho_2} r^3 dr, \qquad H_2 = \frac{2}{a}\int_{\rho_1}^{\rho_2} r dr + a'\int_{\rho_1}^{\rho_2} r^3 dr.$$

Hence

$$W = 4\pi c\int_{\rho_1}^{\rho_2} r dr$$

which is a constant. Thus the attraction on a point within the cavity is null.

4. *Conclusion.* We have supposed the sphere to be of uniform density. It is obvious from the above that we get the same result if the sphere is made up of concentric layers each of constant density, or if the density is a function of ρ. Finally we remark that the analysis employed may be extended readily to hyperbolic space.

In connection with the present note we wish to call attention to a valuable paper* contributed by F. Cajori to the memorial volume "Sir Isaac Newton 1727–1927." It is almost universally held that Newton's delay was due to the fact that the earth's radius was not correctly known at that epoch. Cajori argues conclusively, so we believe, that the real reason was much deeper. It was in fact Newton's inability at that time to show that the attraction of a sphere on an external particle is the same as if the mass of the sphere were concentrated at its center.

YALE UNIVERSITY

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* *Newton's twenty year's delay in announcing the law of gravitation.*

THE EXISTENCE OF THE LEBESGUE-STIELTJES INTEGRAL*

BY R. L. JEFFERY

A definition of a Lebesgue-Stieltjes integral of a function $f(x)$ defined on (a, b) with respect to a non-decreasing function $V(x)$ bounded on (a, b) has been given by Hildebrandt.†

This definition involves the idea of the measurability of f with respect to V. If α is the interval $a'<x<b'$, then $V(\alpha)=V(b'-0)-V(a'+0)$. Let a set E be enclosed in a finite or countably infinite set of non-overlapping open intervals $A\equiv\alpha_1, \alpha_2, \cdots$. Let $V(E)$ be the lower limit of $V(A)=\sum V(\alpha_i)$ for all possible enclosures A. In the same way define $V(CE)$. When

$$V(E) + V(CE) = V(a,b) = V(b) - V(a), \tag{1}$$

the set E is said to be measurable relative to V. If for all real values of l the set for which $f>l$ satisfies (1), then f is measurable relative to V. Hobson‡ gives a definition which involves a different formulation of the same idea. To state this we shall make use of the following correspondence between the points of $\alpha=V(a)\leqq u\leqq V(b)=\beta$ and $a\leqq x\leqq b$. First, if x is a point of discontinuity of V, then x goes by means of $u=V(x)$ into the closed interval $V(x-0)\leqq u \leqq V(x+0)$. There will then correspond to each u on (α, β) at least one value of x on (a, b). If to a value of u there corresponds more than one value of x, then V is constant throughout an interval, and x_u shall be the lower end point of this interval, or the lower bound of points of the interval in case it is open. If $f(x)$ is any function defined on (a, b), then $\psi(u)$ is defined by $\psi(u)=f(x_u)$ and

$$LS\int_a^b f(x)dV(x) = L\int_\alpha^\beta \psi(u)du,$$

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* Presented to the Society, September 7, 1928.

† This Bulletin, vol. 24, pp. 188–190.

‡ *Theory of Functions of a Real Variable*, 3d ed., vol. I, §445.

when the latter exists. Thus the requirement that f be measurable relative to V is replaced by the requirement that ψ be measurable on (α, β). In this note we determine some necessary and sufficient conditions bearing on V under which all Lebesgue measurable functions f are measurable relative to V. These conditions are also necessary and sufficient to insure that all measurable functions f are carried into functions $\psi(u) = f(x_u)$ measurable on (α, β).

As a matter of notation, we shall use A and B to indicate sets of intervals enclosing E and CE respectively. Each set shall be non-overlapping and open, and each interval shall contain at least one point of the set enclosed. E_u shall denote the set into which E_x is carried by means of the relation $u = V(x)$, a point of discontinuity of V going into a closed interval as agreed upon above. If $x_1, x_2, \cdots$ are the countable set of discontinuities of V and s_i the saltus of V at x_i, then, for a given ϵ, $K_x \equiv x_1, x_2, \cdots, x_k$ where $\sum_{i=k+1}^{\infty} s_i < \epsilon/2$. For convenient reference we state some properties* of all sets G which are measurable relative to V.

(1) If E_1 and E_2 belong to G then $E_1 + E_2$ belongs to G.

(2) The Borel measurable sets belong to G.

(3) A necessary and sufficient condition that E be measurable relative to V is that there exist intervals A and B enclosing E and CE such that $V(AB) < \epsilon$.

(4) Any set E on a set of open intervals throughout which V is constant is measurable relative to V.

This follows at once from (3). For evidently E can be put in A where $V(A) = 0$. If then CE is put in any set B, we have $0 \leqq V(AB) \leqq V(A)$, which shows that (3) is satisfied.

THEOREM I. *A necessary and sufficient condition that E_x be measurable relative to V is that E_u be measurable on (α, β).*

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* (1) and (2) are proved by Radon, Wiener Sitzungberichte, vol. 122, pp. 1305 ff. (3) is proved by Bliss, (this Bulletin, vol. 24, p. 13), for continuous monotone functions, but the proof is applicable to any non-decreasing function.

Suppose E_x measurable relative to V. Then by (3) we can determine A_x and B_x so that $V(A_xB_x)<\epsilon$. The open intervals A_x go, by means of $u=V(x)$, into a set of open intervals A_u together with a countable set of points A_u' which belong to E_u, and E_u is on A_u+A_u'. Similarly CE_u is on open intervals B_u and a countable set of points B_u'. But $m(A_uB_u)=V(A_xB_x)<\epsilon$. This, together with the fact that A_u' and B_u' are countable, shows that E_u is measurable.

Suppose E_u is measurable. Let $\overline{E}_u$ be the closed intervals of E_u which come from points of discontinuity of V. $\overline{E}_x$ is then a countable set of points which, on account of (2) is measurable relative to V. Let $e_u=E_u-\overline{E}_u$. If K_u is the set of closed intervals coming from K_x, then e_u is on the open intervals CK_u. Put e_u in A_u and ce_u in B_u where A_u is on CK_u and where $m(A_uB_u)<\epsilon/2$. The open intervals A_u and B_u correspond to open intervals A_x and B_x which contain e_x and ce_x respectively. And since A_u is not on K_u, A_x contains none of the points of the set K_x. Consequently (A_xB_x) contains none of the set K_x, and as a result of this

$$V(A_xB_x) \leqq m(A_uB_u) + \sum_{i=k+1}^{\infty} s_i < \epsilon.$$

It then follows from (3) that e_x is measurable relative to V, and by (1) so also is $E_x=\overline{E}_x+e_x$.

It has been shown by Carathéodory* that the function V can be represented as the sum of two functions $V=\phi+\chi$, where ϕ depends only on the discontinuities of V and where χ is continuous. We can now prove the following theorem.

THEOREM II.† *In order that every measurable set E_x go into a measurable set E_u, it is necessary and sufficient that χ be absolutely continuous.*

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* *Vorlesungen über Reelle Funktionen*, §153.

† The corresponding theorem for continuous functions has been proved by Rademacher, (Monatshefte für Mathematik und Physik, vol. 27 (1916), p. 266). His theorem is the special case of ours where ϕ is identically zero.

The condition is sufficient. Suppose E_x measurable. Let ${}_1E_x$ be the part of E_x which belongs either to the discontinuities of V or to intervals throughout which V is constant. ${}_1E_u$, being a set of closed intervals and a countable set of points, is measurable. We complete the argument by showing that $e_u = E_u - {}_1E_u$ can be put in A_u and ce_u in B_u, where $m(A_uB_u) < \epsilon$. The function χ being absolutely continuous, there exists a $\delta > 0$ such that if D is any set of intervals with $m(D) < \delta$, then $\chi(D) < \epsilon/2$. The set e_x contains none of the points of K_x. Hence, since e_x is measurable and since K_x is a finite set, e_x can be put in A_x and ce_x in B_x, where A_x contains none of the points of K_x, and where $m(A_xB_x) < \epsilon/2$. Since each interval of the set A_x contains at least one point of e_x, and since the end points of B_x are points of e_x it follows that A_x and B_x correspond to open intervals A_u and B_u containing e_u and ce_u respectively. Again, since A_x contains none of the points of K_x, neither does (A_xB_x). Hence,

$$m(A_uB_u) = V(A_xB_x) = \phi(A_xB_x) + \chi(A_xB_x) < \sum_{i=k+1}^{\infty} s_i + \epsilon/2 < \epsilon.$$

The condition is also necessary. Suppose χ is not absolutely continuous. For a given $\epsilon > 0$ let $\chi(\delta)$ be the upper limit of $\chi(\alpha)$ for all possible sets of intervals α for which $m\alpha < \delta$. Obviously, if $\delta_1 < \delta_2$, then $\chi(\delta_1) \leqq \chi(\delta_2)$. And since χ is not absolutely continuous, $\lim_{\delta \to 0} \chi(\delta) = d > 0$. Making use of these facts we now prove the following lemma.

LEMMA I. *There exists on (a, b) a set of points D_x of zero measure which contains no points of discontinuity of V, and no points of intervals throughout which V is constant, and which is carried, by means of $u = V(x)$, into a measurable* set D_u with $mD_u \geqq d$.*

Choose $\delta_1, \delta_2, \cdots$, an infinite sequence of positive numbers approaching zero monotonically, and such that $\sum \delta_i$

* Carathéodory (loc. cit., §512) has considered the analogous problem for continuous monotone functions. By a method different from ours he arrives at the existence of a set D_x of zero measure which goes into a set D_u with outer measure $\geqq d$.

converges. Since $\chi(\delta_i) \geqq d$ for all i, we can find a set of intervals α_x^i such that $\chi(\alpha_x^i) > d - \epsilon$, and $m\alpha_x^i < \delta_i$. The infinite sequence of sets of intervals α_x^i defines, by means of $u = \chi(x)$, an infinite sequence of sets of intervals α_u^i on $\alpha' = \chi(a) \leqq u \leqq \beta = \chi(b)$, where $m\alpha_u^i = \chi(\alpha_x^i) > d - \epsilon$, $(i = 1, 2, \cdots)$. It then follows* that on (α', β') we have a measurable set of points D_u'' where $mD_u'' > d - \epsilon$, and each point of which belongs to an infinite number of the sets of intervals α_u^i. If we remove from D_u'' the countable set which comes by means of $u = \chi(x)$ from the countable set of discontinuities of V, there remains a set D_u' which still has measure $> d - \epsilon$. Let D_x'' be the set on (a, b) whose image, by means of $u = \chi(x)$, is D_u''. Each point of D_u' is on an infinite set of the sets of intervals α_u^i. Consequently each point of the set D_x'' is on an infinite set of the sets of intervals α_x^i. Thus, for $i = l$, where l is as large as we wish, D_x'' is on an infinite set of the intervals $\alpha_x^{l+1}, \alpha_x^{l+2}, \cdots$. Hence $mD_x'' \leqq \sum_{i=l+1}^{\infty} m\alpha_x^i \leqq \sum_{i=l+1}^{\infty} \delta_i$. On account of the convergence of $\sum \delta_i$ and the fact that we can take l as great as we please, it follows that $mD_x'' = 0$. If now we consider a sequence of values of ϵ approaching zero, we get a sequence of sets ${}_1D_x'', {}_2D_x'', \cdots$, where ${}_iD_x''$ has zero measure and ${}_iD_u''$ has measure $\geqq d - \epsilon_i$. It follows that the set D_u' which consists of all the points in any of the sets ${}_iD_u'$ has measure $\geqq d$, and the corresponding set D_x' has measure zero.

It will now be shown that $u = V(x)$ carries D_x' into a set $\overline{D}_u$ measurable on (α, β) with $m\overline{D}_u \geqq d$. On (α', β'), D_u' is measurable and contains none of the points K_u which come from K_x by means of $u = \chi(x)$. Hence D_u' can be put in A_u and CD_u' in B_u where A_u contains none of the finite set K_u, and where $m(A_uB_u) < \epsilon/2$. The intervals A_u and B_u correspond by means of $u = \chi(x)$ to intervals A_x and B_x, where A_x contains D_x' and contains none of the points of K_x,

* W. H. Young, Proceedings of the London Mathematical Society, (2), vol. 2, p. 26. Also Borel, Comptes Rendus, December, 1903.

and where $\chi(A_xB_x) = m(A_uB_u) \leqq \epsilon/2$. But since A_x contains no points of K_x neither does (A_xB_x). Hence

$$V(A_xB_x) = \phi(A_xB_x) + \chi(A_xB_x) \leqq \sum_{i=k+1}^{\infty} s_i + \frac{\epsilon}{2} < \epsilon.$$

Hence D_x' is measurable relative to V. It then follows from Theorem I that $\overline{D}_u$ is measurable on (α, β). And evidently $m\overline{D}_u$ cannot be less than $\chi(D_x') \geqq d$. If we now remove from $\overline{D}_u$ the countable set which comes by means of $u = V(x)$ from intervals throughout which V is constant, we arrive at a set D_u, where $mD_u \geqq d$, and where the corresponding set D_x contains no discontinuities of V and no points of intervals throughout which V is constant, and where $mD_x = 0$.

If N_u is any non-measurable component of D_u, then N_x is part of D_x and $mN_x = 0$. But $u = V(x)$ carries N_x into N_u. Hence V does not carry every set measurable on (a, b) into a set measurable on (α, β). If $f(x)$ is a function such that $f = 1$ on N_x and zero elsewhere on (a, b), then $\psi(u) = f(x_u) = 1$ on N_u and zero elsewhere on (α, β). Thus $\psi(u)$ is not measurable on (α, β). Also, by Theorem I, f is not measurable relative to V. We have the following result.

THEOREM III. *A necessary and sufficient condition that every function f measurable on (a, b) be measurable relative to V is that χ be absolutely continuous. This condition is also necessary and sufficient that $\psi(u) = f(x_u)$ be measurable on (α, β).*

THEOREM IV.* *A necessary and sufficient condition that every function f bounded and measurable on (a, b) possess a Lebesgue-Stieltjes integral in the sense of Hildebrandt, or in the sense of Hobson, is that χ be absolutely continuous.*

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* M. D. Menchoff and Mlle. Bary, Annali di Mathematica, (4), vol. 5, have considered the Lebesgue-Stieltjes integral of f with respect to V where V is absolutely continuous. They show (p. 24) that the integral according to their definition exists whenever f is bounded and measurable. Their discussion is only applicable to the case where V is absolutely continuous.

ZERO-FREE REGIONS OF LINEAR PARTIAL FRACTIONS*

BY MORRIS MARDEN†

1. *Introduction.* The object of this paper is to determine simple regions in the plane which do not contain any zeros of the partial fraction

$$\Phi = \sum_{j=1}^{n} \frac{\alpha_j}{z - a_j},$$

when the α_j's are complex constants.

The case of real α_j's has already been adequately treated by Gauss, Lucas, Jensen, and Bôcher in connection with their study of the derivative of a polynomial and the jacobian of two binary forms. In terms of Φ, their results may be stated as follows.

(a) If all the α_j's have the same sign, there are no zeros of Φ outside of any convex polygon enclosing the points a_j. If all the a_j, in addition, lie on the line-segment AB, there are no zeros of Φ except at points on AB‡.

(b) If all the α_j's have the same sign, and the a_j's are either real or in conjugate imaginary pairs, there are no imaginary zeros of Φ at points outside of all the Jensen circles.‖ In any Jensen circle containing k a_j's, and not overlapping any other Jensen circle, there are at least $k-1$ and at most $k+1$ roots of Φ.¶

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* Presented to the Society, October 27, 1928.

† National Research Fellow, at Princeton University.

‡ The result of which this theorem is an immediate consequence was first stated by Gauss, (Werke, vol. 3, p. 112), in 1816, but rediscovered by Lucas (Comptes Rendus, 1868). This theorem, with reference to the derivative of a polynomial, was first stated by Lucas, Journal de l'École Polytechnique, vol. 46 (1879), p. 8.

‖ See §6 for definition. This part of the theorem was stated without proof by Jensen, Acta Mathematica, vol. 36 (1912), p. 190; proved by Walsh, Annals of Mathematics, vol. 22 (1920), pp. 128–144.

¶ Walsh, ibid.

(c) If $\sum\alpha_j=0$ and if all the a_j corresponding to positive α_j and to negative α_j lie respectively in the non-overlapping circular regions C_1 and C_2, there are no zeros of Φ at points outside of these regions.*

2. *Method.* In the case of complex α_j's as in the case of the real, the zero-free regions of Φ may be readily obtained when $K(\Phi)$ the conjugate imaginary of Φ, is interpreted as the resultant of all the vectors†

$$f_j = \frac{\beta_j}{\bar{z} - \bar{a}_j} \quad \text{and} \quad g_j = -\frac{i\gamma_j}{\bar{z} - \bar{a}_j}; \qquad (j = 1,2, \cdots, n).$$

where $\beta_j+i\gamma_j=\alpha_j$. If we set $a_j-z=r_je^{i\theta j}$, we may write

$$f_j = \frac{\beta_j}{r_j}e^{i(\theta_j+\pi)} \quad \text{and} \quad g_j = \frac{\gamma_j}{r_j}e^{i(\theta_j+\pi/2)}.$$

These equations tell us that, as vectors drawn from z, the first along the line a_jz and the second perpendicular to this line, and having magnitudes contingent only upon r_j and α_j, the f_j and g_j are both independent of choice of axes. A zero-free region of Φ is then one in which the vector

$$K(\Phi) = \sum_{j=1}^{n}(f_j + g_j)$$

or its component in any direction does not vanish.

As the α_j are here complex numbers, it is convenient to regard them as points in a separate α-plane. In this plane it may be assumed without loss of generality, due to the form of Φ, that none of the α_j's lies at the origin. This is equivalent to supposing that, for all j,

$$|f_j| + |g_j| > 0. \tag{1}$$

Since multiplication of Φ by e^{ki} does not affect its zeros, *any*

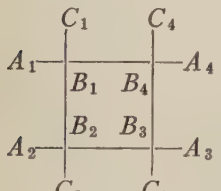
* Bôcher, Proceedings of the American Academy, vol. 40 (1904), pp. 469–484.

† The vector $F_j=f_j+g_j$ may be regarded as the force on a unit particle at z exerted according to the inverse distance law by a particle of mass α_j at a_j. See Bôcher, loc. cit., p. 475, footnote.

theorem true about the zeros of Φ for a given distribution D of the α_j's is likewise true for any distribution D' obtainable from D by a rotation of the α-plane about its origin. Our theorems will however be stated only for simple distributions of the α_j's, the understanding being, of course, that the same theorems hold for somewhat more general distributions.

3. *First Application.* The simplest cases with complex α_j's may be derived from the theorems of §1 by replacing points on the real axis by points on an arbitrary line through the origin. These cases we shall consider, however, by reason of the closing remark of §2, as already accounted for.

Let us assume at first instead that all the points α_j lie in* the first quadrant of the α-plane and the corresponding a_j in* the rectangle $B_1B_2B_3B_4$.



If z is any point in the angle $A_4B_4C_4$, the components of f_k and g_k parallel to A_1A_4 and C_1C_2 will be, respectively,

$$H_k = |f_k| \cos\phi_k + |g_k| \sin\phi_k,$$
$$V_k = |f_k| \sin\phi_k - |g_k| \cos\phi_k,$$

ϕ_k being the angle between f_k and a parallel to A_1A_4.

In order to locate the zeros of Φ in the angle $A_4B_4C_4$, we shall try to find points at which $\sum_1^n H_k = \sum_1^n V_k = 0$.

As $0 \leqq \phi_k \leqq \pi/2$, $H_k \geqq 0$. Because of (1), $H_k = 0$ for all k only in the following cases:

I. $\qquad f_k = \phi_k = 0, \qquad$ all k ;

II. $\qquad g_k = \phi_k - \dfrac{\pi}{2} = 0, \quad$ all k ;

III. $f_k = \phi_k = 0$, some k, $g_k = \phi_k - \pi/2 = 0$, remaining k.

In cases I and II, no zeros of Φ turn up because the V_k are of the same sign and therefore $\sum_1^n V_k \neq 0$.

In case III, on the contrary, where some of the a_j lie on

* In §§3, 4, 5, we shall use all our locative phrases "in," "to the right of," etc. in the wide sense, including thereby the boundary points of the regions considered. In §§6 and 7, we shall revert to the strict sense of these terms.

the line B_1B_4 and the corresponding α_j on the imaginary axis, and the remaining a_j lie on the line B_3B_4 and the corresponding α_j on the real axis, there is a possibility of $\sum_1^n V_k = 0$ and of a zero of Φ at the point B_4.

Similar considerations hold in the remaining angles $A_kB_kC_k$. *There are no zeros of* Φ *in any of the angles* $A_kB_kC_k$, *except perhaps at just one of the points* B_k. *If, furthermore, all the* a_k *lie on* B_1B_4, *there are no zeros to the right of* C_3C_4 *or to the left of* C_1C_2.

4. *Corollaries.* More elegant results for the case that the α_j's are in the first quadrant can now be obtained if the corresponding α_j are required to lie in a circle C of radius R. In the concentric circle C' of radius $2^{1/2}R$, a square S can be inscribed which will circumscribe C and which, when suitably chosen, will include in an angle $A_kB_kC_k$ any point outside C'. Consequently *there are no zeros of* Φ *outside of* C'.

If $n=2$, "outside of C'" must be taken in the strict sense. If a_1 and a_2 are situated on C, $\pi R/2$ units apart, S may have a_1 on one side and a_2 on the adjacent side. Case III of §3 would then permit a zero to be on C', as the example

$$\frac{i}{z-i}+\frac{1}{z-1}=0$$

with the unit circle as C, shows.

If the points a_j just considered are all on the line-segment AB, Φ will have a larger zero-free region. On AB as diagonal a rectangle R can so be constructed as to include in an angle $A_kB_kC_k$ any point outside of the circle Γ having AB as diameter. *Outside of* Γ *there are therefore no zeros of* Φ.

Again "outside of Γ" must be given a strict meaning when $n=2$. If a_1 and a_2 are at A and B, case III arises with its possibility of a zero on Γ, as for instance in the example

$$\frac{1}{z-1}+\frac{i}{z+1}=0,$$

where A: $(-1, 0)$ and B: $(1, 0)$.

5. *Other Applications.* By essentially the methods of §3, the following results may also be deduced:

(a) *Suppose the α_j to consist of two groups G_1 and G_2, the former situated in the first quadrant and the latter in the second quadrant of the α-plane. If the a_j corresponding to G_1 and G_2 lie respectively in the angles $A_1B_1C_2$ and $A_4B_4C_3$, there are no zeros of Φ in the strip $C_1B_1B_4C_4$, save perhaps upon the segment B_1B_4.*

The exceptions may occur only if $G_1=G_{1a}+G_{1c}$ and $G_2=G_{2a}+G_{2c}$, where G_{1a} and G_{2a} have their α_j's on the imaginary axis, G_{1c} and G_{2c} their α_j's on the real axis, and G_{1a}, G_{1c}, G_{2a}, and G_{2c} have their a_j's respectively on the segments A_1B_1, B_1C_2, A_4B_4, B_4C_3.

If $G_{1c}=G_{2c}=0$ (that is, G_{1c} and G_{2c} contain no points), any point of B_1B_4 may be a zero of Φ.

If $G_{1a}\cdot G_{1c}\neq 0$ and $G_{2c}=0$, the point B_1 is the only possible zero of Φ on B_1B_4.

If $G_{1c}=0$ and $G_{2a}\cdot G_{2c}\neq 0$, the point B_4 is the only possible zero of Φ on B_1B_4.

If the lines C_1C_2 and C_3C_4 are coincident, we shall exclude the case $G_{1a}=G_{2a}=0$, but include the cases

$$G_{1a}=0 \text{ and } G_{2a}\cdot G_{2c}=0,$$

$$G_{2a}=0 \text{ and } G_{1a}\cdot G_{1c}=0,$$

and

$$G_{1a}\cdot G_{1c}\cdot G_{2a}\cdot C_{2c}\neq 0,$$

in which cases B_1 is the only possible zero of Φ on B_1B_4.

(b) *Suppose the α_j to consist of four groups $G_k(k=1, 2, 3, 4)$, the group G_k being in the kth quadrant of the α-plane. If the a_j corresponding to G_k lie in the angle $A_kB_kC_k$, there are no zeros of Φ in the rectangle $B_1B_2B_3B_4$.*

We are supposing no two sides of $B_1B_2B_3B_4$ to coincide. A degenerate case of interest is, however, one in which A_1A_4 and A_2A_3 are coincident and the a_j corresponding to G_k lie on A_1B_1 if $k=1$ or 2, and on A_4B_4 if $k=3$ or 4. Then, if not all the α_j's lie on the real axis, there are no zeros of Φ on the segment B_1B_4. The theorems of (a) and (b) also hold for $n=\infty$ provided Φ is absolutely convergent in the above-mentioned zero-free regions.

6. *A Jensen Case.* So far no restrictions have been placed upon the behavior of Φ for real values of z. In this section we shall assume, in order to make Φ real on the real axis, that the a_j and the corresponding α_j are both real or both in conjugate imaginary pairs. We shall also suppose the α_j corresponding to an a_j in the upper half-plane also to be in the upper half-plane, and vice-versa. Furthermore, we shall limit the α_j to the first and fourth quadrants of the α-plane.*

Let us construct in the z-plane the circles (the Jensen circles) having as diameters the line segments joining pairs of conjugate a_j, and the indefinite lines (the Jensen lines) through the pairs of conjugate a_j. If we denote by R the region consisting of all points simultaneously outside of all the Jensen circles and to the left of all the Jensen lines, we may define R by the inequalities

$$(x - b_j)^2 + y^2 - c_j^2 > 0, \quad x - b_j < 0 ;$$

all j corresponding to imaginary $a_j = b_j + ic_j$.

In the sum Φ let us discern two types of terms, one corresponding to real a_j:

$$t_1 = \frac{\beta_j}{z - b_j} = \frac{\beta_j[(x - b_j) - iy]}{(x - b_j)^2 + y^2}, \qquad (\alpha_j = \beta_j + \gamma_j),$$

and the other corresponding to a pair of conjugate a_j:

$$t_2 = \frac{\beta_j + i\gamma_j}{z - (b_j + ic_j)} + \frac{\beta_j - i\gamma_j}{z - (b_j - ic_j)} .$$

The latter has as component parallel to the y-axis:

$$I(t_2) = \frac{-2y[\beta_j\{(x - b_j)^2 + y^2 - c_j^2\} - 2c_j\gamma_j(x - b_j)]}{\{(x - b_j)^2 + (y - c_j)^2\}\{(x - b_j)^2 + (y + c_j)^2\}} .$$

As $\beta_j \geqq 0$, $c > 0$, and $\gamma > 0$, we find in R

$$\operatorname{sgn} I(t_1) = \operatorname{sgn} I(t_2) = -\operatorname{sgn} y.$$

The components of t_1 and t_2 parallel to the y-axis will there-

* See the footnote to §3.

fore not vanish at points in R off the real axis. That is to say, *there are no imaginary roots of* Φ *in* R.

The question of the real roots of Φ in R may be easily settled through a study of the real function Φ on the real axis. Again we shall separate the terms corresponding to real a_j from those corresponding to pairs of conjugate a_j. The first

$$S_1 = \frac{\beta_j}{x - b_j} \quad \text{has} \quad \frac{dS_1}{dx} = -\frac{\beta_j}{(x - b_j)^2};$$

the second

$$S_2 = \frac{\beta_j + i\gamma_j}{x - (b_j + ic_j)} + \frac{\beta_j - i\gamma_j}{x - (b_j - ic_j)},$$

$$\frac{dS_2}{dx} = \frac{2\beta_j\{c_j^2 - (x - b_j)^2\} + 4c_j\gamma_j(x - b_j)}{\{(x - b_j)^2 + c_j^2\}^2}.$$

Therefore $d\Phi/dx < 0$ in R. In addition, as x moves to the right from one a_j to the next a_j, Φ changes from $+\infty$ to $-\infty$. In other words, any interval of the real axis in R not containing any a_j has at most one root of Φ; between two successive a_j in R, there is exactly one root of Φ. In short, *if the number of* a_j *in* R *is* k, *the number of real roots of* Φ *in* R *is* k, $k-1$, *or* $k+1$.*

The above equations also allow us to deduce at once the results for the case in which the α_j corresponding to real a_j lie on the positive real axis, and the remaining α_j are in the second and third quadrants of the α-plane. The above theorems will hold for this case if R in these theorems is replaced by K, the region consisting of all points simultaneously inside of all the Jensen circles and to the left of all the Jensen lines. (K may not contain any points.)

These theorems will also be valid when $n = \infty$ provided Φ converges uniformly in any closed region in R or K not including any points a_j, and all the imaginary a_j lie to the right of a fixed line parallel to the y-axis.

* E. B. Van Vleck has an unpublished proof of Jensen's theorem which uses similar analysis.

7. *The Function* $\eta(z)$. As a final application particularly of the methods of §6, we turn to the function

$$\eta(z) = \frac{1}{z}\left\{\frac{\zeta'(z)}{\zeta(z)} - (1 + \log 2\pi)\right\}$$
$$= \frac{2}{z^2} - \frac{1}{z-1} - \sum_{k=1}^{\infty} \frac{1}{k(z+2k)} + \sum_{-\infty}^{\infty} \frac{1}{\rho_n(z-\rho_n)},$$

where $\zeta(z)$ is the Riemann zeta function, and the ρ_n its zeros.

Concerning the ρ_n, it is known* that

$$\rho_{-n} = \bar{\rho}_n$$
$$0 < R(\rho_n) < 1, \qquad \text{(all } n\text{)},$$
$$R(\rho_n) = \tfrac{1}{2}, \qquad \text{for } |I(\rho_n)| \leqq 300,$$

and that $\rho = \frac{1}{2} \pm Ki$, $K = 14.13472(5)$, are the nearest ρ_n to the real axis.

Let us draw a circle of radius K, center at $(1/2, 0)$ and denote by S the interior of this circle to the left of the imaginary axis. Then in S there are no imaginary zeros of $\eta(z)$. In each of the intervals $(0, -2)$, $(-2, -4)$, $(-4, -6)$, $(-6, -8)$, $(-8, -10)$, and $(-10, -12)$ of the real axis there is exactly one zero, and in the interval $(-12, \frac{1}{2} - K)$ there is at most one zero of $\eta(z)$.

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* See Hutchinson, Transactions of this Society, vol. 27 (1925), p. 49, and references therein. My attention was called to the logarithmic derivative of the zeta function by C. E. Hille.

THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS IN A SPACE OF $(n+1)$ DIMENSIONS*

BY D. R. DAVIS

1. *Introduction.* Among the various types of inverse problems of the calculus of variations are those of Darboux, Hamel, Hirsch, and Kürschak.† Darboux discussed the problem of the plane showing that for a given equation of the form $y''=\phi(x, y, y')$ there exist an infinity of functions $f(x, y, y')$ such that the integral $\int_{x_1}^{x_2} f(x, y, y')dx$ taken along one of the integral curves of the given equations furnishes a maximum or minimum. Hamel found the general type of integral whose minimizing arcs are straight lines. Of the last two Hirsch considers an equation of the type $F(x, y, y', y'', \cdots, y^{(n)})=0$ and Kürschak generalizes this by introducing n independent variables. In both of these cases it was found that a necessary and sufficient condition for a given equation of the type considered to give a solution of a problem in the calculus of variations is that it have its equation of variation self-adjoint. No such restriction was found in Darboux's problem; however, it is well known that every differential equation of the second order for plane curves may be transformed into one whose equation of variation is self-adjoint.

The inverse problem of the calculus of variations for three-dimensional space is treated in my thesis.‡ It is the

* Presented to the Society, San Francisco Section, June 2, 1928.

† Darboux, *Théorie des Surfaces*, vol. 3, §606.

A. Hirsch, *Ueber eine Charakteristische Eigenshaft der Differentialgleichungen der Variationsrechnung*, Mathematische Annalen, vol. 49, p. 49.

J. Kürschak, *Ueber die Transformation der partiellen Differentialgleichungen der Variationsrechnung*, Mathematische Annalen, vol. 56, p. 155.

G. Hamel, *Geometrieen, in denen die Geraden die Kurzesten sind*, Mathematische Annalen, vol. 59, p. 255.

‡ *Inverse problem of the calculus of variations in higher space*, written under the direction of G. A. Bliss, University of Chicago, 1926. Published in the Transactions of this Society, vol. 30 (1928), pp. 710–736.

purpose of this paper to discuss the corresponding problem for a space of $(n+1)$ dimensions.

2. *Fundamental Properties of given Differential Equations.* Let us consider a system of n differential equations of the form

$$H_i(x, y_i, y_i', y_i'') = 0, \quad (i,j = 1, \cdots, n), \tag{1}$$

whose solutions are

$$y_i = y_i(x);$$

these have the derivatives y_i', y_i'' with respect to x. Under the hypothesis that the equations of variation of the given equations (1) form a self-adjoint system,* a function $f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ can be determined such that the given equations are the differential equations for the solutions of the problem of minimizing the integral

$$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx. \tag{2}$$

The self-adjoint conditions which are needed here are summarized in the following theorem which is fully treated in the reference cited below.†

THEOREM. *Necessary and sufficient conditions that the system of differential expressions*

$$J_i(u) \equiv A_{ik}(x)u_k + B_{ik}(x)u_k' + C_{ik}(x)u_k'', \quad (i,k = 1,2, \cdots, n),$$

shall be self-adjoint are

$$C_{ik} \equiv C_{ki},$$
$$B_{ik} + B_{ki} \equiv 2C_{ik}',$$
$$A_{ik} \equiv A_{ki} - B_{ki}' + C_{ki}''.$$

In the above and following expressions, the notation of

* This condition is also necessary; see Theorem II of my thesis, loc. cit., or J. Hadamard, *Leçons sur le Calcul des Variations*, p. 156.

† See my thesis, loc. cit.

tensor analysis is used, that is, whenever two subscripts are alike in two factors of a term, say of the form $A_{ik}u_k$, then the expression represents a sum with respect to the repeated index.

The equations of variation of the system (1) are

$$H_{iy_j'}u_j'' + H_{iy_j'}u_j' + H_{iy_j}u_j = 0. \tag{3}$$

For this system the self-adjoint conditions of the above theorem give respectively the following relations:

$$\begin{aligned} H_{iy_j''} &= H_{jy_i''}, \\ H_{jy_i'} + H_{iy_j'} &= 2(H_{iy_j''})', \\ H_{jy_i} &= H_{iy_j} - (H_{iy_j'})' + (H_{iy_j''})'', \end{aligned} \tag{4}$$

which must be identities in x, y_i', y_i', y_i'''.

The second set of relations (4) assert that each of the functions $H_i (i = 1, \cdots, n)$ is linear in y_k'' $(k = 1, \cdots, n)$, since terms in y_k''' do not occur in the first members. Therefore, the given functions (1) may be written in the form

$$\begin{aligned} H_i = M_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \\ + P_{ij}(x, y_1, \cdots, y_n, y_1', \cdots, y_n')y_j''. \end{aligned} \tag{5}$$

In this notation the first of relations (4) becomes

$$P_{ij} = P_{ji}. \tag{6}$$

From the last two of relations (4) we have

$$H_{jy_i} - H_{iy_j} = [(H_{iy_j''})' - H_{iy_j'}]' = \tfrac{1}{2}(H_{jy_i'} - H_{iy_j'})'. \tag{7}$$

Since the coefficient of y_k''' in the expansion of the second member of this equation must vanish we have the following conditions:

$$H_{jy_i'y_k''} = H_{iy_j'y_k''}. \tag{8}$$

In the notation of (5) these relations become

$$P_{jky_i'} = P_{iky_j'}. \tag{9}$$

From these conditions and (6) it follows that the expression

$P_{jky_i'}$ remains unchanged under all permutations of the indices i, j, k.

From (4_2) with the aid of (5) and (9), we obtain

$$
\begin{aligned}
M_{iy_j'} + M_{jy_i'} &= 2P'_{ij} - P_{iky_j'}y_k'' - P_{jky_i'}y_k'' \\
&= 2(P_{ijx} + P_{ijy_k}y_k' + P_{iy_ky_k'}y_k'') \\
&\quad - P_{jky_j'}y_k'' - P_{jky_i'}y_k'' \\
&= 2(P_{ijx} + P_{ijy_k}y_k').
\end{aligned} \tag{10}
$$

By applying relations (7) to the expressions for H_i given in (5) and with the use of (9), we get

$$
M_{jy_i} - M_{iy_j} = (P_{iky_j} - P_{jky_i})y_k'' + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})'. \tag{11}
$$

If these equations are to be identities in the variables involved, we must have

$$
\left\{
\begin{aligned}
&P_{iky_j} - P_{jky_i} = \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_{y_k'}, \\
&M_{jy_i} - M_{iy_j} = \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_x \\
&\qquad\qquad + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_{y_k}y_k'.
\end{aligned}
\right. \tag{11_a}
$$

The first of these systems may be obtained from (10) with the use of (9) but the second is an independent system of relations which must be satisfied.

The above results may be summarized as follows:

THEOREM 1. *If a system of differential equations of the second order*

$$
H_j(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0, \qquad (j = 1, \cdots, n),
$$

is to have equations of variation which are self-adjoint along every curve $y_i = y_i(x)$, then it must have the form

$$
\begin{aligned}
H_i &= M_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \\
&\quad + P_{ij}(x, y_1, \cdots, y_n, y_1', \cdots, y_n')y_j'',
\end{aligned}
$$

$(i, j = 1, \cdots, n)$ where the functions M_i and P_{ij} satisfy the conditions

$$(12)\qquad \begin{cases} P_{ij} = P_{ji}\,; \quad P_{jky_i'} = P_{iky_j'}, \\ M_{jy_i'} + M_{iy_j'} = 2(P_{ijx} + P_{ijy_k}y_k'), \\ M_{jy_i} - M_{iy_j} = \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_x \\ \qquad\qquad + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_{y_k}y_k', \end{cases}$$

identically in x, y_i, y_i'.

3. *Determination of the Integrand* f *of the Integral* I. For a system of equations $H_j = 0$ which has the form indicated in the above theorem with coefficients satisfying (12), it is possible to determine the integrand f of the integral I in (2) to an arbitrary function of $x, y_1, \cdots, y_n$ such that $H_j = 0$ are the equations of its minimizing curves.

A function $g(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ will first be determined satisfying the relations

$$(13)\qquad g_{y_i'y_j'} = P_{ij}, \qquad (i,j = 1, \cdots, n).$$

The relations (12_2) are necessary and sufficient conditions for the integrability of system (13). The function g will be determined by the integral

$$g = \int_{y_{10},\cdots,y_{n0}}^{y_1',\cdots,y_n'} L_1dy_1' + L_2dy_2' + \cdots + L_ndy_n',$$

where

$$L_1 = \int_{y'_{10},\cdots,y_{n'0}}^{y_1',\cdots,y_n'} P_{11}dy_1' + P_{12}dy_2' + \cdots + P_{1n}dy_n',$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$L_n = \int_{y'_{10},\cdots,y'_{n0}}^{y'_1,\cdots,y_n'} P_{n1}dy_1' + P_{n2}dy_2' + \cdots + P_{nn}dy_n'.$$

If g is a particular solution of the system (13), the most general solution is given by the formula

$$(14)\quad f = g(x,y_1,\cdots,y_n,y_1',\cdots,y_n') + a(x,y_1,\cdots,y_n) + b_k(x,y_1,\cdots,y_n)y_k', \quad (k = 1,\cdots,n),$$

where $a, b_1, \cdots, b_n$ are arbitrary functions of $x, y_1, \cdots, y_n$.

If the given equations are identical with the Euler-Lagrange equations for the integral (2) then the equations

$$\frac{d}{dx}(f_{y_i'}) - f_{y_i} = M_i + P_{ij}y_j'' \tag{15}$$

must be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n''$. The value of f given in (14) must now satisfy the system (15); thus we have

$$\frac{d}{dx}(g_{y_i'} + b_i) - (g_{y_i} + a_{y_i} + b_{ky_i}y_k') = M_i + P_{ik}y_k'',$$

which readily reduce to

$$(b_{ix} - a_{y_i}) + (b_{iy_k} - b_{ky_i})y_k' = M_i + g_{y_i} - [g_{y_i'x} - g_{y_i'y_k}y_k'. \tag{16}$$

If these equations are to be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ the expressions on the right must be independent of y_i' and linear in y_k' when $k \neq i$. Upon differentiating the second members of (16) with respect to y_i' one obtains

$$M_{iy_i'} - g_{y_i'y_i'x} - g_{y_i'y_i'y_k}y_k',$$

which by conditions (12_3), for $i=j$, are identically zero. Differentiation with respect to y_j' $(j \neq i)$, gives

$$b_{iy_j} - b_{jy_i} = M_{iy_j'} + g_{y_iy_j'} - g_{y_i'y_j} - g_{y_i'y_j'x} - g_{y_i'y_j'y_k}y_k'.$$

With the aid of (13) and (12_3) this system reduces to

$$b_{iy_j} - b_{jy_i} = \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'}) + g_{y_iy_j'} - g_{y_i'y_j}. \tag{17}$$

If in these equations we again take the partial derivatives, with respect to y_j', the functions on the right yield the following expressions:

$$\tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_{y_j'} + P_{ji}y_i - P_{ij}y_j. \tag{18}$$

By setting $k=j$ in the first system of (11_a) we see that the expressions (18) are each identically zero. Thus we have shown that the second members of (16) are, as indicated

by the expressions on the left, independent of y_i and linear in y_k for $k \neq i$.

Upon substituting the values of the first members of (17) in (16) we see that the system (16) which are partial differential equations for a, $b_1, \cdots, b_n$ may be replaced by the following:

$$(19)\qquad \begin{aligned} b_{jv_i} - b_{iv_j} &= g_{v_i' v_j} - g_{v_i v_j'} - \tfrac{1}{2}(M_{iv_j'} - M_{jv_i'}), \\ b_{jx} - a_{v_j} &= M_j + g_{v_j} - g_{v_j' x} - g_{v_j v_k'} y_k' \\ &\quad - \tfrac{1}{2}(M_{jv_k'} - M_{kv_j'}) y_k' . \end{aligned}$$

In (19_1) there are $n(n-1)/2$ equations and in (19_2) n equations in the $n+1$ variables x, $y_1, \cdots, y_n$. The determinant of the matrix of the functional expressions on the right in (19_1) is skew symmetric. Solutions for the system (19) exist if the totality of $n(n+1)/2$ equations are compatible.

We shall now prove the following theorem.

THEOREM 2. *A necessary and sufficient condition that there exist a solution for a system of differential equations of the form*

$$(20)\qquad b_{jv_i} - b_{iv_j} = \phi_{ij}, \quad b_{jx} - a_{v_j} = \theta_j,$$

where a, ϕ_{ij}, b_j, θ_j, $(i, j = 1, 2, \cdots, n)$, *are functions of* $x, y_1, \cdots, y_n$ *and* $\phi_{ij} = -\phi_{ji}$, *is that*

$$(21)\qquad \phi_{ijx} - \theta_{jv_i} + \theta_{iv_j} \equiv 0$$

identically in $x, y_1, \cdots, y_n$ *for every pair of values of* i *and* j.

That the condition is necessary is evident. In order to prove that it is also sufficient consider the given equations (20) for fixed values of i and j, namely, $i = p$, $j = q$. For these values of i and j system (20) gives

$$(22)\quad b_{qv_p} - b_{pv_q} = \phi_{pq}, \quad a_{v_q} - b_{qx} = -\theta_q, \quad b_{px} - a_{v_p} = \theta_p.$$

Now since

$$\phi_{pqx} - \theta_{qv_p} + \theta_{pv_q} \equiv 0,$$

identically in x, y_p, y_q, there exists a solution, say a,

b_p, b_q, of equations (22)*. Now let $j=r$, where r is any other value of j than q. The substitution of the functions a, b_p, b_r in the equations for $i=p$, $j=r$ gives

$$(23)\quad b_{ry_p} - b_{py_r} = \phi_{pr}, \quad a_{y_r} - b_{rx} = -\theta_r, \quad b_{px} - a_{y_p} = \theta_p.$$

Since the last of these equations is identical with the last equation of (22), it is satisfied by the functions a, b_p. Solving for the derivatives of b_r in the first two equations, we obtain

$$b_{ry_p} = \phi_{pr} + b_{py_r}, \quad b_{rx} = \theta_r + a_{y_r},$$

which are compatible provided that

$$\phi_{prx} + b_{py_rx} = \theta_{ry_p} + a_{y_ry_p}.$$

By employing the last of equations (22) this equation becomes

$$\phi_{prx} - \theta_{ry_p} + \theta_{py_r} = 0,$$

which is identically zero by hypothesis. A similar condition prevails for $i=s$, $j=q$, where s is any other value of i than p. It follows that the remaining $(n+3)(n-2)/2$ equations are compatible for a solution of the given equations for $i=p$, $j=q$. Hence, there exists a solution of the given system.

System (20) is identical with (19) if ϕ_{ij} and θ_j represent the second members of (19) respectively. Relations (21) applied to the second members of (19) give the following system:

$$(24)\quad 0 = (M_{iy_j} - M_{jy_i}) - \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_x + \tfrac{1}{2}(M_{jy_k'} - M_{ky_j'})_{y_i}y_k' + \tfrac{1}{2}(M_{ky_i'} - M_{iy_k'})_{y_j}y_k'.$$

These equations are identities in x, $y_1, \cdots, y_n$, $y_1', \cdots, y_n'$, provided the following system is identically zero:

$$(25)\quad (M_{jy_k'} - M_{ky_j'})_{y_i}y_k' + (M_{ky_i'} - M_{iy_k'})_{y_j}y_k' - (M_{jy_i'} - M_{iy_j'})_{y_k}y_k' \equiv 0,$$

* See H. Weber, *Die Partiellen Differentialgleichungen*, vol. 1, pp. 221, etc.

which is obtained by replacing the first two expressions in parenthesis of (24) by their value obtained from the last system of (12).

It is readily verified that those terms of (25) for which any two of the indices i, j, k are equal vanish identically. It remains to show that the terms for which $k \neq i \neq j$ will also vanish. For this purpose consider the last of the self-adjoint relations (12). Let this system be written successively for $j=p$, $i=r$; $j=q$, $i=p$; $j=r$, $i=q$. If these three equations are differentiated with respect to y_q', y_r', y_p' respectively and the results added, we obtain equations (25) except for the factor y_k'. Hence, equations (25) are identically zero in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$.

Since the conditions of Theorem 2 are satisfied there exists a solution of the system (19). Let a, b_i, b_j be a particular solution of system (19); then the most general solution will be

$$
\begin{aligned}
&a(x, y_1, \cdots, y_n) + u(x, y_1, \cdots, y_n), \\
&b_i(x, y_1, \cdots, y_n) + v_i(x, y_1, \cdots, y_n), \\
&b_j(x, y_1, \cdots, y_n) + v_j(x, y_1, \cdots, y_n), \qquad (i < j),
\end{aligned} \tag{26}
$$

where the functions u, v_i, v_j satisfy the relations

$$
v_{jy_i} - v_{iy_j} = 0, \quad u_{y_j} - v_{jx} = 0, \quad v_{ix} - u_{y_i} = 0, \quad (i>j). \tag{27}
$$

The general solution of the system (19) may now be written in the form

$$
\begin{aligned}
&a(x, y_1, \cdots, y_n) + u(x, y_1, \cdots, y_n), \\
&b_k(x, y_1, \cdots, y_n) + v_k(x, y_1, \cdots, y_n), \\
&\qquad\qquad (k = 1, \cdots, n).
\end{aligned} \tag{28}
$$

Relations (27) are necessary and sufficient conditions that the expression

$$
u + v_k y_k', \qquad (k = 1, \cdots, n),
$$

should be the total derivative of an arbitrary function

$t(x, y_1, \cdots, y_n)$. In view of (28), the integrand f of (14) takes the form

$$(29)\quad f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + a(x, y_1, \cdots, y_n) + b_k(x, y_1, \cdots, y_n)y_k' + \frac{d}{dx}t(x, y_1, \cdots, y_n),$$

where $a, b_1, \cdots, b_n$ are solutions of the system (19) and t is an arbitrary function of $x, y_1, \cdots, y_n$.

THEOREM 3. *If a system of differential equations of the form*

$$H_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0, \qquad (i = 1, \cdots, n),$$

has equations of variation

$$H_{iy_j''}u_j'' + H_{iy_j'}u_j' + H_{iy_j}u_j = 0,$$

which form a self-adjoint system along every curve $y_i = y_i(x)$, then there exists an integral of the form

$$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

having the equations $H_i(x, y_j, y_j', y_j'') = 0$ as its Euler equations. The most general such integral has an integrand

$$f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + a(x, y_1, \cdots, y_n) + b_k(x, y_1, \cdots, y_n)y_k' + \frac{d}{dx}t(x, y_1, \cdots, y_n)$$

where g is a particular solution of (13), *$a, b_1, \cdots, b_n$ are a particular solution of system* (19), *and t is an arbitrary function of $x, y_1, \cdots, y_n$.*

THE UNIVERSITY OF OREGON

ON THE MAPPING OF THE QUADRUPLES OF THE INVOLUTORIAL G_4 IN A PLANE UPON A STEINER SURFACE*

BY ARNOLD EMCH

1. *Introduction.* Castelnuovo has definitely shown† that every plane involution may be mapped uniformly upon a rational surface. As may be expected, and as the author has shown in case of the involution of sextuples,‡ from such a mapping process arise interesting properties of certain configurations and curves which reflect geometric properties of the corresponding surface, and conversely. In this paper the involution induced by the group

$$G_4 \equiv \begin{pmatrix} \pm x_1, & \pm x_2, & \pm x_3 \\ x_1, & x_2, & x_3 \end{pmatrix}$$

is investigated from this standpoint.

In what follows I shall denote the involutorial quadruple in the plane (x) merely by G_4. To construct this, let $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$ be the coordinate triangle, and $B(x_1, x_2, x_3)$ a generic point. Join B to A_1, A_2, A_3 and construct the fourth harmonic lines to BA_1, BA_2, BA_3 with respect to the pairs of sides A_1A_2, A_1A_3; A_2A_3, A_2A_1; A_3A_1, A_3A_2. These three lines intersect in the triangle $B_1(-x_1, x_2, x_3)$, $B_2(x_1, -x_2, x_3)$, $B_3(x_1, x_2, -x_3)$, which together with $B(x_1, x_2, x_3)$ form the G_4. If we construct for each B_i the symmetric G_6 and denote it by (B_i), we obtain the configuration of the octahedral group

$$G_{24} \equiv \begin{pmatrix} \pm x_i, & \pm x_k, & \pm x_l \\ x_1, & x_2, & x_3 \end{pmatrix},$$

* Presented to the Society, February 23, 1929.

† *Sulla razionalità delle involuzioni piane*, Mathematische Annalen, vol. 44 (1894), pp. 125–155.

‡ *On the mapping of the sextuples of the symmetric substitution group G_6 in a plane upon a quadric*, this Bulletin, vol. 33 (1927), pp. 745–750.

consisting of the four sextuples (B), (B_1), (B_2), (B_3), which lie each on a conic as is well known.*

2. *Mapping on the Steiner Surface.* The substitutions of the G_4 outside of identity, expressed as collineations, are the three involutorial perspectives

$$I_1 \equiv (\rho x_1' = -x_1, \quad \rho x_2' = x_2, \quad \rho x_3' = x_3),$$
$$I_2 \equiv (\rho x_1' = x_1, \quad \rho x_2' = -x_2, \quad \rho x_3' = x_3),$$
$$I_3 \equiv (\rho x_1' = x_1, \quad \rho x_2' = x_2, \quad \rho x_3' = -x_3),$$

which leave the quadruple G_4 invariant. If we propose to construct those integral rational homogeneous functions in (x) which are invariant under these perspectives, we must form homogeneous functions of the squares of the variables: $F(x_1^2, x_2^2, x_3^2)$. The simplest set of these is evidently $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0$, of which type there are only three linearly independent. These conics form a net with $A_1A_2A_3$ as a common self-polar triangle. Next in order is the quartic

$$(1) \quad b_1x_1^4 + b_2x_2^4 + b_3x_3^4 + c_1x_2^2x_3^2 + c_2x_3^2x_1^2 + c_3x_1^2x_2^2 = 0,$$

which may be reduced by a collineation to the form

$$(2) \quad a_1x_2^2x_3^2 + a_2x_3^2x_1^2 + a_3x_1^2x_2^2 + a_4(x_1^4 + x_2^4 + x_3^4) = 0,$$

so that all quartics of the webb (2) may be expressed linearly by the relations

$$(3) \quad \rho y_1 = x_2^2x_3^2, \quad \rho y_2 = x_3^2x_1^2, \quad \rho y_3 = x_1^2x_2^2,$$
$$\rho y_4 = x_1^4 + x_2^4 + x_3^4.$$

From (3) it is evident that to the quadruple G_4 in (x) corresponds in the space (y) uniquely a point. The locus of these points is easily obtained as the Steiner surface Γ_4

$$(4) \quad y_1^2y_2^2 + y_2^2y_3^2 + y_3^2y_1^2 - y_1y_2y_3y_4 = 0.$$

Hence we may state the following theorem.

* Emch, A., *Some geometric applications of symmetric substitution groups*, American Journal of Mathematics, vol. 45 (1923), pp. 192–207.

THEOREM 1. *By the transformation* (3) *the Steiner surface* (4) *is mapped upon a plane, so that to a generic point on the surface correspond four points* B, B_1, B_2, B_3 *of a self-projective quadruple in the group of involutions* I, I_1, I_2, I_3.

3. *Properties of Quadruples.* To a plane section of Γ_4 by $a_1y_1+a_2y_2+a_3y_3+a_4y_4=0$ corresponds in (x) the quartic (2). In particular to a section of Γ_4 by a plane through the triple-point A_4 of Γ_4 corresponds the quartic

$$a_1x_2^2x_3^2 + a_2x_3^2x_1^2 + a_3x_1^2x_2^2 = 0, \tag{5}$$

and to the section by $y_4=0$ the quartic

$$x_1^4 + x_2^4 + x_3^4 = 0. \tag{6}$$

The quartic (5) is the transform of the conic

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0, \tag{7}$$

by the quadratic transformation $\rho x_1' = x_2x_3$, $\rho x_2' = x_3x_1$, $\rho x_3' = x_1x_2$. To the conic (7) corresponds on Γ_4 a conic which is the residual intersection by the quadric $a_1y_2y_3+a_2y_3y_1+a_3y_1y_2=0$. When a point B lies on (7), then all points of the quadruple G_4 determined by it lie on it. Now two generic points determine the conic (7), so that we have the following obvious theorem.

THEOREM 2. *Two arbitrary distinct quadruples* G_4 *lie on a definite conic of the net* (7).

A tangent plane at a generic point P of Γ_4 cuts the surface in two conics K_1 and K_2, whose four intersections lie at P and on A_4A_1, A_4A_2, A_4A_3. To these conics correspond in (x) two conics on the quadrangle corresponding to P. Obviously, any conic in (x) may be one of these conics. The product of the two conics in (x), corresponding to K_1 and K_2, must have the form (2). Hence when $a_1x_1^2+a_2x_2^2+a_3x_3^2=0$ is one of the conics, the other $b_1x_1^2+b_2x_2^2+b_3x_3^2=0$ must be such that their product will have the form (2). This is the case when $b_1=1/a_1$, $b_2=1/a_2$, $b_3=1/a_3$, so that their product,

$(a_1x_1^2+a_2x_2^2+a_3x_3^2)\,(a_2a_3x_1^2+a_3a_1x_2^2+a_1a_2x_3^2)=0$, when expanded in the form (2), becomes

$$(8)\quad a_1(a_2^2+a_3^2)x_2^2x_3^2+a_2(a_3^2+a_1^2)x_3^2x_1^2 + a_3(a_1^2+a_2^2)x_1^2x_2^2+a_1a_2a_3(x_1^4+x_2^4+x_3^4)=0.$$

The plane to whose section with Γ_4 corresponds in (x) the degenerate quartic (8) is therefore

$$(9)\quad a_1(a_2^2+a_3^2)y_1+a_2(a_3^2+a_1^2)y_2 + a_3(a_1^2+a_2^2)y_3+a_1a_2a_3y_4=0.$$

As the coordinates of this plane are $\rho u_1=a_1(a_2^2+a_3^2)$, $\rho u_2=a_2(a_3^2+a_1^2)$, $\rho u_3=a_3(a_1^2+a_2^2)$, $\rho u_4=a_1a_2a_3$, the envelope of the tangent-planes of the Steiner surface, by elimination of the a's, is easily obtained as

$$(10)\qquad 4u_4^3-(u_1^2+u_2^2+u_3^2)u_4+u_1u_2u_3=0,$$

a cubic of class 3, as is well known.

Consider now a generic line g in (y) cutting Γ in four points P_1, P_2, P_3, P_4. To these correspond in (x) four quadruples G_4^1, G_4^2, G_4^3, G_4^4, which together form the base points of the pencil of quartics corresponding to the plane sections through g. Through g we can draw three tangent planes to Γ_4 according to (10), which touch Γ_4 in T_1, T_2, T_3. To these T's correspond in (x) three quadruples D_1, D_2, D_3. The tangent planes t_1, t_2, t_3 cut Γ_4 in three pairs of conics, of which each passes through the four P's and one of the points T, and to which correspond in (x) also three pairs of conics. Each of the latter contains all four quadruples G_4^i, $i=1, 2, 3, 4$, and one of the quadruples D. Hence the quadruples G_4 and D are as entities precisely in the relation of a complete quadrangle with its three diagonal points.

The analogy of the quadruples G and D with a complete quadrangle leads to a very simple geometric proof that the Steiner surface is of class 3. Through the quadruples G we can pass three pairs of conics $K(G_4^1, G_4^2)$, $K(G_4^3, G_4^4)$; $K(G_4^1, G_4^3)$, $K(G_4^2, G_4^4)$; $K(G_4^1, G_4^4)$, $K(G_4^2, G_4^3)$, such that each pair contains all four quadruples. Each of these pairs

has a quadruple D as base points, and to each corresponds on Γ_4 two conics through P_1, P_2, P_3, P_4 and one of the points T. The plane of these conics touches Γ_4 at T. Hence there are three such tangent-planes through g to the Steiner surface. These results may be stated in the following form.

THEOREM 3. *The quadruples of the involutorial G_4 in the plane (x) may be arranged in groups of seven according to a complete quadrangle array. Any two quadruples in (x) determine such a configuration. Thus there are ∞^4 such configurations. To each such configuration correspond on the Steiner surface the four intersections of a line g with Γ_4 and the three points of tangency of the tangent-planes from g to Γ_4.*

4. *Quadruples on Quartics.* To a point on Γ_4:

$$y_1,\ y_2,\ y_3,\ (y_2^2 y_3^2 + y_3^2 y_1^2 + y_1^2 y_2^2)/(y_1 y_2 y_3),$$

corresponds in (x) the quadruple

$$[(y_2y_3)^{1/2},\ (y_3y_1)^{1/2}, (y_1y_2)^{1/2}]\ ;\ [-(y_2y_3)^{1/2}, (y_3y_1)^{1/2},\ (y_1y_2)^{1/2}]\ ;$$
$$[(y_2y_3)^{1/2}, -(y_3y_1)^{1/2}, (y_1y_2)^{1/2}]\ ;\ [\ (y_2y_3)^{1/2}, (y_3y_1)^{1/2}, -(y_1y_2)^{1/2}].$$

To a point on $x_1=0$ corresponds on Γ_4, $\rho y_1 = x_2^2 x_3^2$, 0, 0, $\rho y_4 = x_2^4 + x_3^4$, that is, a definite point on A_4A_1 in (y). To this same point, corresponds in (x) a quadruple which lies on $x_1=0$, and which, from the form of the equation $x_2^4 - (y_1/y_4)x_2^2x_3^2 + x_3^4 = 0$, is easily seen to have the form $(0, x_2, x_3)$; $(0, x_2, -x_3)$; $(0, x_3, x_2)$; $(0, x_3, -x_2)$. The third and fourth of these points are obtained by projecting the inverse of (x_1, x_2, x_3) and $(x_1, x_2, -x_3)$ in the quadratic involution $x_i' = 1/x_i$ from A_1 in (x) upon $x_1=0$. The first two points, each counted twice, must be considered as a degenerate involutorial quadrangle. The same is true of the other two.

Now a generic line l in (x) cuts a quartic C_4 in four points; consequently to l corresponds on Γ_4 a space quartic L_4, since it cuts the curve D_4, corresponding to C_4, and its plane in four points. As l cuts each of the sides of the coordinate triangle in one point, L_4 cuts each A_4A_1, A_4A_2, A_4A_3 in (y), say in B_1, B_2, B_3. To L_4 correspond in (x) besides l the three

associated lines l_1, l_2, l_3 in the involution, which together form an involutorial quadrilateral L whose vertices lie two by two on the sides of the coordinate triangle. If $a_1x_1+a_2x_2+a_3x_3=0$ is the line l, then the pole of l with respect to the coordinate triangle $A_1A_2A_3$ is $E(a_1a_3,\ a_3a_1,\ a_1a_2)$ and its quadratic transform $F(a_1,\ a_2,\ a_3)$. The triangular polar of F is $m=a_2a_3x_1+a_3a_1x_2+a_1a_2x_3=0$. This determines another involutorial quadrilateral $M=mm_1m_2m_3$ in (x) whose vertices lie again by pairs on the sides of $A_1A_2A_3$. To these vertices correspond on Γ_4 the same points B_1, B_2, B_3 as in the case of L. To the quadrilateral M corresponds on Γ_4 a space quartic M_4 through $B_1B_2B_3$. The plane β of these points cuts Γ_4 in a quartic B_4, and L_4 and M_4 each cut B_4 in a fourth point outside of B_1, B_2, B_3. The plane β has the form

$$\beta = a_1^2(a_2^4 + a_3^4)y_1 + a_2^2(a_3^4 + a_1^4)y_2 + a_3^2(a_1^4 + a_2^4)y_3 - a_1^2a_2^2a_3^2y_4 = 0. \tag{11}$$

Denoting the coordinates of this plane by u_1, u_2, u_3, u_4, its envelope is easily found as the class-cubic

$$4u_4^3 - (u_1^2 + u_2^2 + u_3^2)u_4 - u_1u_2u_3 = 0. \tag{12}$$

This is, however, different from the Steiner class-cubic. The plane β cuts Γ_4 in a quartic B_4 to which corresponds in (x) the quartic N:

$$a_1^2(a_2^4 + a_3^4)x_2^2x_3^2 + a_2^2(a_3^4 + a_1^4)x_3^2x_1^2 + a_3^2(a_1^4 + a_2^4)x_1^2x_2^2 - a_1^2a_2^2a_3^2(x_1^4 + x_2^4 + x_3^4) = 0. \tag{13}$$

N passes through the 12 vertices of the quadrilaterals L and M in (x), and moreover through the involutorial quadrangles corresponding to the fourth intersections of L_4 and M_4 with B_4. These results may be stated as follows.

THEOREM 4. *There exists in (x) a system of quartics which cut the sides of the fundamental triangle of the involution in 12 points of two complete quadrilaterals. To these correspond in (y) two space quartics through the same three points $B_1B_2B_3$*

on the double lines of Γ_4. *The locus of the planes* β *is a certain class-cubic.*

To the general invariant quartic of the involution

$$(14)\quad b_1x_1^4 + b_2x_2^4 + b_3x_3^4 + c_1x_2^2x_3^2 + c_2x_3^2x_1^2 + c_3x_1^2x_2^2 = 0$$

corresponds on Γ_4 a space quartic cut out by the cone

$$(15)\quad b_1y_2^2y_3^2 + b_2y_3^2y_1^2 + b_3y_1^2y_2^2 + y_1y_2y_3(c_1y_1 + c_2y_2 + c_3y_3) = 0,$$

which has A_4A_1, A_4A_2, A_4A_3 in (y) as nodal lines. When $b_1 = b_2 = b_3 = c_4$, then by means of Γ_4, (15) reduces to $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$, which verifies the results in case of a plane quartic on Γ_4.

Concerning the base points of a pencil of quartics of the general involutorial type we may state the following fact.

THEOREM 5. *The* 16 *base points of a pencil of involutorial quartics in* (x) *form* 4 *involutorial quadruples which lie two by two on* 6 *involutorial conics.*

5. *Flexes and Double Tangents.* If we draw the three tangent-planes from a line g to Γ_4, they cut Γ_4 in three pairs of conics through $(P_1P_2,\ P_3P_4)$; $(P_1P_3,\ P_2P_4)$; $(P_1P_4,\ P_2P_3)$. With every couple P_iP_k is associated just one such conic, just as in (x) there is just one conic through the quadruples G_4^i and G_4^k. As two of the points P determine the remaining two on the line joining them, it is evident that *there is just one such conic through two generic points* P_i *and* P_k *on* Γ_4.

A generic plane cuts Γ_4 in a rational quartic to which corresponds in (x) a quartic C_4 without double points. This is clear, since to a double point of L on A_4A_1 in (y) correspond in (x) four distinct points on A_2A_3, and to every other point of L corresponds a proper involutorial quadruple on C_4. The Hessian of C_4 is a sextic H_6 which as a covariant of C_4 cuts C_4 in 6 involutorial flex-quadruples. Hence to H_6 corresponds on Γ_4 a sextic which cuts the quartic L, corresponding to C_4, in six points. These form 15 couples P_iP_k and through each of these couples passes a conic on Γ_4. Hence we may state the following theorem.

THEOREM 6. *The* 24 *flexes of a quartic* C_4 *in* (x) *lie on* 15 *conics, of which each contains* 8 *of the* 24 *flexes.*

This follows, of course, also from the fact that any two generic involutorial quadruples lie on a conic.

For a plane of the pencil $y_1+y_2+y_3+\lambda y_4=0$, the corresponding quartic C_4 in (x) becomes symmetric, so that with every point on C_4 are associated 5 other points which together lie on a conic. *Hence in this case, in addition to the* 15 *conics of the flexes there are* 4 *other conics of which each contains* 6 *of the flexes.*

The 28 double tangents of C_4 may be grouped in 7 involutorial quadrilaterals, each with two involutorial quadrangles of points of tangency, so that the 56 points of tangency form 14 involutorial quadrangles.

THEOREM 7. *The* 28 *double tangents of the* C_4 *form* 7 *involutorial quadrilaterals whose* 56 *points of tangency lie* 8 *by* 8 *on* $\binom{14}{2}=91$ *conics.*

6. *Tropes.* There are four tangent planes touching Γ_4 along conics. To these correspond in (x) four couples of coincident conics, which must have the form $(a_1x_1^2+a_2x_2^2+a_3x_3^2)^2 = a_1^2x_1^4+a_2^2x_2^4+a_3^2x_3^4+2(a_1a_2x_1^2x_2^2+a_2a_3x_2^2x_3^2+a_3a_1x_3^2x_1^2)=0$. But for a quartic corresponding to a plane section there must be $a_1^2=a_2^2=a_3^2$, which for (a_1, a_2, a_3) gives the possible solutions (a, a, a); $(-a, a, a)$; $(a, -a, a)$; $(a, a, -a)$. Consequently the four tropes of the Steiner surface are

$$
\begin{aligned}
2(y_1+y_2+y_3)+y_4&=0,\\
2(y_1-y_2-y_3)+y_4&=0,\\
2(-y_1+y_2-y_3)+y_4&=0,\\
2(-y_1-y_2+y_3)+y_4&=0.
\end{aligned}
\tag{16}
$$

7. *Quartics on* G_{48}. When (x) and (x') are connected by the quadratic involution $x_i'=1/x_i$, then the corresponding points $P(y)$ and $P'(y')$ on the map Γ_4 are also birationally connected. The corresponding involution on Γ_4 has the form

$$(17) \qquad \delta y_1' = y_2 y_3, \quad \delta y_2' = y_3 y_1, \quad \delta y_3' = y_1 y_2,$$
$$\delta y_4' = y_1'^2 + y_2'^2 + y_3'^2,$$

considering that $y_4 = (y_2^2 y_3^2 + y_3^2 y_1^2 + y_1^2 y_2^2)/(y_1 y_2 y_3)$. If we form the line coordinates $p_{ik} = y_i y_k' - y_k y_i'$ of PP', we obtain a rational congruence, whose intersection with a complex is a ruled surface which cuts Γ_4 in an invariant curve of the involution $(y) \rightleftarrows (y')$. To these correspond in (x) invariant curves of the involution $(x) \rightleftarrows (x')$. Such curves are for example the sextics

$$(18) \qquad x_1^2(x_2^2 - x_3^2)^2 + x_2^2(x_3^2 - x_1^2)^2 + x_3^2(x_1^2 - x_2^2)^2 = 0,$$

which is the locus of couples of corresponding points (x), (x') on the tangents of the class-conic $u_1^2 + u_2^2 + u_3^2 = 0$, and

$$(19) \qquad x_1^2(x_2^2 + x_3^2)^2 + x_2^2(x_3^2 + x_1^2)^2 + x_3^2(x_1^2 + x_2^2)^2 + \lambda x_1^2 x_2^2 x_3^2 = 0.$$

Both are invariant in the Cremona group G_{48}, which is the product of the quadratic involution and the octahedral group G_{24}.

The curve (18) has the vertices of the fundamental triangle and the four invariant points of the quadratic involution as double points and is of the genus three type.

8. *Clebsch and Lüroth Curves.* The quartics which are invariant under the octahedral group G_{24} have the form

$$(20) \qquad \Sigma x_i^4 + 6\lambda \Sigma x_i^2 x_k^2 = 0.$$

Some of these curves are rather well known.* The covariant S is in the pencil, and the parameter λ for S has the value $\lambda' = (\lambda^2 - 2\lambda^3 - \lambda^4)/(6\lambda^4)$. The question whether there are any Clebsch and Lüroth curves in the pencil can easily be answered by determining those values of λ for which the so-called Clebsch determinant of order 6 vanishes.

In case of the quartic (20), this determinant reduces to the form

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* Ciani, E., *I varî tipi possibili di quartiche piane piu volte omologico-armoniche*, Rendiconti di Palermo, vol. 13 (1899), pp. 347–373.

$$\lambda^3 \begin{vmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{vmatrix} = 0. \tag{21}$$

The roots are $\lambda=0$, $\lambda=1$, $\lambda=-1/2$. The first two roots are three- and two-fold, respectively. For $\lambda=0$ we obtain the Dyck curve for which the covariant S becomes indeterminate. For $\lambda=1$ the apolar conic has the form $a(u_1^2-u_2^2)-b(u_2^2-u_3^2)=0$, and (20) becomes

$$C_4 = (x_1+x_2+x_3)^4+(-x_1+x_2+x_3)^4 + (x_1-x_2+x_3)^4+(x_1+x_2-x_3)^4=0,$$

$$S = (x_1+x_2+x_3)(-x_1+x_2+x_3)\cdot(x_1-x_2+x_3)(x_1+x_2-x_3)=0.$$

For $\lambda=-1/2$, the apolar conic simply assumes the form $u_1^2+u_2^2+u_3^2=0$, and

$$C_4 = x_1^4+x_2^4+x_3^4-3(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2)=0,$$

$$S = x_1^4+x_2^4+x_3^4+7(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2)=0.$$

These Clebsch and Lüroth curves invariant under the octahedral collineation group seem to be new.

The construction of these curves, outside its tediousness, does not present particular difficulties. Any tangent t_1 of the apolar conic cuts S in four points from which four other tangents t_2, t_3, t_4, t_5 may be drawn. These together with the first form one of the ∞^1 pentagons inscribed to the Lüroth curve. By means of such a pentagon both curves may be represented in the well known manner:

$$C_4 = \Sigma a_i t_i^4 = 0, \qquad S = \Sigma b_{ijkl} t_i t_j t_k t_l = 0.$$

The University of Illinois

ON A CERTAIN METRIC ASPECT OF PLANE PROJECTIVE TRANSFORMATIONS*

BY LULU HOFMANN

1. *Introduction and Results.* This paper contains the study of a special type of metric relations between projective planes. To bring out clearly the point of view from which the planes are considered, we first briefly summarize the principal known facts about the subject in general, remarking that by "metric" relations we refer merely to properties based on the ordinary euclidean measure of a straight line segment and an angle.

Let π and π' be two projectively related planes.† At a general point of $\pi(\pi')$ there are ∞^1 angles and on a general line ∞^1 segments which preserve their size under the transformation. These angles and these segments form an involution at the point and on the line.

There exist, however, in π and π' two distinguished pairs of corresponding points, the focal points, and two distinguished pairs of corresponding lines, the equi-segmental axes, such that all corresponding angles at these points and all corresponding segments on these lines are congruent. In either plane, these points and lines lie symmetrically with regard to the vanishing line. If the distance of the focal points from the vanishing line is c in π and c' in π', then the distance of the equi-segmental axes from this line is c' in π and c in π'. The quantities c and c' are called the parameters of the transformation, and the line joining the focal points in either plane is called the focal axis.

Thus in two projective planes, the metric properties of

* Presented to the Society, October 27, 1928.

† For the following see H. J. S. Smith, *Memoir on the focal properties of homographic figures*, Collected Works, vol. 1, p. 545–602. We assume with Smith throughout this paper that the projective transformation considered is a general one, and we exclude the affine case.

corresponding segments on corresponding lines and of corresponding angles at corresponding points are known. In this paper, we start out from a new combination of elements, namely we study the metric properties of *corresponding segments at corresponding points.* We choose an arbitrary pair of corresponding points P_0 and P_0' in π and π' and examine those segments radiating out from P_0 which preserve their length as they are transformed into segments radiating out from P_0'. In other words, *after choosing a fixed pair of corresponding points P_0 and P_0', we determine the locus of the points P of π such that the segment PP_0 is congruent to the transformed segment $P'P_0'$.*

The locus of the points P of π such that

$$PP_0 = P'P_0' \tag{1}$$

where P_0 is fixed, is found to be a bipartite quartic Q of genus one. Its two branches* lie on opposite sides of the vanishing line; they intersect only at the infinite point of this line, thus making it a double point, and each there has the corresponding equi-segmental axis for its asymptote.

The second double point of Q lies at the fixed point P_0. It is a node, a cusp, or an isolated double point, according as there are two, one, or no real line elements at P_0 which are equal to their corresponding elements at P_0'. The points of π belonging to the second class form a bipartite sextic which separates those of the first from those of the third class.

The quartic Q is tangent to the circles about P_0 as center at the cyclic points. On any such circle there are therefore four, and on a general algebraic curve of the nth order† $4n$ finite points obeying condition (1).

* For the sense in which we use the word *branch* we refer, for example, to the definition of *complete branch* given by H. G. Zeuthen in the first paragraph of his paper *Sur les différentes formes des courbes planes du quatrième ordre*, Mathematische Annalen, vol. 7 (1874), pp. 420–432. In translation: "We denote by *complete branch* the part of a plane curve which can be described in a continuous manner by a point which, after having passed once through every fixed point, returns to its initial position."

† A curve not passing through the cyclic points and the infinite point of the vanishing line.

Aside from all metric considerations, Q may be described in a purely projective manner as the locus of the ∞^1 double points determined by the projectivity of π and π', as these planes are laid onto one another in the ∞^1 ways possible such that P_0 and P_0' coincide.

The extension to the case where corresponding segments at corresponding fixed points are required not to be equal but to form an arbitrary fixed ratio $k>0$, so that $PP_0 = k \cdot P'P_0'$, furnishes nothing new. It is identical with the original problem for the transformation with the parameters c and $\sqrt{k}\cdot c'$.

2. *Fundamental Properties of the Locus Q, its Double and Infinite Points, its Real Asymptotes, and its Branches.* We represent the projective relation between π and π' in the canonical form,* choosing the focal axis and the vanishing line in either plane as the xy and $x'y'$ axes. The transformation formulas then are

$$x' = \frac{cc'}{x}, \qquad y' = \frac{c'y}{x}; \tag{2}$$

c and c' are the parameters of the transformation; the focal points of $\pi[\pi']$ have the coordinates $(\pm c, 0)$ $[(\pm c', 0)]$ and the equi-segmental axes the equations $x = \pm c'[x' = \pm c]$.

Let $P_0(x_0, y_0)$ and $P_0'(x_0', y_0')$ be the fixed pair of corresponding points. Then the locus of the points P such that

$$PP_0 = P'P_0'$$

is obtained by replacing in

$$(x - x_0)^2 + (y - y_0)^2 = (x' - x_0')^2 + (y' - y_0')^2$$

the primed coordinates by the unprimed ones according to the formulas (2) of transformation. After ridding the resulting expression of its denominators and writing it in the homogeneous coordinates x, y, t, instead of x, y, we obtain

$$x_0^2 x^2[(x - x_0t)^2 + (y - y_0t)^2] - c'^2t^2[c^2(x - x_0t)^2 + (xy_0 - yx_0)^2] = 0. \tag{3}$$

* See the above cited paper by Smith, p. 561.

The locus (3) *of the points P fulfilling condition* (2) *is a quartic curve.* We call it Q. It is obvious from (3) that Q *has double points at* (x_0, y_0) *and* $(x=t=0)$, *that is, at the fixed point* P_0, *and at the infinite point of the vanishing line* (*y-axis*).

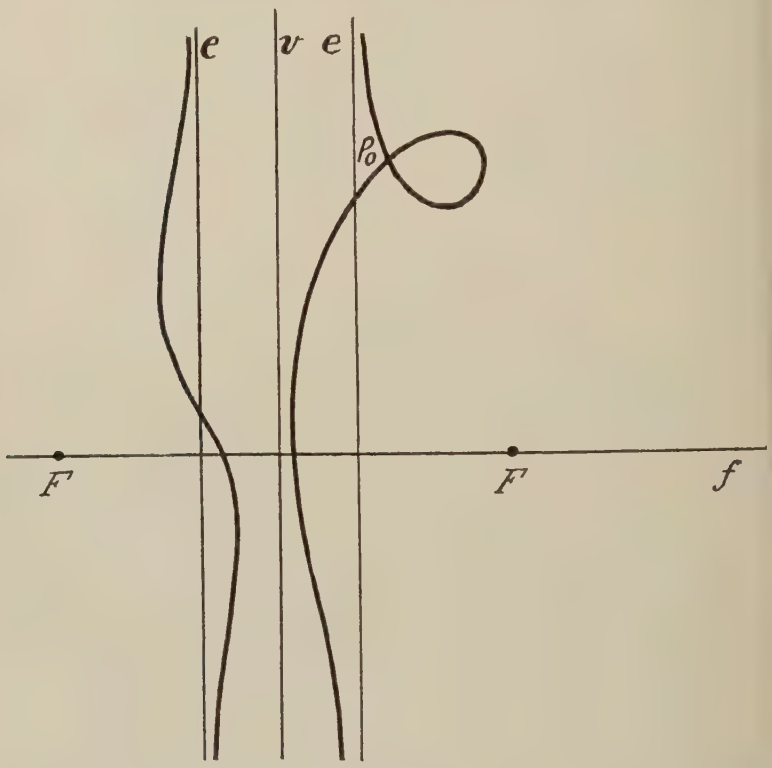

FIG. 1. SCHEMATIC FIGURE OF Q (Focal points F; focal axis f; vanishing line v; and equi-segmental axes e).

The substitution of $x = \pm c't$ into (3) furnishes two expressions with the factor t^3. It *follows that Q is tangent to the two equi-segmental axes at infinity and therefore always has a node at the infinite point of the vanishing line.*

Besides this node, Q has in common with the vanishing line the points $(0, \pm ic)$* and with the line at infinity the

* These points correspond to the cyclic points of π'.

cyclic points. Thus, since it does not have any real finite points on the vanishing line, nor any real infinite points except $(x=t=0)$, and since it has two real distinct asymptotes, parallel to and lying on opposite sides of the vanishing line, it follows that Q *consists of two branches separated by the vanishing line and intersecting only at the infinite point of this line.* Obviously *the branches are odd* (they have only one point of intersection).

If we consider a segment PP_0 not only with regard to its end points but with regard to all the points which make it up, we have to distinguish between the two branches of Q. For a segment with its end point P on that branch of Q which lies on the same side of the vanishing line as P_0, is transformed *point by point* to the segment $P'P_0'$; while, if P lies on the other branch, the segment PP_0 contains a point of the vanishing line and corresponds therefore point by point not to $P'P_0'$ itself but to the complementary infinite segment.

3. *The Intersections of Q with the Circles about P_0 and the Character of this Point as a Double Point. The Sextic S. Properties of Symmetry of Plane Projectivities.* It is obvious from the first term of (3) that Q *not only contains the cyclic points but is there tangent to the circles about P_0 as center.* It follows that Q has in common with any such circle four finite points; when the radius of the circle is sufficiently large, they are all real, as is evident from the character of Q. There is only one imaginary circle which intersects Q three times at each of the cyclic points.

If r is an arbitrary length, there are always exactly four points at this distance from P_0 which are transformed into points at the same distance from P_0'.

To find the four points of π belonging to a certain distance r, independently of Q, we invert for a moment the hitherto considered order of transformation, regarding π' as the original and π as the transformed plane. A circle of radius r about P_0' is transformed into a certain conic of π. The four points in question are found by intersecting this conic with the circle of radius r about P_0 itself.

If we take for r a vanishing quantity dr, the conic in π becomes a vanishing ellipse with P_0 as a center* and therefore has in common with the concentric circle of radius dr two real distinct, coincident, or imaginary diameters. This means *there are two real distinct, coincident, or imaginary line elements at P_0 which are equal to the corresponding elements at P_0', and accordingly P_0 is a node, a cusp, or an isolated double point for Q.* We denote the points of the three different types as N-, C-, and I-points.

We determine the character of P_0 analytically, by demanding that it be a triple point of intersection with Q for the line $y-y_0=\alpha(x-x_0)$ and investigating when, under this condition, α assumes two real distinct, coincident, or imaginary values. Calculation shows that these cases occur according as

$$(4) \qquad c'^2x_0^4t_0^2 + c'^2x_0^2t_0^2(y_0^2 + c^2t_0^2) - x_0^6 - c^2c'^4t_0^6$$

is greater than, equal to, or less than zero.

Expression (4) equated to zero and considered as an equation in x_0, y_0, t_0† shall be referred to as the sextic S. S is the locus of the C-points and separates the N-points from the I-points.

For the intersections of S with the focal axis we have $x_0=\pm c'$, $\pm(cc')^{1/2}$, $\pm i(cc')^{1/2}$. Let us denote by Γ and γ the greater and smaller of the quantities c and c'. It is easily seen that both when $|x_0|<\gamma$ and when $|x_0|>\Gamma$,(4) is negative, while it is positive when $|x_0|=c'$, $y_0\neq 0$. This indicates the division of π into the regions N and I as shown in Fig. 2.

We easily deduce the main properties of S by equating (4) to zero. *S is symmetric with regard to the vanishing line and the focal axis.* At the infinite point of the vanishing line, it has four coinciding points of intersection with a general line, and six with the vanishing line itself and with

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* Disregarding infinitesimals of higher order than dr. See Smith, p. 559.

† We have again replaced the non-homogeneous coordinates by homogeneous ones. The suffix zero merely indicates that the respective point appears in the character of the fixed point of our problem.

the focal axis. Thus these two lines are the only tangents of S at its quadruple infinite point. Referring to the symmetric character of the curve, one can show that this point, besides being the only infinite one is also the only multiple one of S. It follows that *S consists of two branches which each have a point of self contact at the infinite point of the vanishing line. The first branch intersects the focal axis at the points $x = \pm\gamma$ and is tangent at infinity to the vanishing line. The second intersects the focal axis at the points $x = \pm\Gamma$ and is tangent at infinity to the line at infinity itself.*

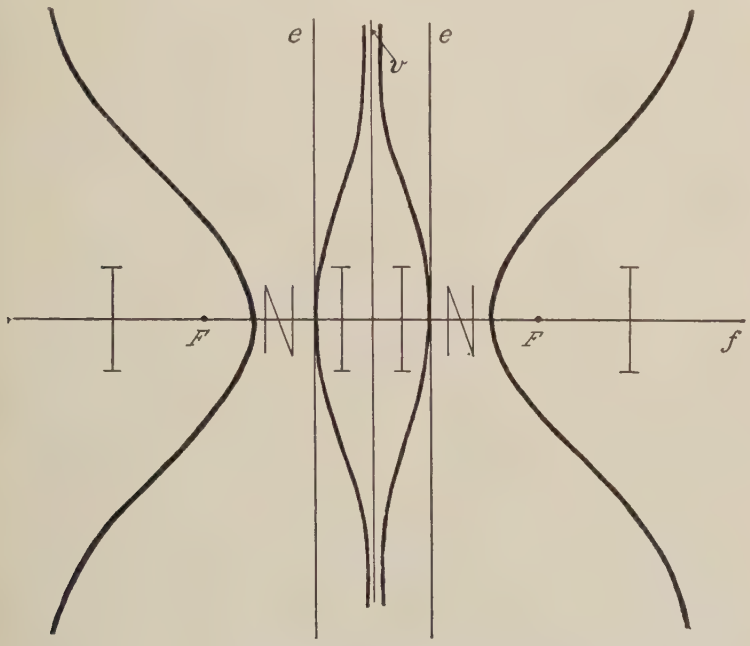

FIG. 2. SCHEMATIC FIGURE of S.

We have stressed the symmetric character of S. We add the remark that *two curves $Q^{(1)}$ and $Q^{(2)}$ belonging to points $P_0^{(1)}$ and $P_0^{(2)}$ which lie symmetrically with regard to the vanishing line or the focal axis, are mutually symmetric with regard to the respective line.* This includes as a special statement that *the Q-curve belonging to a point P_0 on the focal*

axis, is itself symmetric with regard to this axis. The metric point of view brings out the symmetric properties of plane projectivities.

4. *Interpretation of Q as π and π' Coincide and Proof that it is of Genus One.* Let π' be laid on π in such a manner that P_0 and P_0' coincide. There will be two more points, say P_1 and P_2 of π which coincide with the corresponding points of π', thus with P_0 forming the three double points of the transformation. By definition, P_1 and P_2 lie on Q; for, being a double point of the transformation, $P_1 = P_1'$ ($P_2 = P_2'$) as a point of π has the same distance from P_0 that it has from P_0' as a point of π'. Thus, *Q is the locus of the ∞^1 double points determined by the projectivity as π' is laid on π in the ∞^1 ways possible such that P_0 and P_0' coincide.*

Choosing one definite such position of π and π' with the double points P_0, P_1 and P_2, let us consider the line l through P_0 and P_1. It cuts Q in a fourth point (P_0 counting as two), say P_1^*. The point P_1^* belongs, as a double point of the transformation, to that position of π and π' which is obtained from the one considered by rotating π' through an angle of 180° about the perpendicular to l which lies in π and passes through P_0. Accordingly, P_1 and P_1^* can coincide only at the point P_0 or at infinity. On the other hand, the coincidence of P_1 and P_1^* is equivalent with the existence of a double point of Q, so that double points of Q can occur only at P_0 and at infinity. But we know that Q has only one infinite double point, namely, at $x = t = 0$. Thus we have proved that *Q has exactly two double points and is therefore of genus one.*

5. *Degenerations of Q and Specializations of S.* To study the cases in which Q degenerates, we let P_0 assume all possible positions excepting those on the vanishing line and the line at infinity. The character of our problem makes it natural to consider only such finite points P_0 which are again transformed into finite points P_0'.

Owing to the fact that Q consists of two branches lying on opposite sides of the vanishing line and each having one of

the equi-segmental axes for its asymptote, the only possible degenerations of the curve are those in which it contains one or both of these axes as a part. Calculation shows that in the first case P_0 lies on one of the equi-segmental axes $(x_0 = {}_{(-)}^{+} c')$ and in the second case at one of the focal points $(x_0 = {}_{(-)}^{+} c,\ y_0 = 0)$. The corresponding equations for Q are

$$(x_{(\mp)}c't)\left[(x_{(\mp)}c't)(x^2 - c^2t^2) + y\left\{x(y - 2y_0t)_{(\pm)}c'yt\right\}\right] = 0,$$
$$(x^2 - c'^2t^2)\left[(x_{(\mp)}ct)^2 + y^2\right] = 0.$$

When P_0 lies on one of the equi-segmental axes, Q consists of the respective axis and a cubic, which is made up of an oval through P_0 and an odd branch with the other axis for its asymptote. The branches lie on opposite sides of the vanishing line and contain the respective focal points.

When P_0 lies at one of the focal points, Q consists of the absolute lines through this point and both equi-segmental axes.

The degenerations of Q are caused by special positions of P_0 and occur in every projectivity, that is for every value of the parameters c and c'. S, however, depends on these parameters only. It is evident from its constitution that it cannot degenerate, but that it specializes when $c = c'$. In this case it becomes unipartite and adds to its quadruple singularity at infinity two nodes at the focal points. On the contrary, the equality of c and c' has no essential effect on a general Q-curve.

6. *Extension of the Problem to the Case of Corresponding Segments forming a General Fixed Ratio $k > 0$.* We have so far investigated the locus of those points of π whose distances from a fixed point P_0 remain unchanged under the transformation. The investigation immediately extends itself to the case in which we require these distances to be changed according to any fixed ratio $k (k > 0)$: $PP_0 = k \cdot P'P_0'$. Then (3) assumes the form

$$(3)' \quad x_0^2 x^2\left[(x - x_0t)^2 + (y - y_0t)^2\right] - kc'^2t^2\left[c^2(x - x_0t)^2 + (xy_0 - yx_0)^2\right] = 0.$$

But this is the equation corresponding to the original problem ($k=1$) for a projectivity with the parameters c and $\sqrt{k}\cdot c'$. The new curves Q_k and S_k are therefore of exactly the same type as Q and S. The ∞^1 curves Q_k belonging to the same point P_0 form a pencil whose base points are P_0 itself and the infinite point of the vanishing line, each counted four times, and the cyclic points and the points $(0, \pm ic)$, each counted twice.

7. *Summary.* In conclusion, we repeat the fundamental results of the preceding discussion in a form no longer distinguishing π before π', but stressing the obvious symmetry of the problem with regard to both planes.

In two projective planes there is associated with every fixed pair of corresponding points and every value of a positive quantity k a pair of bipartite quartics of genus one corresponding in the projectivity which are the loci of all the pairs of corresponding points whose distances from the respective fixed points form the ratio k. Moreover there is associated with every value of k a pair of bipartite sextics corresponding in the projectivity which are the loci of all those pairs of corresponding points at which there exists one real pair of corresponding line elements forming the ratio k. These sextics separate the points of their planes where there are two such pairs from the points where there is none.

The problem discussed in this paper immediately suggests two new ones. First, the problem corresponding by duality: the study of the pairs of corresponding lines forming congruent angles with two corresponding fixed lines. Secondly, the extension to spaces of higher dimensions. These problems lead to curves of the fourth class and hypersurfaces of the fourth order, but do not appear to furnish essentially new results.

COLUMBIA UNIVERSITY

LANDAU ON NUMBER THEORY

Vorlesungen über Zahlentheorie. By Edmund Landau. Volume I: *Aus der elementaren und additiven Zahlentheorie.* xii+360 pp. Volume II: *Aus der analytischen und geometrischen Zahlentheorie.* vii+308 pp. Volume III: *Aus der algebraischen Zahlentheorie und über die Fermatsche Vermutung.* vii+341 pp. Leipzig, S. Hirzel, 1927.

These three excellently printed and arranged volumes form an addition of the highest importance to the literature of the theory of numbers. With them, the reader familiar with the basic elements of the theory of functions of a real and complex variable, can follow many of the astonishing recent advances in this fascinating field. His interest is enlisted at once and sustained by the accuracy, skill, and enthusiasm with which Landau marshals the analytic facts and simplifies as far as possible the inevitable mass of details.

Part I of Volume I gives a treatment of the elements of the theory of numbers up to the theory of quadratic residues and of the Pellian equation. This occupies a little over 50 pages. In Part II is first deduced the elementary inequality

$$a_1\xi/\log \xi < \pi(\xi) < a_2\xi/\log \xi$$

for the number of primes $\pi(\xi)$ not greater than ξ, after which follows the proof of the theorem of Brun (1919) that, if the number of "twin primes" p, $p+2$ is infinite, these are at least so infrequently distributed that the series of reciprocals

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\cdots$$

converges, in contradistinction with the series of the reciprocals of all primes, which diverges. The reader is then introduced in the same part II to characters $\chi(a)$ mod k, and the associated Dirichlet series, by means of which is established Dirichlet's classical theorem that every arithmetical progression $l+ks(s=1, 2, \cdots)$ contains infinitely many primes if l and k are relatively prime.

All of this material in Parts I and II, which acquaints the reader with some of the most interesting ideas in number theory, occupies slightly less than 100 pages.

In Part III are obtained the classical theorems concerning the representation of an integer as the sum of two, three, or four squares, and in Part IV are developed the classical results concerning the class number of binary quadratic forms.

Part V presents that part of the recent extraordinary series of researches, *Some Problems of "Partitio Numerorum,"* Parts I–V (1920–1925), by Hardy and Littlewood, which lies in the direction of Goldbach's conjecture that every even integer >2 can be expressed as the sum of two odd

primes. *Under the hypothesis* that not only Riemann's function $\zeta(s)$, but all of the similar functions defined by Dirichlet series, $L(s)$, have no zero for s with real part $>\Theta$ with Θ a definite constant $<\frac{3}{4}$, it is proved that every sufficiently great odd integer is representable as the sum of three odd primes, and that "almost all" even positive integers are the sum of two odd primes.

The sixth and last part of Volume I develops the part of the results of Hardy and Littlewood bearing on Waring's theorem (first proved by Hilbert, 1909), that any integer can be expressed as the sum of a fixed number g of nth powers, $g=g(n)$, in which they go a considerable distance toward determining the nature of the number theoretic function $G(n)\leqq g(n)$ representing the number G of powers sufficing for all large integers, and the function $G_1(n)\leqq G(n)$ representing the number G_1 sufficing for almost all large integers; for example, it is proved that

$$G(n) \leqq (n-2)2^{n-1}+5\,.$$

It is interesting that Landau's beautiful earlier result $G(3)\leqq 8$ (1909) escapes the analysis of Hardy and Littlewood who prove only $G(3)\leqq 9$.

At the end of this part is found Vinogradoff's brief proof (1922) of Waring's theorem, involving the use of Weyl's approximation theorems (1916), occupying in all less than 20 pages.

In Part VII, Volume 2, the principal result (Littlewood, 1924) is that the difference between the prime number function $\pi(x)$ and $\int_2^x (du/\log u)$ is of the order of

$$xe^{-\alpha(\log x\log\log x)^{1/2}}$$

at most. This result, as well as the analogous fact for any $\pi(x;k,1)$ is proved in a direct and simplified manner by Landau. Here the comparison with the earlier result of de la Vallée Poussin that this difference is of the order

$$xe^{-\alpha(\log x)^{1/2}}$$

at most, measures the progress made since the appearance of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen* (1909).

Part VIII of the same volume takes up thoroughly the number theoretic function $A(x)$, defined as the number of integer pairs (u, v) for which

$$u^2+v^2\leqq x,$$

that is, the number of points with integer cartesian coordinates within or on the boundary of the circle of radius $x^{1/2}$ and with center at the origin in the u, v plane. The fact that $A(x)-\pi x$ is of the order of $x^{1/2}$ at most is trivial. The major question is to determine as far as possible the order θ of this difference, the above trivial result showing that $\theta\leqq 1/2$. Here the facts established are $\theta\geqq 1/4$ (Hardy and Landau, 1915) and $\theta<1/3$ (Sierpinski, 1906; van der Corput, 1923). Similar results are deduced for other curves.

The first part of Volume 3 (Part XI) expounds the classical theory of algebraic numbers and ideals. Here the principal new development

to be found is the proof of Thue's theorem (1908) that if $g(x, y)$ is an irreducible homogeneous polynomial of degree $n>2$ with integral coefficients, the equation $g(x,y)=a$ for a given integral value of a has only a finite number of solutions. The main purpose of this part is to develop as much of the theory of ideals which is necessary to establish the known results concerning Fermat's famous conjecture that the equation

$$x^n + y^n = z^n$$

has no integral solution x, y, z with $xyz \neq 0$ for $n>2$. In particular the proof of Kummer's theorem that the conjecture holds for $n=p$, a regular prime, is given in full. The concluding part deals with some recent results of Furtwängler (1912), Wieferich (1909), Mirimanoff (1910), and Vandiver (1914, 1919).

The mathematical world owes a great debt of gratitude to Professor Landau for rendering accessible so many of the recent splendid achievements in the theory of numbers.

G. D. Birkhoff

THREE BOOKS ON WAVE MECHANICS

Four Lectures on Wave Mechanics. By Erwin Shrödinger. London and Glasgow, Blackie and Son, 1928. viii+53 pp. Price 5 s.

Collected Papers on Wave Mechanics. By Erwin Schrödinger. Translated from the 2d German edition by J. F. Shearer and W. M. Deans. London and Glasgow, Blackie and Son, 1928. xiii+146 pp. Price 25 s.

Selected Papers on Wave Mechanics. By Louis de Broglie and Léon Brillouin. Translated by W. M. Deans. London and Glasgow, Blackie and Son, 1928. 151 pp. Price 15 s.

The wave mechanics proposed by de Broglie and Schrödinger has assumed during the short space of two years such a dominant role in the field of atomic physics that these three books in English by the authors of the new theory will be eagerly welcomed by physicists and mathematicians in England and America.

Schrödinger's *Four Lectures on Wave Mechanics* were delivered at the Royal Institution last March. In them are explained the ideas underlying the wave mechanics and some of the more important applications of these ideas. The exposition is somewhat popular in form, the mathematical details of computation being omitted from the text. The first lecture unfolds the relation between geometrical optics and classical dynamics as exhibited by the correspondence between Fermat's principle and the principle of Maupertuis (least action). It is this correspondence which suggests that classical methods are no more applicable to micro-mechanics than the ray methods of geometrical optics are to problems in diffraction and which leads in atomic problems to the substitution of a wave equation for the Hamilton-Jacobi equation of macro-mechanics. The success of a physical theory is measured by the agreement of its predictions with experi-

ment, and in the succeeding lectures applications of the theory are made to the hydrogen atom (including the case of a nucleus of finite mass), the simple harmonic oscillator and the rotator. The physical interpretation of the ψ function proposed by the author is shown to lead to correct selection and polarization rules and to correct intensity ratios for the emitted radiation. Finally, the perturbation theory is developed and from it the dispersion formula is deduced.

The serious student of the new mechanics will find a complete account of the subject in the *Collected Papers*. These comprise practically all the author's papers on wave mechanics, including the one in which he shows the equivalence of his theory with the matrix dynamics of Heisenberg, Born and Jordan. With one exception all of them appeared in the *Annalen der Physik* and all were published during the two years 1926 and 1927. As the author remarks in his preface, the results contained in the later papers were unknown to him when the earlier ones were written, and consequently the development is not as logical as it might be made if the book were rewritten in the light of his present knowledge. Nevertheless his exposition of the subject will probably appear clearer to the reader than those contained in several books which have been hurriedly thrown together by others in the attempt to supply the demand for information on the new theory.

Schrödinger's ideas originated in those of de Broglie, who was the first to suggest that material particles might have the properties of a wave group. The *Selected Papers on Wave Mechanics* include several of the fundamental papers of de Broglie as well as papers by Brillouin on both the matrix dynamics of Heisenberg and Dirac and the wave dynamics of de Broglie and Schrödinger. The last paper in the series contains a valuable comparative study by Brillouin of the classical Bose-Einstein and Fermi-Dirac statistics.

These three volumes taken together provide an excellent exposition of those theories of atomic physics which have proved most successful up to the present time. The translations of the last two have been carefully prepared, the originals having been followed as closely as English idiom would permit.

LEIGH PAGE

FRAENKEL ON GRUNDLEGUNG DER MENGENLEHRE

Zehn Vorlesungen über die Grundlegung der Mengenlehre. (Wissenschaft und Hypothese, XXXI). By Adolph Fraenkel. Leipzig and Berlin, B. G. Teubner, 1927. x+182 pp.

This book embodies a series of ten lectures delivered in Kiel, June 8–12, 1925, at the invitation of the *Kant-Gesellschaft, Ortsgruppe Kiel.* The objects to be accomplished by its publication seem to be two-fold: (1) To render intelligible to non-workers in the field, especially philosophers, past and present views on the "Foundations," and (2) to stimulate reflection amongst the mathematical public concerning the foundations of mathematical thought and methods.

With regard to (1) it seems to the reviewer that the author has been singularly successful. We have here in no sense a formal text, the informal nature of the lectures having been preserved, with desirable additions made as a result of the subsequent discussion, together with literary references, etc. And yet the treatment is not superficial. The author plunges at once into a discussion of the notions of "set", equivalence of sets and cardinal numbers, order types, the continuum problem, the "diagonal procedure", antinomies and paradoxes. Of the last are given the (Zermelo-) Russell Paradox, the Burali-Forti Antinomy and Richards' Paradox. The consequences of the antinomies in mathematical thought are briefly mentioned—the failure to give a definition of the "set" notion less "offensive" than the Cantorean, in whose looseness seems to lie the germ of such paradoxes as Russell's—the suggestion that the fault be ascribed to logic, not to mathematics—the resignation of such men as Dedekind and Frege to what they considered the inevitable results of the antinomies.

The remainder of the lectures is devoted to the efforts of various schools so to revise the guiding rules of mathematical technique that the antinomies cannot arise. An exposition is given of the views of the Brouwer school (Intuitionism), as well as of the modified intuitionism of Poincaré, and the changes in the structure of the mathematical edifice that follow an adoption of the intuitionist views. There are also references to the system of Russell and Whitehead.

By far the greater part of the book (from page 58 on), however, is given over to the axiomatic foundation of the Mengenlehre begun by Zermelo. A pleasing feature of this part of the work is the way in which the author leads up to outstanding unsolved problems—certainly the mathematician who reads will *feel* the urge to apply himself to their solution, whether or not he actually yields to the temptation. Another pleasing feature is the occasional discussion of much debated matters, such as the "choice axiom", the non-predicative procedure and the excluded third. The axioms are seven in number. One of these is the choice axiom. Another axiom definitely implies the use of the logical law of the excluded third. And the non-predicative method seems to be indispensable for the axiomatic development.

The author's justification of these inclusions is exceedingly interesting from any point of view, be it philosophical, mathematical or purely logical. Thus, in regard to the choice axiom, after a historical summary of its use, its formulation, and criticisms directed against it, the author maintains "This logical principle has certainly at least a character of evidence and necessity equivalent to that which one attributes to many other axioms that one is accustomed to recognize as indispensable for the foundations of arithmetic, analysis and geometry; it was approved even by one so close to the intuitionist viewpoint as Poincaré. With equal right, then, to that with which one rejects the choice axiom, one could arbitrarily deny other important fundamental principles, and so in the future ban important parts of mathematics." The placing of the choice principle in the axioms is then compared to the assumption of the parallel axioms by the Greeks. This analogy leads the reviewer to wonder if the denial of the choice principle might not lead to fruitful results as did the denial of the parallel axiom. A non-euclidean geometry proved to be a happy conception, and might there not possibly be some significance in a non-Zermeloan Mengenlehre which asserts the existence of sets that cannot be well-ordered?* On a historical basis, at least, one would seem justified in maintaining that the denial of "important fundamental principles" is not necessarily an iconoclasm.

What would probably comprise the tenth lecture (the division of topics occurs in pairs of lectures) is devoted to a discussion of consistence, independence and categoricalness of axiomatic systems in general as well as of the particular system presented for the Mengenlehre. Although the system which Fraenkel gives does away with the existing paradoxes and antinomies, its consistency is an open question. From the point of view of him who accepts the axiomatic foundation this is a great defect—indeed, it can be said that if ever a consistency proof were needed, it is needed in this particular instance, since the very purpose of the adoption of the axiomatic method in this case is to avoid contradiction.

The reviewer heartily recommends these lectures to any who wish a general summary of the state of the "Foundations," as well as to those who wish a good introduction for mature students. The book is finished off with an unusually complete bibliography comprising eight pages, and this is amplified by up-to-date additions made during the period of proof-reading.

R. L. WILDER

* See Church, *Alternatives to Zermelo's assumption*, Transactions of this Society, vol. 29 (1927), p. 178.

GILLAIN ON EGYPTIAN SCIENCE

La Science Égyptienne. L'Arithmétique au Moyen Empire. By O. Gillain, avec une préface de H. Bosmans, S. J. Brussels, 1927. xvi+326 pp.

This work is one of the publications of the Fondation Égyptologique Reine Élizabeth, and like the other works bearing the same imprint it represents a standard of scholarship as well of typography which commends it both to students and to general readers. Not the least of its merits is the preface written by the late and widely lamented Père Bosmans, a scholar and a "gentleman of the old school," whose knowledge of the history of mathematics, as also of medieval ecclesiastical literature, was the result of the labors of a long and useful life.

The title of the book does not indicate very clearly the nature of the work, which is chiefly a treatise upon and quite largely a translation of the Rhind (Ahmes) Papyrus, but with a brief consideration of other manuscripts in the Introduction and in Chapter IV. In one way such a work may be thought to be superfluous in-as-much as we have two editions of Eisenlohr's translation, the much better edition of Peet, and the still better and more elaborate one by Dr. Chace, the second volume of which is just appearing. In another sense, however, it is to be welcomed, since it gives to French readers who may not be familiar with German or English a good idea of the oldest mathematical treatise of any extent that has come down to us.

Since the nature of this papyrus is now so well known to American and British readers, both through the editions above mentioned and through recent reviews, it is sufficient at this time to call attention to the general line of treatment followed by M. Gillain. His Introduction (pp. 1–22) sets forth the nature of Egyptian mathematics of the period, describes the measures then in use, mentions the fantastic theories of the Pyramid measurers, and concludes with a brief history and description of the work of Ahmes. He also refers to the earlier fragments of a mathematical nature, including the Moscow manuscript made known by Turaev (Touraieff) in 1917, and soon to be published in translation. As to the Pyramid theorists, M. Gillain expresses the very sane opinion, "La Grande Pyramide est sans contredit, admirable, mais ses constructeurs n'étaient point sorciers."

The work is divided into four chapters, as follows: Chapter I,—Elementary computations, in which are explained the general methods used in what we call the four fundamental operations, both with integers and with fractions, illustrative examples being selected from the Ahmes text. Chapter II,—The tables of fractions, in which a comparison is made between the cumbersome method used by the Egyptians in, say, dividing 37 by $1+2/3+1/2+1/7$,—a problem which the Egyptian calculator might have had to solve,—and the simple division of 37 by the equivalent fraction 97/42, which might conceivably be required in our day. This is followed by the usual tables of $2/(2n+1)$ for values of n from 1 to 50, and by examples from the text illustrating its use. Chapter III,—Problems, in which the

author considers such topics as the rule of three, false position, proportion, progressions, and roots. Chapter IV,—The spirit of arithmetic, in which he briefly discusses other sources than the Rhind Papyrus and considers Egyptian calculation in general and the nature of the Egyptian mind as revealed by the arithmetical sources known to us. His conclusion is clear and succinct,—"Telle que nous la connaissons, l'arithmétique égyptienne n'est ni plus ni moins qu'une physique des nombres,"—it never reached the metaphysical stage.

The chief merit of the work lies in the fact that the author has set forth, for French scholars, the general nature of Egyptian arithmetic in about the 17th century B.C., or a little before, illustrating his points by translations. These translations are chiefly if not wholly from the Peet edition of the Rhind Papyrus. The interpretation of Ahmes and of Egyptian arithmetic in general, shows a considerable range of reading from such standard writers as Bobynin, Eisenlohr, Favaro, Griffith, Gunn, Hultsch, Loria, Neugebauer, Révillout, Schack-Schackenburg, and Sethe.

As to the interpretation of the obscure passages in Ahmes, the student will do better to consult Dr. Chace's monumental work, and as to a bibliography of the subject M. Gillain's list is not in the same class as the remarkable one by Professor Archibald which is published in volume I of the same treatise.

One thing that frequently strikes the reader of scientific works in French, rather more than in other languages, is the carelessness shown in proofreading. Possibly the trouble is primarily due to French chirography, which renders it difficult for compositors to distinguish between certain letters; possibly the French scholar sacrifices niceness of typographical detail to niceness of expression; and possibly the habit of author proofreading, which did not characterize the early products of the press has not been as fully established in the Gallic regions as in those under Teutonic or British intellectual influence. Whatever may be the cause, the work of M. Gillain suffers, like so many other works in the same language, from this defect. Thus we have such spelling as Karpinsky for Karpinski, Piazzi Smith for Piazzi Smyth, Schackenbourg as well as Schackenburg, Brunschwigg for Brunschvicg, together with Sepher ha-mmispar and Gottingsche, to select only a few examples. These are mentioned not primarily as criticisms of M. Gillain's work, but rather as a problem in race psychology which is here illustrated. As already stated, the work itself has distinct merit and will be very helpful to French and Belgian students.

David Eugene Smith

SHORTER NOTICES

Modern Researches on the Singularities of Functions Defined by Taylor's Series. By S. Mandelbrojt. Edited by E. R. C. Miles. The Rice Institute Pamphlet, Volume XIV, No. 4, pp. 225–352.

The material collected in this pamphlet is a result of a series of lectures delivered by the author at the Rice Institute during the academic year 1926–27. The purpose and scope are best described by a short paragraph taken from a foot-note on the first page of the book:

"The present type of research, which began with the famous thesis of Hadamard, has broadened considerably during the past few years. Complete proofs of all the theorems cannot be compressed in the space available in a short treatise, so the author treats only of such theorems as seem to give unity to the theory, and of the latter theorems proofs are given, for the greater part, in detail. The reader will find a large bibliography, as well as an enumeration of nearly all the results on the singularities of Taylor's series in Hadamard and Mandelbrojt: *La Série de Taylor et son Prolongement Analytique*, Scientia, No. 41, also in the author's volume of the Mémorial des Sciences Mathématiques. The present treatise may be regarded as complementary to, as well as an elaboration of, some parts of the works just mentioned."

The first four chapters deal with that part of the theory which has now become classical, treating such material as the theorems of Hadamard and Hurwitz on the composition of singularities and a necessary and sufficient condition that a function should be meromorphic on its circle of convergence. In Chapters V and VI the author launches into the more recent developments of the theory, proving first an interesting theorem of his own regarding power series which have on their circle of convergence at least one singularity not a pole, and then giving various generalizations and applications. Chapter VI again returns to classical theory, presenting the theory of order of singularities. In criticism of this chapter we call attention to the fact that although the first section defines and discusses fractional differentiation and integration, no mention is made of its connection with the Hadamard operator defined in the second section. A reader meeting the subject for the first time (and it is believed that the book will in general be valuable for such readers) may naturally inquire why fractional differentiation is introduced at all. It is not until the close of the chapter, twenty pages farther along that one reads the statement, without proof, that if one of these operators yields a continuous function of finite deviation so too does the other. Chapters VIII and IX deal with the fundamental theorems of Faber and Fatou and with the author's generalizations of the latter. Chapters X and XI center around Dr. Mandelbrojt's contributions to the theory and related theorems. The final chapter deals with series having the circle of convergence as a cut, following the method introduced by Hadamard and amplified by Fabry, Leau and others.

The reviewer believes that the book will be a welcome addition to the treatises on the subject. Mr. Miles and those responsbile for the Pamphlet are to be congratulated upon the careful way in which the lectures are edited. Misprints are rare, and the general high character of the Pamphlet, established in earlier numbers dealing with mathematical subjects, is maintained. The consistency with which references are given to unproved theorems and to the proofs of facts employed in the demonstrations is highly commendable. If this principle were followed more generally, mathematical literature would be more readable.

D. V. Widder

Theory of Probability. By William Burnside. Cambridge University Press, 1928. 106+xxx pp. Short table of Error-Integral.

This *Theory of Probability* was edited for the press by A. R. Forsyth from a manuscript which Burnside practically completed before his death. In barely a hundred pages, it treats a great variety of topics in a decidedly unique and interesting manner. It is not a book on statistics. Burnside's interest is in probability itself. First, he develops in somewhat extended form relations between probabilities of a general nature associated with n conditions, some of which are to be satisfied and some not—connecting his primal idea of equal likelihood with these conditions. Then he proceeds to the discussion of typical problems, mainly algebraic in their origin. Analysis is freely used—of course, it is indispensible—but the reader gets the impression that Burnside tries to view a problem first in an algebraic setting. Thus the difference equation is more in evidence than the differential equation. This is a distinctive feature. It makes the book of special value to readers who have cultivated mainly the analytic side.

The book is fairly well organized, in spite of the fact that Burnside may not have considered it in its final form. A greater unification of topics involving the Gaussian law would have been desirable,—this law appears on pages 42, 44, 52, 73, 88. On page 73, the validity of this law as the best approximation is not so clear, as the elementary probability designated by x is made very small—this leading naturally to the Poisson Exponential, obtained on page 45. Burnside is not satisfied with the usual statement of the assumption of equal likelihood. He insists upon "assuming each two of the n results equally likely," instead of "assuming all the n results to be equally likely," see page 101. Burnside's phrasing appears a little more specific—although it seems difficult to conceive how all can be equally likely if some two of the results are not equally likely. In Chapter VII, which deals with the theory of errors, assumptions are given which lead respectively to the arithmetic mean, the median, and the average of the least and the greatest measurement. The arithmetic mean is found acceptable "as any other assumption would imply that either an excess of positive errors or an excess of negative errors has occurred." If here by "positive errors" the *number* of positive errors is meant, the median would be indicated rather than the arithmetic mean. The argument in Chapter VI follows conventional lines, including an artificial assumption regarding the a priori

probability at the top of page 85, made so as not to violate the first equation on page 84. Not many disturbing typographical errors were found—but the radicals at the foot of page 89 and at the top of page 90 should be in the denominators, and on page 91, the coefficient of the square of X should be inverted. On page 64 in the last line, $s/2$ should be replaced by $(s/2)-1$, and $(s/2)-1$ by $(s/2)-2$. Here and at the foot of page 98 mention might have been made of Pearson's *Tables of the Incomplete Gamma Function* for computational completion of the problems. At the foot of page 35 it is assumed that p does not equal $1/2$.

To epitomize the content of Burnside's book is somewhat difficult. The last chapters deal with the probability of causes (with interesting exemplification of Bayes's Theorem), geometrical probability, and the theory of errors. As early as page 22, problems on "runs" appear, and these under the title "duration of play" constitute the principal part of Chapter III, and are again taken up on page 72. These problems are more fundamental than they at first seem. They are especially fascinating in a book so full of good material.

E. L. Dodd

Drei Abhandlungen über die Auflösung der Gleichungen von Leonhard Euler (1738, 1764, 1790). Translated by Samson Breuer. 94 pp. Leipzig, Akademische Verlagsgesellschaft, 1928. Ostwald's Klassiker, Nr. 226.

Students interested in the history of the theory of equations will find here, reproduced in translation from the Latin into German, three treatises by one of the greatest mathematicians of the eighteenth century, who still held to the belief that the algebraic solution of the general quintic equation and equations of still higher degrees was possible. Knowing the extraordinary dexterity of Euler in analytical processes, we are impressed by his statement that though his article did not contain a complete solution, it indicated the form which a root must take, and perhaps it would be "of greater service to others and lead them finally to the desired goal." Euler continues: "Since an equation of any degree comprehends all equations of lower degree, the solution of an equation of any degree must involve all the processes for equations of lower degree." Euler assumed as given by induction that the resolvent of the general quintic is a quartic. In the second treatise Euler remarks that if p, q, r, s are the roots of the quartic resolvent, then the general form of a root of a quintic equation would seem to be $x=f+\sqrt[5]{p}+\sqrt[5]{q}+\sqrt[5]{r}+\sqrt[5]{s}$, but he remarks that this form is not sufficiently sharp as it would seem to lead to more than five roots of the quintic. Euler assumes next $x=f+A\sqrt[5]{p}+B\sqrt[5]{p^2}+C\sqrt[5]{p^3}+D\sqrt[5]{p^4}$, and is led to solvable equations, including those previously solved by De Moivre. In the third article, written in 1776 and printed in 1790, Euler assumed the coefficients to depend upon two or more parameters and then derived numberless forms of equations of all degrees, which are solvable by algebra. Breuer, the translator of these articles, has supplied much bibliographical detail and many notes aimed to assist the reader.

Florian Cajori

The Logic of Modern Physics. By P. W. Bridgman. New York, Macmillan, 1927. 14+228 pp.

The author begins by calling attention to the fact that the accepted classical concepts in physics were found inadequate to explain the experimental facts obtained in testing Einstein's deductions from his relativity theory. He points out the necessity of studying our concepts to prevent a similar failure in the future. By concept he means a set of operations. A lengthy study of the operations involved in measuring distances follows and the concept of an astronomical and an atomic distance is arrived at in terms not only of actual operations but also in terms of hypotheses involved. Following this mode of thought, concepts of space, time, identity, velocity, force, mass, energy, electricity, etc. are discussed in detail. In connection with light, the author concludes that "from the point of view of operations, it is meaningless or trivial to ascribe physical reality to light in intermediate space, and light as a thing travelling must be recognized to be a pure invention." It must follow that finite velocity of light does not exist and likewise Einstein's definition of simultaneity. This necessarily results from a conception that a reality can be only when a phenomenon exists. Difficulties arise which appear as great as those which phenomenalism tries to remove. Finally, the last chapter is devoted to a study of Special Views of Nature in which he is led to accept the postulate that physical laws are formally simple.

The book is an earnest attempt at formulation of concepts in physics based on operations with due regard for the hypotheses that may be involved. It should be found welcome to mathematicians and philosophers as well as experimental physicists.

A. F. Kovarik

Quellenhefte zur Elementarmathematik: Band I, *Elementargeometrie*, Heft 1: *Der Stoff bis zur Untersekunda.* By Kuno Fladt. Leipzig and Berlin, B. G. Teubner, 1928. vi+73 pp.

This is a source book on the history of geometry in which are reproduced selections from such parts of historic sources as have retained their place in elementary instruction down to the present time. No obsolete material is included. The booklet is a companion to another publication by the same author, written as a geometric text for pupils about fourteen years of age, in the Untersekunda. After a brief historical introduction there are given, in the source book, extracts, in German translation and in modern symbols, of such authors as Euclid, Archimedes, Proclus, Heron and some more modern writers, all accompanied with suitable historic notes. An attractive feature among the illustrations is the part of Raphael's School of Athens which shows the geometrical and astronomical group, and also the reproduction of a recently described mosaic, said to have been taken from the ruins of Herculaneum, and showing the death of Archimedes. The booklet represents an effort to utilize in elementary instruction the interest attached to the history of mathematics.

Florian Cajori

Einführung in die Theoretische Physick. By Max Planck. 1. *Einführung in die Allgemeine Mechanik.* 4th ed. viii+226 pp., 43 figures. R.M. 6—; bound, R.M. 8—. 2. *Einführung in die Theorie der Elektrizität und des Magnetismus.* 2d ed. viii+210 pp., 13 figures. R.M. 6—; bound, R.M. 8—. Leipzig, Hirzel, 1928.

These two volumes form the fourth and second editions respectively of two of the component parts of a work designed, when completed, to offer an introduction to the whole field of classical theoretical physics. The author's object is apparently to provide a series of texts which can be used with profit by the independent reader as well as by the university student. It would seem that in these two books the aim is well achieved.

The volume on mechanics is developed mainly from the standpoint of the author's pedagogical thesis that students beginning theoretical physics usually have less trouble with the mathematical discussion than with the physical interpretation. In line with this view the subject is not presented as a complete logical structure but the development is from the very simple to the more complicated. Elegant proofs of theorems give way to interpretative methods even at the expense of space. In formal construction the work is divided into the usual parts, viz. mechanics of a particle and mechanics of a system of particles. The treatment in the former case is not particularly noteworthy save that the author takes the concept of *force* as fundamental, ignoring the usual difficulties confronting the beginner in this connection. An interesting feature of the second part is the treatment of the Hamilton-Jacobi equation and its integration, so important in modern atomic theory.

The author has followed a somewhat different method in the volume on electricity and magnetism. There the treatment is severely logically deductive in nature rather than experimentally inductive. This is deliberately chosen to preserve unity of presentation and to emphasize the few fundamental concepts instead of the many diverse applications. The classical theory of Maxwell is followed with principal emphasis on the concepts of energy density and energy flow. Indeed the electric and magnetic fields are defined in terms of the former and the field equations are deduced by the application of the laws of Joule and Poynting and the conservation of energy. Electrostatics and magnetostatics are treated as special cases of the more general field relations (that is, no change with the time). Electric currents are treated as illustrations of "stationary" fields, viz. fields in which the electric and magnetic intensities do not change with the time but in which there is a continual change of electrical energy into heat. The "non-stationary" phenomena which form the general case of electrodynamics are treated in the last 60 pages of the book. Here the development follows classical lines very closely, and there is practically no mention of the electron theory. The difficulties of the original Maxwell-Hertz theory applied to moving media are merely mentioned and but a page or two devoted to Lorentz's solution.

The style is clear and the type is good. No serious errors have been detected. In both volumes the figures are rather poorly drawn.

R. B. Lindsay

Einführung in die Axiomatik der Algebra. By H. Beck. Berlin, Walter de Gruyter, 1926. 194 pp.

This book is a presentation from the view point of formal logic, of the fundamental operations and theorems of algebra. Chapter one discusses the usual axioms of algebra and the immediately resulting theorems. Articles ten and eleven of this chapter deal in particular with natural numbers. The next four chapters define the usual algebraic operations for point sets, vectors and matrices, and discuss the validity of the axioms of Chapter one in the fields of these elements. It should not be surprising that these axioms do not all hold. For, although two axioms may be alike in form, they are different axioms if they deal with different elements, unless of course they only state relations between properties common to both sets of elements. For example the statement, p. 19, that A_2 failing and I_4 holding for point-sets implies the independence of these axioms, has no significance. This merely shows that the point-set analogs of A_2 and I_4 are inconsistent. Notwithstanding this criticism, the reviewer considers these chapters a worthwhile parallelling of these various algebras.

Chapters six to ten inclusive treat in a brief and attractive way the theory of linear equations, linear vector forms, linear and quadratic forms, and matrices and determinants. Chapter eleven gives a brief axiomatic introduction to the theory of finite groups. The last chapter discusses the genetic structure of algebra upon the basis of Peano's five axioms. The treatment is condensed, well organized and very readable.

GLENN JAMES

Beyond the Electron. By Sir J. J. Thomson. Cambridge University Press, 1928. 43 pp.

The contents of this highly interesting booklet is a lecture given at Girton College and includes mathematical appendices bearing on points involved. The incentive was obtained from the experiments of the author's son, Professor G. P. Thomson, showing that a moving electron is always accompanied by a series of waves; a result also obtained from the experiments of Davisson and Germer. Because of these experiments the old idea that an electron is merely a charged particle must be given up. In his characteristically lucid manner, Sir Joseph goes step by step in forming pictures of two physical things: the transmission of energy and the propagation of waves and presents ultimately a view of the electron with a dual structure, "one part of this structure, that where the energy is located, being built up of a number of lines of electric force, while the other part is a train of waves in resonance with the electron and which determine the path along which it travels."

In view of the attempts of physicists to reconcile phenomena explainable in some cases by the undulatory theory and in other cases by the corpuscular conceptions of the physical entities: light, X-rays, electrons, etc., the booklet should prove of great interest to mathematicians and physicists alike.

A. F. KOVARIK

Aufgabensammlung zur Funktionentheorie. By Konrad Knopp. Volume II: *Aufgaben zur höheren Funktionentheorie.* Sammlung Göschen, No. 878. Berlin and Leipzig, Walter de Gruyter, 1928. 143 pp.

This little book is the continuation of Volume I (*Aufgaben zur elementaren Funktionentheorie*, No. 877) which appeared under the same title in 1923. Like all other books published by Dr. Knopp, large as well as small ones, the present volume is a new contribution, small, but significant, to the widely known reputation of its author as an excellent scholar and talented teacher. The difficult task of selecting from the immense material of the modern theory of functions the problems just within the reach of a beginner is here masterfully accomplished. A student who will go faithfully through 183 problems of Vol. I and 187 problems of Vol. II will almost imperceptibly find himself in possession of a technique which is absolutely necessary for starting an advanced study; at the same time, he will acquire general information as to the present situation of the subject. He will find problems (of different grades of difficulty) on power- and Laurent-series, on singular points of analytic functions and calculus of residues, on factorization and partial fraction expansion of entire and meromorphic functions, on simple and double periodic functions, on analytic continuation and behavior of power series on their circles of convergence, on Riemann surfaces and conformal transformations. The solutions which follow the problems are condensed enough to be valuable even to students who use them. It is the reviewer's opinion that the two volumes of the *Aufgabensammlung* together with the two volumes of Knopp's *Funktionentheorie* (Sammlung Göschen, Nos. 668, 703) can be successfully used in this country in teaching the first course of the theory of functions of a complex variable.

J. D. Tamarkin

L'Évolution des Idées Géométriques dans la Pensée Grecque: Point, Ligne, Surface. By Federigo Enriques. Translated by Maurice Solovine. Paris, Gauthier-Villars, 1927.

This short treatise contains in outline the history of ancient geometry as a basis for an exposition of the philosophy of Greek geometry and the relation of ancient to modern thought. The merit of this tract lies in the exhibition of the philosophic movement. The historic part reveals certain omissions which would have necessitated the re-phrasing of some parts of the discussion relating to the evolution of the logic of geometry. Thus, in outlining Egyptian geometry, the author fails to point to the fact which has become evident in recent years that early Egyptian writers had advanced further than was formerly supposed and had reached results which as far as Greek testimony indicates, were not transmitted to ancient Greece. Thus the Greeks did not know of the remarkable approximation to the area of the circle, contained in the Rhind papyrus. More striking yet, is the content of another Egyptian papyrus (the Moscow papyrus) presumably of equal antiquity, which, like a flash of lightning in the darkness, momentarily illumines the obscure past. It contains the correct computation of the volume of the frustrum of a square pyramid. Had Enriques had in mind

this startling achievement, his discussion of the origin of the theorem relating to the volume of a pyramid would have been somewhat different. His discussion of the critique of Pythagorean concepts of points due to Parmenides and Zeno of Elea is excellent, as is also his account of the development of the infinitesimal analysis of Democritus and Archimedes, and his comparison of it with modern views. There is another historical point to which it may be worth while to direct attention. Bertrand Russell and A. N. Whitehead refer to Weierstrass as the first to banish the fixed infinitesimal from the differential and integral calculus. Enriques mentions Weierstrass, but also Cauchy and Dini. We wish to remind the reader of the historical fact that the fixed infinitesimal was banished, in the works of Benjamin Robins in 1735, of Colin Maclaurin in 1742, and of Simon Lhuilier in 1786, though, of course, these men had not reached the arithmetization of the theory of limits of the time of Weierstrass.

FLORIAN CAJORI

Invariants of Quadratic Differential Forms. By O. Veblen. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 24.) Cambridge University Press, 1927. viii+102 pp.

This book has been prepared to replace Tract No. 9 of the same series, *Invariants of Quadratic Differential Forms*, by J. E. Wright, which has been out of print for a number of years.

As pointed out in the preface, the presentation is formal in character as the space would permit of only the simplest applications. In Chapter I certain preliminaries are taken up such as the notation, the Kronecker deltas and then application to theorems on determinants. Chapter II is entitled *Differential Invariants*. Here the author considers such concepts as *n-dimensional space, coordinate system, invariant, tensor*, in a manner which commends the highest praise. Nowhere else have I seen these ideas so carefully presented. Chapter III considers quadratic differential forms and the theory of covariant differentiation and the curvature tensor. Chapter IV is devoted to euclidean geometry. The development is carried through largely for an n-dimensional space and in terms of a coordinate system not assumed to be Cartesian. In this fashion the expressions for a number of the metrical invariants of Riemannian geometry are introduced. Chapter V is given to a study of the problem of Christoffel concerning the equivalence of two quadratic differential forms. Chapter VI deals principally with the geometry of paths through the powerful method of "normal coordinates." A short historical section appears at the end of each chapter which serves to point out the chief sources of material.

In the opinion of the reviewer this book is exceedingly well done. The careful formulation of the underlying concepts and the meticulous phrasing of the definitions make this little book an invaluable one to a beginner in the subject, and for the same reason, it seems to me it must afford the adept a high degree of aesthetic enjoyment.

J. H. TAYLOR

NOTES

At a meeting held in New York City on December 27, 1928, a new scientific society, the Acoustical Society of America, was formed to bring together workers in all branches of pure and applied acoustics. Dr. Harvey Fletcher, of the Bell Telephone Laboratories, was elected president.

The Vienna Academy of Sciences has awarded its Lieben prize, for the best mathematical work by an Austrian during the preceding three years, to Professor Karl Menger, for his memoirs on the theory of dimensions.

The Accademia Pontificia dei Nuovi Lincei has announced that it will award a prize of 10,000 lire for an unpublished critical work on the theory of quanta. Competing manuscripts, which may be in English, French, German, Spanish, Italian, or Latin, should reach the Academy by October 30, 1929. Competition is open to all except members of the Academy.

The Royal Academy of Naples has awarded its biennial prize to Professor Enea Bortolotti, for his memoir entitled *Geometria degli spazi riemanniani*.

The gold medal of the Royal Astronomical Society has been awarded to Professor Ejnar Hertzsprung, of the Leyden Observatory, for his work in stellar astronomy.

The Royal Society of Edinburgh has awarded its Gunning Victoria Jubilee prize for the period of 1924–28 to Professor E. T. Whittaker, for his contributions to mathematical science and the promotion of mathematical research in Scotland.

Professors A. A. Michelson, of the University of Chicago, and R. A. Millikan, of the California Institute of Technology, were awarded gold medals by the Society of Arts and Sciences at a meeting in New York City, February 22, 1929.

The American Institute of Electrical Engineers has made the following awards: the Edison gold medal for achievement in electrical science to Dr. F. B. Jewett, for his pioneer research and development work in connection with the theory and practice of voice transmission; the Lamme medal to Mr. Allan Bertram Field "for the mathematical and experimental investigation of eddy current losses in large slot-wound conductors in electrical machinery."

Professor Aurel Stodola, of the Zurich Technical School, has been elected correspondent of the Paris Academy of Sciences in the section of mechanics.

Professors Oskar Perron and Heinrich Tietze, of the University of Munich, have received the title of Geheimer Regierungsrat.

Professor F. Hausdorff, of the University of Bonn, has been elected a member of the Moscow Mathematical Society.

The Council of the Royal Society of London has recommended for election as fellow Professor G. R. Goldsbrough, of Armstrong College, Newcastle-on-Tyne.

Mr. William Bowie, of the United States Coast and Geodetic Survey, has been elected correspondent of the Paris Academy of Sciences in the section of geography and navigation.

Dr. W. J. Humphreys, of the United States Weather Bureau, has been elected a corresponding member of the Meteorological Society of Hungary.

The University of Oxford has conferred the degree of doctor of science on Professor Oswald Veblen, of Princeton University.

Associate Professor Leopold Vietoris, of the University of Innsbruck, has been appointed professor of mathematics at the Vienna Technical School.

Professor Léon Brioullin, of the Collège de France, has been appointed to the newly established professorship of physical theories at the Sorbonne.

Dr. Maurice Fréchet, maître de conférences at the Sorbonne, has been appointed "professeur sans chair" by the Faculty of Sciences of the University of Paris.

Dr. Edwin Feyer, of the Breslau Technical School, has been promoted to an associate professorship.

Associate Professor R. Furch, of the University of Rostock, has been promoted to a professorship.

Dr. Ferenz Jüttner has been appointed to an associate professorship of theoretical physics at the University of Freiburg i. Br.

Associate Professor Wolfgang Krull, of the University of Freiburg i. Br., has been appointed professor of mathematics at the University of Erlangen.

Dr. Karl Loewner, of the University of Berlin, has been appointed an associate professor of mathematics at the University of Cologne.

Dr. B. L. van der Waerden, of the University of Göttingen, has been appointed professor of mathematics at the University of Groningen.

Professor Enea Bortolotti has been appointed professor of analytic projective geometry at the University of Cagliari.

Professor Luighi Brusotti, of the University of Cagliari, has been transferred to the professorship of analytic geometry at the University of Pisa.

Professor G. Juvet, of the University of Neuchâtel, has been appointed professor at the University of Lausanne, as successor to Professor L. Maillard, retired.

Professor R. Nevanlinna, of the University of Helsingfors, lectured at the Zurich Technical School during the year 1928–29, in place of Professor Hermann Weyl, on leave of absence in the United States.

The following have been admitted as docents: Dr. F. Löbell, at the Stuttgart Technical School; Dr. Robert Remak, at the University of Berlin.

The following graduate courses in mathematics are announced for the summer of 1929.

University of California, Berkeley, intersession, May 20–June 29.—By Professor T. M. Putnam: Theory of algebraic equations and of infinite series, Geometric introduction to the theory of functions.—By Professor D. N. Lehmer: Metric differential geometry. Summer session, July 1–August 10: Professor B. A. Bernstein: Elementary algebra for advanced students.—Professor E. R. Hedrick: Analytic geometry of space, Functions of a complex variable.—Professor James Pierpont: Non-euclidean geometry.—Professor Hermann Weyl: Representations of groups, applications and representations of groups to quantum physics.

University of California at Los Angeles, July 1–August 10.—Professor Harriet E. Glazier: Foundations of arithmetic.—Professor M. W. Haskell: Advanced geometry, Geometric introduction to the theory of functions.—Professor Sophia H. Levy: The Teaching of mathematics.

University of Chicago, first term, June 17–July 24; second term, July 25–August 30.—By Professor H. E. Slaught: Differential equations, Elliptic integrals.—By Professor A. C. Lunn: Relativity, Lattices and crystal groups.—By Professor A. B. Coble (Illinois): Analytic projective geometry, Algebraic geometry.—By Professor L. M. Graves: Introduction to analysis, Functions of lines.—By Professor R. W. Barnard: Theory of equations, Metric differential geometry.—By Professor W. Bartky: Theoretical mechanics, Modern theories of analytic differential equations.—By Professor C. C. MacDuffee (Ohio State): Theory of numbers, Theory of algebraic numbers, Linear algebras.—By Professor H. S. Everett: Advanced calculus.

University of Colorado, first term, June 18 to July 21; second term, July 23 to August 24.—By Professor Light: Teacher's course in mathematics; History of mathematics; Partial differential equations.—By Professor Kempner: Algebraic analysis; Projective geometry; Differential equations. Second term.—By Professor Light: Statistics; Teacher's course (repeated); Partial differential equations (continued). —By Professor Kempner: Projective geometry (continued); Differential equations (continued).—By Professor Kendall: Theory of equations.

Columbia University, July 8 to August 16.—By Professor G. D. Birkhoff: Mathematical elements of art; Introduction to relativity.—

By Professor W. B. Fite: Differential equations.—By Professor J. F. Ritt: Theory of numbers.—By Professor B. O. Koopman: Functions of a real variable.

Cornell University, July 6 to August 16.—By Professor Virgil Snyder: Teachers' course; Projective geometry.—By Professor F. R. Sharpe: Advanced analytic geometry.—By Professor W. A. Hurwitz: Advanced calculus.—By Professor W. B. Carver: Theory of numbers. —By Professor C. F. Craig: Elementary differential equations. Reading and research will be directed by Professors J. I. Hutchinson, Snyder, Sharpe, Hurwitz, Carver, D. C. Gillespie, Craig, and C. F. Roos.

University of Illinois, June 17 to August 10.—By Professor G. A. Miller: The theory of numbers.—By Professor R. D. Carmichael: Partial differential equations.—By Professor E. B. Lytle: Teachers' course; Theory of equations and determinants.—By Professor H. Levy: Geometric transformations.—By Dr. V. A. Hoersch: Advanced calculus.—By Dr. F. C. Ogg: Projective geometry.

University of Indiana, June 13 to August 9.—By Professor S. C. Davisson: Theory of functions of a complex variable; Differential equations; Theory of equations.—By Professor D. A. Rothrock: Partial differential equations; Advanced calculus.—By Professor H. E. Wolfe: College geometry; Analytic mechanics.—By Professor C. B. Hennel: General mathematics; Analytic geometry.

University of Iowa, first term, June 8 to July 19.—By Dr. M. A. Nordgaard: Subject matter and teaching of mathematics.—By Dr. Conkwright: Ordinary differential equations; Theory of numbers.—By Professor Wylie: Celestial mechanics; Mathematics of finance; Descriptive astronomy.—By Professor Ward: Modern geometry.—By Professor Chittenden: Advanced calculus; Orthogonal functions.—By Professor Rietz; Actuarial theory and practice; Statistics. Second term, July 22 to August 23.—By Mr. McCoy: Matrices and determinants.—By Dr. Nordgaard: The history of mathematics.—By Professor Ward: Modern geometry; Differential equations.—By Professor Reilly: Algebra for high school teachers; Linear difference equations.

Johns Hopkins University, July 1 to August 9.—By Dr. John Williamson: Theory of functions of a complex variable.

University of Kansas, first term, June 12 to July 20.—By Professor C. H. Ashton: Advanced calculus; seminar.—By Professor U. G. Mitchell: Projective geometry I; Teachers' course in mathematics; seminar. Second term, July 22 to August 17.—By Professor Mitchell: Projective geometry II; History of mathematics; seminar.

University of Michigan, June 24 to August 16.—By Professor J. W. Bradshaw: Higher algebra; Projective geometry.—By Professor P. Field: Vector analysis; Applied mathematics, engineering problems.—By Professor W. B. Ford: Advanced calculus; Infinite series with special reference

to Fourier series.—By Professor T. H. Hildebrandt: Theory of functions of a real variable; Partial differential equations.—By Professor L. C. Karpinski: Teaching of geometry; History of mathematics.—By Professor T. R. Running: Empirical formulas.—By Professor H. C. Carver: Advanced mathematical theory of statistics.—By Professor L. A. Hopkins: Analytic mechanics; Celestial mechanics.—By Professor N. H. Anning: Differential equations.—By Professor C. J. Coe: Integral equations.—By Professor J. A. Nyswander: Theory of probability; Finite differences.—By Professor R. L. Wilder: Foundations of mathematics.—By Mr. N. C. Fisk: Graphical methods.—By Mr. D. K. Kazarinoff: Aerodynamics.

University of Minnesota, first term, June 18 to July 27.—By Professor Dunham Jackson: History of ancient and modern mathematics.—By Professor Elizabeth Carlson: Differential equations. By Professorial Lecturer J. V. Uspensky: Theory of numbers.—By Professor Jackson: Fourier, Legendre and Bessel series.—By Professors Raymond Brink, Dunham Jackson, and Professorial Lecturer J. V. Uspensky: Reading in advanced mathematics. Second term, July 29 to August 31.—By Professorial Lecturer J. V. Uspensky: Recent developments in the mathematical theory of probability.—By Professor A. L. Underhill and Professorial Lecturer J. V. Uspensky: Reading in advanced mathematics.

Ohio State University, June 18 to August 30.—By Professor S. E. Rasor: The teaching of mathematics; Theory of functions of a complex variable; Advanced calculus.—By Professor A. D. Michal: Continuous groups; Tensor analysis.—By Professor Grace Bareis: Projective geometry.

University of Pennsylvania, July 1 to August 10.—By Professor H. H. Mitchell: Galois theory of equations.—By Professor J. R. Kline: Functions of a complex variable.—By Professor F. D. Murnaghan, of Johns Hopkins University: Inversive geometry; Linear differential equations.—By Professor J. M. Thomas: Integral invariants.

University of Pittsburgh.—By Professor K. D. Swartzel: Functions of a complex variable; Teaching of mathematics.—By Professor F. A. Foraker: Modern synthetic geometry; Solid analytic geometry.—By Associate Professor Taylor: Advanced calculus; Functions of a real variable.—By Assistant Professor Culver: Theory of equations.

Stanford University, June 20 to August 31.—By Professor W. A. Manning: Group theory; Theory of functions.—By Professor G. T. Whyburn (Texas): Point-set theory.

University of Texas, first term, June 4 to July 15.—By Professor R. L. Moore: Functions of real variables; Foundations of geometry.—By Professor E. L. Dodd: Probability; Analytic functions.—By Professor H. J. Ettlinger: Differential equations; Ruler and compass constructions.—By Professor H. S. Vandiver: Number theory; Advanced calculus.—By Professor A. E. Cooper: Advanced calculus; Theory of equations.—By Professor P. M. Batchelder: Teaching problems in mathematics. Second term, July 15 to August 26.—By Professor R. L. Moore: Functions of real vari-

ables (continued); Non-euclidean geometry.—By Professor H. J. Ettlinger: Partial differential equations; Definite integrals.—By Professor E. G. Keller: Advanced calculus.

University of Wisconsin, July 1 to August 9.—By Professor R. W. Babcock: Vector analysis.—By Professor H. W. March: Differential equations; Definite integrals.—By Professor E. B. Skinner: Differential geometry; Finite groups; Infinite series. Special nine weeks session for graduates, July 1 to August 30.—By Professor M. H. Ingraham: Higher algebra; Theory of approximations.—By Professor Warren Weaver: Theory of relativity; Advanced electrodynamics; Complex variable theory. (Only one of the two last named courses will be given, the choice depending upon the demand.)—By Professor J. H. Van Vleck of the Department of Physics: Introduction to atomic theory and line spectra; Dielectric and magnetic media; Quantum mechanics and chemistry. (Only one of the two last named courses will be given, the choice depending upon the demand.)

The following forty-nine doctorates with mathematics or mathematical physics as major subject were conferred during 1928; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

A. A. Albert, Chicago, August, *Algebras and their radicals and division algebras.*

H. E. Arnold, Yale, June, *The rational space quintic curve of the second species and its relation to the rational plane quartic curve.*

May M. Beenken, Chicago, June, *Surfaces in five-dimensional space.*

T. C. Benton, Pennsylvania, June, *On continuous curves which are homogeneous except for a finite number of points.*

A. H. Blue, Iowa, July, *On the structure of sets of points of classes one, two, and three.*

G. B. Briggs, Princeton, June, *On types of knotted curves.*

P'ei Yuan Chou, California Institute of Technology, June, theoretical physics, *The gravitational field of a body with rotational symmetry in Einstein's theory of gravitation.*

L. W. Cohen, Michigan, June, *On subsets of separable metric space homeomorphic with subsets of the linear continuum.*

H. A. Davis, Cornell, June, physics, *Involutorial transformations belonging to a linear complex.*

H. A. DoBell, Cornell, February, industrial organization, *On the geometry of the triangle.*

D. C. Duncan, California, May, astronomy, *Rational quintic curves autopolar with respect to a finite number of conics.*

J. M. Earl, Minnesota, July, physics, *Polynomials of best approximation on an infinite interval.*

J. J. Gergen, Rice, June, *Generalized lacunae, On Taylor's series admitting the circle of convergence as a cut, On accessible points on the boundary of a three dimensional region.*

A. O. Hickson, Chicago, August, *An application of the calculus of variations to boundary-value problems.*

E. L. Hill, Minnesota, June, major physics, minor mathematics, *Quantum mechanics of the rotational distortion of spin multiplets in molecular spectra.*

Rosa L. Jackson, Chicago, August, *The boundary-value problem of the second variation for parametric problems in the calculus of variations.*

R. L. Jeffery, Cornell, June, physics, *The sequences of functions which define a definite integral containing a parameter.*

Marie M. Johnson, Chicago, August, *Tensors of the calculus of variations.*

B. W. Jones, Chicago, June, *Representation by positive ternary quadratic orms.*

E. G. Keller, Chicago, August, *On the origin of a planet from a ring system.*

G. H. Keuligan, Johns Hopkins, June, *Vibrations of an elongated* U-*bar.*

M. S. Knebelman, Princeton, June, *Collineations and motions in generalized spaces.*

Mark Kormes, Columbia, January, *On basis sets.*

Lincoln LaPaz, Chicago, August, *An inverse problem of the calculus of variations.*

W. T. MacCreadie, Cornell, February, physics, *On the stability of the motion of a viscous fluid.*

Morris Marden, Harvard, June, *On the location of the roots of the jacobian of two binary forms and of the derivative of a rational function.*

W. L. Moore, Illinois, June, mathematical physics, *On the geometry of the Weddle surface.*

D. C. Morrow, Chicago, June, *The determination of all quaternary quadratic forms which represent every positive integer.*

F. W. Perkins, Harvard, February, *On the oscillation of harmonic functions.*

J. W. Peters, Johns Hopkins, June, *Invariants of sets of points under inversion.*

O. J. Peterson, Michigan, June, *On the rational plane quintic with three cusps.*

C. G. Phipps, Minnesota, July, physics, *Problems in approximation by functions of given continuity.*

Allie W. Richeson, Johns Hopkins, June, *Pentagons inscribed in circles.*

W. C. Risselman, Minnesota, July, physics, *Approximation to a given function by means of polynomials in another given function.*

V. B. Rojansky, Minnesota, August, major physics, minor mathematics, *The Stark effect of hydrogenic atoms in the new quantum mechanics.*

W. E. Roth, Wisconsin, June, applied mathematics, *A solution of the matrix equation* $P(X)=A$.

C. A. Rupp, Chicago, June, *An extension of Pascal's theorem to space of r dimensions.*

N. E. Rutt, Pennsylvania, June, *Concerning the cut points of a continuous curve when the arc curve contains exactly n independent arcs.*

S. A. Schelkunoff, Columbia, May, *On certain properties of the metrical and generalized metrical groups in linear spaces of n dimensions.*

A. A. Shaghoian, California, May, analytical mechanics, *Solution of homogeneous linear difference equations by means of infinite determinants.*

C. D. Smith, Iowa, February, *On generalized Tchebycheff inequalities in mathematical statistics.*

F. E. Smith, Catholic, June, physics and philosophy, *The triangles in- and circumscribed to the triangular symmetric rational quartic curve.*

Dan Sun, Chicago, August, *Projective differential geometry of quadruples of surfaces with points in correspondence.*

E. L. Thompson, Chicago, June, *Systems of two differential equations from the Lie group standpoint.*

Morgan Ward, California Institute of Technology, June, physics, *The foundation of general arithmetic.*

Marie J. Weiss, Stanford, June, *Primitive groups which contain substitutions of prime order p and of degree* $6p$ *or* $7p$.

C. O. Williamson, Chicago, August, *Stability of an airplane with rotating propeller.*

D. W. Woodard, Pennsylvania, June, *Two dimensional analysis situs, with special reference to the Jordan curve-theorem.*

Ko-Chuen Yang, Chicago, August, *Various generalizations of Waring's problem.*

Princeton University has received from its Class of '87 the sum of $200,000 to endow the chair of astronomy held by Professor H. N. Russell, a member of that class. The class also presented a portrait of Professor Russell to the University.

Professor G. H. Hardy lectured at the University of Iowa, March 23, 1929, on *Hilbert's Logic*, and at Columbia University, April 17–18, on *Prime Numbers*.

Professor and Mrs. E. B. Van Vleck will sail from San Francisco on August 15, for a trip around the world. Professor Van Vleck will retire from active teaching in June, 1929.

Professor Hermann Weyl, of the Zurich Technical School, who was appointed Visiting Lecturer for this Society for the present year, has delivered lectures recently at Harvard University (April 29–May 3), Chicago (May 7), Illinois (May 8–10), The Rice Institute (May 20–22), California Institute of Technology (May 31–June 3), Stanford University (June 7–10). The subjects of his lectures have been: *Gravitation and the electron*, *Consistency of mathematics*, *The spherical symmetry of atoms*, *The theory of groups and quantum mechanics*, and *The symmetric and the linear group*.

Dr. Gregory Breit, mathematical physicist of the Department of Terrestrial Magnetism, has been assigned for a year to work in atomic physics in various laboratories and universities of Europe.

Dr. Elizabeth B. Cowley, of Vassar College, has been promoted to a professorship of mathematics.

Professor Werner Heisenberg, of the University of Leipzig, will lecture at the University of Chicago on theoretical physics, during the summer quarter.

Dr. A. O. Hickson, of Brown University, has been appointed assistant professor of mathematics at Duke University.

Dr. M. S. Knebelman has been appointed assistant professor of mathematics at Princeton University.

Mr. E. R. C. Miles, of Rice Institute, has been appointed assistant professor of mathematics at Duke University.

Professor C. H. Richardson, formerly of Georgetown College, now professor of mathematics at Bucknell University, was recently elected by the Faculty of Bucknell University to serve as head of the department of mathematics for the current academic year.

Dr. C. H. Smiley of the University of Illinois, has been appointed Guggenheim Fellow for 1929–30.

Dr. W. H. Taylor has been appointed head of the department of mathematics at Alabama College.

Associate Professor Robert Torrey, of the University of Mississippi, was granted leave of absence for the first semester of the present academic year. Professor Torrey was injured in a train wreck last June.

Professor G. T. Whyburn, recently appointed at Johns Hopkins, has been appointed Guggenheim Fellow for next year, and will have leave of absence from Johns Hopkins.

Professor J. Boussinesq, member of the section of mechanics of the Paris Academy of Sciences, died February 19, 1929, at the age of eighty-six.

Professor Luighi Carlini, of the Technical School of Udine, died December 28, 1928.

Mr. A. K. Erlang, of the Copenhagen Telephone Company, known for his work in the applications of mathematics to telephone problems, died February 3, 1929, at the age of fifty-one.

Professor Johannes von Kries, of Freiburg i. Br., the distinguished physiologist, editor of the German edition of Helmholtz's *Physiological Optics* and author of *Die Prinzipien der Wahrscheinlichkeitsrechnung*, died December 30, 1928, at the age of seventy-five.

Professor Samuel Oppenheim, of the department of astronomy of the University of Vienna, died August 15, 1928, at the age of seventy.

Professor P. H. Puiseux, honorary astronomer of the Paris Observatory, died September 28, 1928.

Professor Charles Riquier died at Caen in January, 1929, at the age of seventy-eight.

Professor Giovanni Russo, of the Technical School at Catanzaro, died June 4, 1928, at the age of seventy-seven.

Mr. W. B. Alcock, senior fellow of Emmanuel College, Cambridge, died January 18, 1929, at the age of seventy-one.

Professor A. W. Bickerton, of Canterbury College, Christchurch, New Zealand, known for his impact theory of cosmic evolution, died January 23, 1929, at the age of eighty-seven.

Mr. T. H. Blakesley, honorary secretary of the London Physical Society, known for his work in the theory of alternating currents, died February 13, 1929, at the age of eighty-one.

Mr. M. J. M. Hill, of University College, London, died January 11, 1929, at the age of seventy-two.

Mr. A. W. Prater, of the University of California at Los Angeles, died April 16, 1929.

Professor W. H. Sherk, of the University of Buffalo, died in January, 1929.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

AUERBACH (F.). Lebendige Mathematik. Breslau, Hirt, 1928.

BARRAU (J. A.). Figuren en hare coordinaten. Groningen, Wolters, 1928. 19 pp.

VAN DEN BERG (F.). De multi-sektie van den hoek in de vlakke meetkunde op 5 verschillende wijzen aangetoond. Rotterdam, the author, 1927.

BLASCHKE (W.). Vorlesungen über Differentialgeometrie. Band 3. Berlin, Springer, 1929.

BREUER (S.). See EULER (L.).

BROWN (O. E.). See NAZINOFF (P. S.).

BURNSIDE (W. S.) and PANTON (A. W.). The theory of equations with an introduction to the theory of binary algebraic forms. Edited by M. W. J. Fry. 7th edition. Volume 2. London, Longmans, 1928.

CAJORI (F.). A history of mathematical notations. Volume 2: Notations mainly in higher mathematics. Chicago, Open Court, 1929. 18+367 pp.

COURANT (R.). Vorlesungen über Differential-und Integralrechnung. Band 2: Funktionen mehrerer Veränderlicher. Berlin, Springer, 1929. 360 pp.

VAN DIEREN (C. M.). Nulstelsels in het platte vlak. (Dissertation, Utrecht.) Utrecht, Drukkerij Zuidam, 1928. 77 pp.

VAN DITMARSH (A. J.). Afbeelding van de lijnelementen van een vlak op den complex der Raaklijnen van een monoïde. (Dissertation, Utrecht.) Zutphen, Nauta, 1928. 29 pp.

DÜSING (K.). Leitfaden der Kurvenlehre. Mit zahlreichen Anwendungen aus der Technik von E. Preger. 3te vermehrte Auflage, bearbeitet von E. Wilde. Leipzig, Jänecke, 1928. 8+143 pp.

EULER (L.). Drei Abhandlungen über die Auflösung der Gleichungen. (1738, 1764, 1790.) Übersetzt und herausgegeben von S. Breuer. (Ostwalds Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1928. 94 pp.

EUWE (W.). Differentiaal-invarianten en partiëelle differentiaalvergelijkingen uit de tensorrekening. (Dissertation, Amsterdam.) Amsterdam, H. J. Paris, 1928. 12+84 pp.

FORD (W. B.). College algebra. Revised edition. New York, Macmillan, 1928. 6+278 pp.

FRY (M. W. J.). See BURNSIDE (W. S.).

HAAG (J.). Cours complet de mathématiques élémentaires. Tome 3. Paris, Gauthier-Villars, 1928. 346 pp.

HARDING (A. M.) and MULLINS (G. W.). Plane trigonometry. New York, Macmillan, 1928. 8+118 pp.

HASSE (H.) und SCHOLTZ (H.). Die Grundlagenkrisis der griechischen Mathematik. Charlottenburg, Panverlag Kurt Mentzner, 1928. 72 pp.

HOESTRA (H. P.). Onderzoek van eenige bilineaire congruenties van kubische ruimtekrommen door eene afbeelding op het puntenfeld. (Dissertation, Utrecht.) Utrecht, van Druten, 1928. 80 pp.

HOHEISEL (G.). Partielle Differentialgleichungen. (Sammlung Göschen.) Berlin, de Gruyter, 1928. 159 pp.

VON HOLZHAUSEN (W.). Logik und Zweckreinheit im neudeutschen Schachproblem. Berlin, de Gruyter, 1928. 114 pp.

KÜHNE (O.). Die mathematische Schule in der Nationalökonomie. Band I, 1ter Teil: Die italienische Schule bis 1924. Berlin, de Gruyter, 1928.

LENNES (N. J.). College algebra. New York, Harper, 1928. 14+301 pp.

LIETZMANN (W.). Aus der Mathematik der Alten. Leipzig, Teubner, 1928.

LÖFFLER (E.). Ziffern und Ziffernsystem. Teil 1. Leipzig, Teubner, 1928. 2+54 pp.

LORIA (G.). Storia delle matematiche. Volume I: Antichita, medio evo, rin ascimento. Torino, S.T.E.N., 1929. 497 pp.

MULLINS (G. W.). See HARDING (A. M.).

NAZIMOFF (P. S.). Applications of the theory of elliptic functions to the theory of numbers. Translated from the Russian by A. E. Ross. Mimeographed by O. E. Brown. Evanston, 1928. 146 pp. (On sale at the University of Chicago Book Store: $3.00.)

OSGOOD (W. F.). Lehrbuch der Funktionentheorie. 5te Auflage, 1ter Band, und 2te Auflage, 2ter Band, 1te Lieferung. Leipzig, Teubner, 1928, 1929. 14+818+8+307 pp.

PANTON (A. W.). See BURNSIDE (W. S.).

PESCI (G.). Lezioni di calcolo numerico, grafico e meccanico. Livorno, Autografia della Reale Accademia Navale, 1928.

PREGER (E.). See DÜSING (K.).

ROSS (A. E.). See NAZIMOFF (P. S.).

ROTHE (R.). Höhere Mathematik für Mathematiker, Physiker und Ingenieure. Teil II. Leipzig, Teubner, 1929. 8+201 pp.

RUTGERS (J. G.). See SCHUH (F.).

SCHMEHL (C.). Die Jacobische Erzeugungsweise der Flächen zweiten Grades. Giessen Roth, 1929.

SCHOLZ (H.). See HASSE (H.).

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SCHWANK (F.). See VIVANTI (G.).

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STAMMLER (G.). Begriff, Urteil, Schluss. Untersuchungen über Grundlagen und Aufbau der Logik. Halle a. S., Niemeyer, 1928.

STRACHAN (R. C.). A table of hyperbolic radians, adapted to the derivation of hyperbolic functions from trigonometric tables. New York, the author, 1928.

STRAUCH (J.). Die primitiven Wurzeln im allgemeinen Zahlkörper

(Dissertation, Leipzig.) Leipzig, Druckerei der Werkgemeinschaft, 1927. 11 pp.

SWETT (G. W.). Construction of alignment charts. New York, Wiley, 1928. 98 pp.

THOMPSON (A. J.). Logarithmetica Brittanica. Being a standard table of logarithms to twenty decimal places. Part 4: Numbers 40,000 to 50,000. (Tracts for Computers, No. 16.) Cambridge, University Press, 1929. 6+100 pp.

VIVANTI (G.). Elemente der Theorie der linearen Integralgleichungen. Übersetzt und mit Anmerkungen versehen von F. Schwank. Hannover, Helwingsche Verlagsbuchhandlung, 1929.

WILDE (E.). See DÜSING (K.).

PART II. APPLIED MATHEMATICS

ANGENHEISTER (G.). See GEIGER (H.).

BACON (R.). Opera hactenus inedita. Fasciculus VIII: Questiones supra libros quatuor phisicorum Aristotelis, nunc primum edidit F. N. Delorme collaborante Robert Steele. Oxford, Clarendon Press, 1928. 22+284 pp.

—— The Opus Majus of Roger Bacon. A translation by R. B. Burke. 2 volumes. Philadelphia, University of Pennsylvania Press, and London, Oxford University Press, 1928. 13+418+6+422 pp.

BARKHAUSEN (H.). Elektronen-Röhren. 3te Auflage. Band 2: Röhrensender. Leipzig, Hirzel, 1928. 6+121 pp.

BEAVER (J. L.). Elements of alternating currents and alternating current apparatus. 2d edition. New York, Longmans, 1928.

BOSLER (J.). Cours d'astronomie, Faculté des Sciences de Paris. Tome 3: Astrophysique. Paris, Hermann, 1928. 5+723 pp.

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BUSH (V.). Operational circuit analysis. With an appendix by N. Wiener. New York, Wiley, 1929. 10+392 pp.

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DAHL (O. G. C.). Electric circuits. Theory and applications. Volume 1: Short calculations and steady-state theory. New York, McGraw-Hill, 1928. 10+390 pp.

DEBYE (P.). See SOMMERFELD (A.).

DELORME (F. N.). See BACON (R.).

Enrera (J.). Polarisation diélectrique. Paris, Les Presses Universitaires de France, 1928. 172 pp.

Falkenhagen (H.). Quantentheorie und Chemie. Leipzig, Hirzel, 1928. 142 pp.

Föppl (O.). See Geiger (H.).

Forsterling (K.). Lehrbuch der Optik. Leipzig, Hirzel, 1928. 12+610 pp.

Fowler (R. H.). Statistical mechanics. The theory of the properties of matter in equilibrium. Cambridge, University Press, 1929. 570 pp.

Geiger (H.) und Scheel (K.), herausgegeben von. Handbuch der Physik. Band 6: Mechanik der elastischen Körper. Bearbeitet von G. Angenheister, A. Busemann und O. Föppl. Redigiert von R. Grammel. Berlin, Springer, 1928. 12+632 pp.

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Giorgi (G.). Che cos è elettricità? Roma, Paolo Cremonese, 1928. 136 pp.

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Haas (A.). Wave mechanics and the new quantum theory. Translated by L. W. Codd. London, Constable, 1928. 18+124 pp.

Hänert (L.). Geschütz und Schuss. Berlin, Springer, 1928. 7+359 pp.

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Hart (I. B.). An introduction to advanced heat. London, Bell, 1928. 7+336 pp.

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Immler (W.). Flugzeugnavigation. München, Oldenbourg, 1928.

Janet (C.). La classification hélicoidale des éléments chimiques. Beauvais, Imprimerie Départementale de l'Oise, 1928. 77 pp.

Jennings (C.) and Tobutt (R. L. W.). Army mathematics. Oxford, University Press, 1928. 2+224 pp.

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Pettersson (H.). Künstliche Verwandlung der Elemente (Zertrümmerung der Atome). Aus dem Schwedischen übersetzt von E. Kirsch. Berlin, de Gruyter, 1929.

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Schneider (E.). See Schleede (A.).

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Smekal (A.). See Riecke (E.).

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—— Probleme der modernen Physik, Arnold Sommerfeld zum 60. Geburtstag gewidmet von seinen Schülern. Herausgegeben von P. Debye. Leipzig, Hirzel, 1928. 221 pp.

Steele (R.). See Bacon (R.).

Sullivan (J. W. N.). The bases of modern science. London, Benn, 1928. 10+246 pp.

Swain (G. F.). Festigkeitslehre. Autorisierte Übersetzung von A. Mehmel. Berlin, Springer, 1928. 12+630 pp.

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THE MARCH MEETING IN NEW YORK

The two hundred sixty-eighth regular meeting of the American Mathematical Society was held at Columbia University, on Friday and Saturday, March 29–30, 1929, extending through the usual morning and afternoon sessions. The attendance included the following one hundred thirty-two members of the Society:

C. R. Adams, A. A. Albert, R. L. Anderson, R. G. Archibald, H. E. Arnold, Bamforth, Bellamy, A. A. Bennett, Benton, A. C. Berry, William James Berry, William Johnston Berry, Birkhoff, Bowden, B. H. Camp, G. A. Campbell, Carlitz, Cope, L. P. Copeland, Cramlet, Dehn, Demos, Doak, Doermann, Dresden, Eisenhart, Emch, Engstrom, Feld, Fite, D. A. Flanders, Fort, R. M. Foster, Philip Franklin, Frink, Gale, Gehman, Gergen, Gill, Gronwall, C. C. Grove, G. H. Hardy, Hazeltine, Hebbert, Hildebrandt, L. S. Hill, Hille, Himwich, Hofmann, Hollcroft, Huber, R. L. Jackson, R. A. Johnson, Kaplan, Kasner, O. D. Kellogg, Kenny, Kholodovsky, Rose Klein, Kline, Knebelman, Koopman, J. P. Kormes, Mark Kormes, Kunte, Lamson, Langman, Lefschetz, Lehr, Littauer, Litzinger, Lubin, MacColl, MacInnes, Marden, Meder, Michal, E. J. Miles, H. H. Mitchell, Molina, C. N. Moore, T. W. Moore, Richard Morris, Mullins, F. H. Murray, C. A. Nelson, Newman, Ore, F. W. Owens, H. B. Owens, Parsons, Pell-Wheeler, Pfeiffer, Pooler, Post, R. G. Putnam, Raudenbush, Reddick, M. S. Rees, Rider, Ritt, Robertson, Seely, Serghiesco, Sheffer, Simons, Singer, C. E. Smith, P. A. Smith, S. E. Smith, M. H. Stone, Swartzel, Tamarkin, Evan Thomas, J. M. Thomas, J. E. Thompson, Trjitzinsky, Veblen, Wahlert, H. E. Webb, D. W. Weeks, Weida, Weinstein, Weisner, Weyl, Widder, Wiener, W. J. Willis, J. W. Young, Mabel M. Young, Margaret M. Young, Zippin.

There was no meeting of the Council or of the Board of Trustees. A telegram of greeting was received from the meeting of the Society held at the same time in Chicago, and was acknowledged appropriately.

The afternoon sessions on Friday and Saturday were devoted to a Symposium on Fourier Series. The following papers were presented: Friday, March 29, *Modern work in the theory of ordinary trigonometric series*, by Professor G. H. Hardy, with discussion by Professors Einar Hille and C. N. Moore, and Dr. T. H. Gronwall; Saturday, March 30, *Fourier series and almost periodic functions from the standpoint*

of the theory of groups, by Professor Hermann Weyl, with discussion by Professors Norbert Wiener, T. H. Hildebrandt, Philip Franklin, and M. H. Stone. Ex-President Birkhoff presided at the Symposium, relieved during the discussion on Saturday afternoon by Professor Kasner.

The morning sessions were devoted to the reading of short papers. Vice-President Young presided on Friday morning, relieved by Professor Ritt, and Professor H. H. Mitchell on Saturday morning. Titles and abstracts of the papers read at these sessions follow below. The papers of Bamforth (except his joint paper with Birkhoff), Browne, Cope (second paper), Féraud, Gronwall (second paper), Hickey, Lubin, Michal (second paper and joint paper with Peterson), Peterson, Pierpont, Rutledge, Seely, Vandiver, Walsh, Wiener were read by title. Dr. Féraud was introduced by Professor Birkhoff, Miss Hickey by Professor Michal, Mr. Hurwitz by Professor Ritt, Mr. Thielman by Professor Michal, and Mr. Whitney by Professor Pierpont.

1. Professor C. R. Adams: *Note on integro-q-difference equations.*

The equation considered is one in which the unknown function $f(x)$ is subjected simultaneously to a q-difference operator of the nth order and to an integral operator of Volterra type. Certain facts about the solutions of the equation of first order when $|q| \leqq 1$ follow from the work of Picone, Popovici, Nalli and Tamarkin; these writers not unnaturally confined themselves to the real domain. The main purpose of this paper is to investigate the existence of solutions in the complex domain for the equation of order n including the case of $|q|<1$ and the essentially different case of $|q|>1$. Assuming the known functions in the equation to be analytic we find formal series solutions. When $|q|>1$ the existence of analytic solutions is shown by direct proof of convergence. In the case of $|q|<1$ results are obtained both by a generalization of Tamarkin's work and by a method of successive approximation; these methods also are employed in the real domain when the known functions are subject to less stringent hypotheses.

2. Dr. I. M. Sheffer (National Research Fellow): *The linear functional operators and equations associated with sets of polynomials.*

The writer has considered in a previous report certain algebraic properties of sets of polynomials $P: \{P_n(x)\}$, $n=0, 1, \cdots$, where the degree of $P_n(x)$ does not exceed n. Now we consider certain linear functional

operators and equations which arise when we turn to the analytic aspect of such sets. An operator L is defined as carrying the identity set I into P. We can express L as a linear differential operator of infinite order. With L we consider a second operator $\mathcal{L}$, and we investigate the characteristic numbers and characteristic functions of these operators, the problem of expanding "arbitrary" analytic functions in terms of these characteristic functions, and the question of solving the functional equations which are determined by the operators.

3. Professor C. N. Moore: *On the absolute convergence of the double Fourier series.*

In this paper it is shown that if a function of two variables which is of bounded variation also satisfies at a point a condition which is a generalization of the Lipschitz condition for functions of one variable, the double Fourier series corresponding to this function will be absolutely convergent at the same point. This theorem is a generalization to double Fourier series of a criterion found by Zygmund (Journal of the London Mathematical Society, vol. 3 (1928), p. 194) for the ordinary Fourier series.

4. Dr. F. R. Bamforth (National Research Fellow) and and Professor G. D. Birkhoff: *Divergent series and singular points of ordinary differential equations.*

Consider the system of differential equations $dx_i/dt = X_i(x_1, x_2, x_3)$ in which the X_i are analytic in x_i for $|x_i| \leqq r$, $r>0$ and $X_i(0, 0, 0) = 0$, $i = 1, 2, 3$. Let the equations of variation be $dy_i/dt = \sum_{j=1}^{3} c_{ij} y_j$ where the constants c_{ij} are the values of $\partial X_i/\partial x_j$ evaluated at $x_1 = x_2 = x_3 = 0$. Let the roots m_1, m_2, and m_3 of the equation $|c_{ij} - m\delta_{ij}| = 0$ be such that m_1 and m_2 are greater than zero, m_3 is less than zero, and there exists among them no relationship of the form $m_i = p_1 m_1 + p_2 m_2 + p_3 m_3$, where the p_i are integers, positive or zero, whose sum is greater than 1. In this paper it is shown that under these hypotheses, in a certain neighborhood of the origin $x_1 = x_2 = x_3 = 0$, the system of differential equations $dx_i/dt = X_i(x_1, x_2, x_3)$ is equivalent to the system $dz_i/dt = m_i z_i$, $i = 1, 2, 3$, by means of a transformation of the type $z_i = z_i(x_1, x_2, x_3)$ where the functions $z_i(x_1, x_2, x_3)$ have the properties that they are zero for $x_1 = x_2 = x_3 = 0$, are of class C^∞, and are analytic at every point (x_1, x_2, x_3) which does not lie on a certain set of three surfaces passing through the origin.

5. Dr. F. R. Bamforth: *A boundary-value problem for a system of ordinary differential equations not linear in the parameter.*

In matrix notation in which small letters will stand for vectors of order n, capital letters for square matrices of order n, and Greek letters for numbers, the boundary-value problem studied may be written in the form $y' = y(A + B_1/(\lambda - \beta_1) + B_2/(\lambda - \beta_2))$, $My(a) + Ny(b) = 0$, where $\beta_1 \neq \beta_2$. The elements of the matrices A, B_1, B_2 are real single-valued, continuous

functions of x on the interval $a \leqq x \leqq b$, λ is a parameter, and the elements of the matrices M and N as well as β_1 and β_2 are real numbers. For boundary-value problems of the above form, definitions of adjointness and of definite self-adjointness are given. For definitely self-adjoint boundary-value problems questions relative to the existence of characteristic parameter values and the possibility of expanding an arbitrary vector in terms of the characteristic solutions are considered.

6. Dr. F. R. Bamforth: *A boundary-value problem for a system of differential equations of the second order.*

In a paper presented by the author at the Chicago meeting of the society last spring, a set of normal forms was derived for boundary-value problems connected with systems of ordinary differential equations which are linear in a parameter λ. This paper brought to light certain boundary-value problems of general type which had never been studied. The present paper is a study of one of these problems, which may be written in the form $y_1' = (a_{11}(x)+\lambda)y_1+a_{12}(x)y_2$, $y_2' = \lambda(x)y_1-(a_{11}(x)-\lambda)y_2$, $m_{11}y_1(a)+m_{12}y_2(a)+n_{11}y_1(b)+n_{12}y_2(b)=0$ $m_{21}y_1(a)+m_{22}y_2(a)+n_{21}y_1(b)+n_{22}y_2(b)=0$. Under certain hypotheses on the coefficients, it is found that there exists a denumerable infinitude of characteristic numbers. Furthermore, sufficient conditions for the simultaneous expansion of a pair of fairly arbitrary functions in terms of the characteristic solutions are given.

7. Mr. B. P. Gill: *An analog for algebraic functions of the Thue-Siegel theorem.*

In this paper the following theorem is proved. Let $\theta(t)$ be an algebraic function of t defined by an irreducible equation of degree $n \geqq 2$. Then there exist only a finite number of rational functions $x(t)/y(t)$, (x and y polynomials in t), such that, if the difference $\theta(t)-x(t)/y(t)$ be expanded about infinity in a series of descending powers of t, the initial term of the series is of degree less than $-2n^{1/2}\times$the degree of y. Applications are indicated to indeterminate algebraic equations in which the unknowns represent polynomials.

8. Mr. Solomon Hurwitz: *Algebraic first integrals of algebraic differential equations.*

It was shown by Fuchs that if an algebraic differential equation $f(x, y, y')=0$, or $y'=u(x, y)$, has a general solution $V(x, y)=c$, where V is an algebraic function of x and y, then there exists a general solution $R(x, y, u)=c$, R being a rational function of x, y, and u. In the present paper, it is shown that if an algebraic differential equation of the nth order $f(y^{(n)}, y^{(n-1)}, \cdots, y', y, x)=0$, or $y^{(n)}=u(y^{(n-1)}, \cdots, y', y, x)$, has an algebraic first integral $V(y^{(n-1)}, \cdots, y', y, x)=c$, then there exists a first integral $R(y^{(n-1)}, \cdots, y', y, x, u)=c$, where R is a rational function of the quantities in parentheses.

9. Dr. T. F. Cope (National Research Fellow): *Sufficient conditions for a weak minimum for a special case of the problem of Mayer with variable end points.*

The problem considered is that of finding conditions which suffice for a curve $y_i=y_i(x)$, $x_1\leqq x\leqq x_2$, $i=1, \cdots, n$, to minimize the first of a set of functions $f_\rho[x_1, x_2, y_j(x_1), y_j(x_2)]$, $\rho=1, \cdots, r\leqq 2n+2; j=1, \cdots, n$, in the class of similar curves which make $f_2, \cdots, f_r$ vanish and besides satisfy the differential equations $\theta_\alpha(x, y_j, y_j')=0$, $\alpha=1, \cdots, m<n$. First necessary conditions, namely the Euler equations and transversality conditions, were deduced by Bliss (Transactions of this Society, vol. 19 (1918), p. 305). In the author's Chicago dissertation, presented to this Society last year at the April meeting in Chicago, another necessary condition was found, which is essentially that the characteristic numbers of a boundary-value problem associated with the second variation must be greater than or equal to zero. If x_1 and x_2 are fixed, it is shown in the present paper that sufficient conditions for a weak relative minimum can be deduced from the necessary conditions mentioned above, the comparison curves being of class C'.

10. Dr. T. F. Cope: *A set of sufficient conditions for a minimum for the problem of Mayer with variable end points for specialized comparison arcs.*

The problem considered is that of the preceding paper. If the comparison curves are defined suitably, it is proved that the necessary conditions mentioned in the first paper give rise to conditions which are sufficient for a weak relative minimum even when x_1 and x_2 as well as $y_i(x_1)$, and $y_i(x_2)$ are permitted to vary. The boundary-value problem associated with the second variation in this case is of an essentially more complicated type than that of the first paper, and special methods are required to discuss it.

11. Dr. T. H. Gronwall: *Straight line geodesics in Einstein's parallelism geometry.*

The conditions that in an Einstein parallelism geometry the straight lines and the geodesics shall coincide are set up, and the resulting partial differential equations for the Z's completely integrated in the case of three dimensions.

12. Professor D. V. Widder: *On the composition of the singularities of functions defined by factorial series.*

The purpose of the present paper is to determine the form of the coefficients c_n in the series $F(s)=\sum c_n n!/[s(s+1)\cdots(s+n)]$ so that the function $F(s)$ which is defined by the series shall have singularities at most at the points $\alpha+\beta$, where the points α are the singularities of the function $f(s)=\sum a_n n!/[s(s+1)\cdots(s+n)]$, the points β those of the function

$\phi(s)=\sum b_n n!/[s(s+1)\cdots(s+n)]$. It is found under certain specified conditons that c_n may be taken as $a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$, the nth coefficient of the power series expansion of the product of the two power series $\sum a_n s^n$ and $\sum b_n s^n$. The result is obtained by expressing $f(s)$ and $\phi(s)$ as generating functions of the form $\int_0^\infty e^{-st}\alpha(t)dt$, and applying a result previously obtained and presented to the Society by the author.

13. Dr. W. J. Trjitzinsky: *On composition of singularities.*

Several theorems on composition of singularities of the type analogous to Hadamard's multiplication theorem are derived. The method is based essentially on complex integration, and on the possibility of deforming a contour unless singularities of a certain integrand tend to coincide. Let α and β be the respective singularities of two functions. Among the theorems obtained are two about a third function, in the first theorem with singularities of the form α with the exponent β, and in the second theorem with singularities of the form $(a\alpha\beta+b\alpha+c\beta+d)/(e\beta+f)$. An extension of the latter theorem, which is a generalization of Hadamard's theorem, is given for functions of two variables, similar to Haslam-Jones' extension of the ordinary multiplication theorem.

14. Professor O. D. Kellogg: *Singular manifolds among those of an analytic family.*

This paper will appear in full in an early issue of this Bulletin.

15. Dr. C. M. Cramlet (National Research Fellow): *Invariants of the n-ary q-ic differential form.*

The general differential form may be written as $a_{\rho_1\cdots\rho_q}dx^{\rho_1}\cdots dx^{\rho_q}$, where the $a_{r_1\cdots r_q}$ are functions of the n variables $x^1,\cdots,x^n$. Subjecting the x's to an analytic transformation and assuming the form to be invariant implies a sequence of tensors. The problem of obtaining these tensors is reduced to a problem of an affine connection Γ^a_{bc} and a tensor $a_{r_1\cdots r_q}$, where the Γ^a_{bc} are functions of position and are given explicitly in terms of the $a_{r_1\cdots r_q}$ and their derivatives. Covariant differentiation may then be defined as in affine geometry and a complete set of tensors are the $a_{r_1\cdots r_q}$ and their successsive covariant derivatives, and the curvature tensor B^r_{stu} and its covariant derivatives. The equivalence of two forms is stated in terms of a finite number of these tensors. The condition under which the present solution can be effected is that the discriminant of the form is non-vanishing. It is expected that this condition can be replaced by the condition that there exists at least one non-vanishing algebraic invariant of the tensor $a_{r_1\cdots r_q}$. Some interpretations of the significance of the tensors are given.

16. Professor A. D. Michal: *Projective functional invariants.* Preliminary communication.

This paper is concerned with the construction of functionals that are

the function-space analogs of the projective invariants of n-dimensional affinely connected manifolds. A generalization of Weyl's projective curvature tensor is obtained.

17. Professor A. D. Michal: *Tensor analysis in function space under functional transformations with differentials of the third kind.*

In an earlier paper (American Journal of Mathematics, vol. 50 (1928), pp. 473–517) the author has shown how to construct a theory of tensors in function space. In the present paper this theory is generalized and made a more harmonious whole by the employment of functional transformations with differentials of the third kind, that is, (1): $\delta z^i = {}_i'z^{(i)}\delta\bar{z}^i + {}'z_\alpha{}^i\delta\bar{z}^\alpha$. The differential functional forms are taken in the general normal form with generalizations in the differentials of their coefficients appropriate to (1).

18. Professor A. D. Michal and Mr. T. S. Peterson: *The invariant theory of functional forms under the group of linear functional transformations of the third kind.*

The invariant theory of special functional forms under the Fredholm group was initiated by A. D. Michal (American Journal of Mathematics, vol. 50 (1928)). Further contributions to this subject were given by T. S. Peterson (this Bulletin, vol. 34 (1928), p. 416). This paper is concerned with the development of an invariant theory of still more general "algebraic" functional forms in addition to the more general underlying group, that of linear functional transformations of the third kind.

19. Mr. H. P. Thielman: *On new integral addition theorems for Bessel functions and series of the hypergeometric type.*

Although Bessel functions and hypergeometric series have been studied in great detail, the literature seems to have very little to say about their integral addition theorems. This paper derives a large number of such addition theorems by employing Volterra's theories of functions of composition.

20. Miss Maude I. Hickey: *Representations of groups of linear functional transformations.*

This paper begins the systematic study of representations of groups of linear functional transformations in one function by means of linear functional transformations in a sequence of functions of several variables.

21. Mr. T. S. Peterson: *Note on the law of transformation of the nth resolvent.*

This note supplements in full the law of transformation of the nth resolvent as defined in an earlier paper (see T. S. Peterson, this Bulletin, vol. 34 (1928), p. 416).

22. Professor James Pierpont: *On the attraction of spheres in elliptic space.*

This paper has appeared in full in the May-June number of this Bulletin.

23. Professor George Rutledge: *On the relation between de la Vallée-Poussin summability and the cardinal series of interpolation.*

W. L. Ferrar (Proceedings of the Royal Society of Edinburgh, vol. 45 (1925), p. 269) has proved that, under certain very general conditions, the cardinal series of interpolation is summable by the method of de la Vallée-Poussin to the sum of the Gauss series of even order, whenever the latter series is convergent. Restricting conditions are inherent in Ferrar's method of proof, but do not in fact exist. The Gauss series of even order is identically equal, at any stage of summation, to the Lagrange polynomial of even degree, and when the latter is written in the form (previously presented by the author) $x\prod_{k=1}^{n}(1-x^2/k^2)\{f(0)/x+\sum_{r=1}^{n}\binom{2n}{n-r}\binom{2n}{n}^{-1}(-1)^r \cdot [f(r)/(x-r)+f(-r)/(x+r)]\}$, it becomes at once apparent that the sum within the braces is nothing other than the sum of $n+1$ terms of the sum (VP) of the cardinal series. Thus summability (VP) of the cardinal series and convergence of the Lagrange polynomial of even degree (and hence of the series of Gauss and Stirling, of even order) involve exactly the same limit and are equivalent without restriction.

24. Professor P. R. Rider: *On the distribution of the ratio of mean to standard deviation in small samples.*

The ratio of the mean of a sample (measured from the mean of the universe) to the standard deviation of the sample plays an important part in a number of statistical problems such as determining the probability that the mean of the sample does not deviate from the mean of the universe by more than a specified amount, comparing two mean values, and finding the sampling errors of regression coefficients. The distribution of this ratio for the case of a normal universe has been completely derived. The derivation, however, depends upon certain assumptions which do not hold for other than normal universes. The present paper derives the distribution of the ratio for certain discrete universes, with particular reference to a rectangular universe. Incidentally the distributions of other statistical parameters (mean, median, extreme average, least variate, range, and variance) for small samples are discussed.

25. Professor B. H. Camp: *Some corollaries of Bayes' theorem.*

If the incomplete Beta function is inserted as the unknown frequency function in Bayes' theorem, two results follow immediately. One is that, if m successes have occurred in n trials, the probability of a success on the

next trial is quite arbitrary, as it should be, dependent on the character of the universe from which the n trials were selected. This result compels attention to the indeterminateness of the theorem. What is frequently only a tacit assumption is now so introduced as to stand out baldly in the concluding formula. Another result is that, if, as sometimes happens, one actually does know something about the original universe from which the material under investigation was selected, this knowledge can be utilized. A further discussion is made to answer the question as to precisely what assumptions with regard to the universe the actuary and the physicist are tacitly making when they assume that the true values are near those found by experiment.

26. Professor Oystein Ore: *Remarks on the arithmetic theory of abelian fields.*

This paper contains an application of the author's arithmetic theory of Galois fields (Mathematische Annalen, vol. 100) to abelian fields. The complete system of prime-ideal types is determined, and a new proof is given for the theorem of Kronecker that all abelian fields are cyclotomic fields.

27. Dr. A. A. Albert (National Research Fellow): *The rank function of any simple algebra.*

The rank function of the general simple algebra A over F is considered. We define the central field H of A to be the field of all elements of A commutative with every element of A, and represent A as a normal algebra over H. Then the order of A over H is $q=t^2r^2$, and the rank function of A is the rank function of an associated total matric algebra A' over an extended field H', has degree tr, is irreducible in the function field $K=H(\xi_1, \xi_2, \cdots, \xi_q)$ of its coefficients, and has the symmetric group with respect to K. Then, if the order of H with respect to F is s, the order of A over F is st^2r^2, and the rank function of A in F has degree str and is irreducible in the function field of its coefficients.

28. Dr. A. A. Albert: *The structure of algebras which are the direct products of rational generalized quaternion division algebras.*

Attempts were made, before Cecioni constructed his normal division algebras, to construct new division algebras in sixteen units by considering algebras which were the direct products of any two rational generalized quaternion division algebras. Such attempts are here shown to have been futile, as such a direct product is never a division algebra. The resulting structure of these direct products is also considered and the theorems generalized to direct products of any number of rational generalized quaternion division algebras.

29. Mr. H. T. Engstrom: *The determination of the common index divisor of an algebraic field in some general cases.*

It is shown that certain types of prime-ideal decomposition of a rational prime p in an algebraic field of nth degree determine the power of p dividing the common index divisor of the field. An explicit formula for this power of p is found. In the general case, a maximum is established for the common index divisor in terms of the degree of the field.

30. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem and unbounded continuous curves.*

The author has previously analysed the relation of bounded continuous curves to the Janiszewski-Mullikin theorem. He now shows that a cyclicly connected unbounded continuous curve satisfying this theorem is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected point set. He constructs the inverse of the given curve from which it appears that the curve is a generalized cylinder (open at both ends); the curve is called a cylinder-tree, and is shown to be equivalent to a tree of elements which are its simple closed subcurves. The author discusses the case that the unbounded curve is not cyclicly connected.

31. Professor T. R. Hollcroft: *Linear systems of surfaces on a hypersurface in four dimensions.*

A linear system of surfaces on a given hypersurface in four dimensions is defined by the intersections of a linear system of hypersurfaces with the given hypersurface. The investigation includes a study of the properties of the general linear system, complete linear systems, pencils, nets and webs of a linear system, systems of surfaces on a given hypersurface cut out by its polar hypersurfaces. Linear systems of surfaces in ordinary space have been extensively studied. These systems appear as special cases of the above when the order of the given hypersurface is unity. Many of the theorems established for linear systems of three-space surfaces hold also for these more general linear systems.

32. Professor R. G. Putnam: *Note on Riesz's fourth condition for elements of accumulation.*

One of the conditions which elements of accumulation must satisfy in the abstract sets considered by F. Riesz is the following: An accumulation element A of a set E is determined by the subsets of E of which A is an accumulation element. In a Hausdorff space one of the conditions which neighborhoods must satisfy is: To every pair of distinct points x and y there correspond neighborhoods U_x and U_y which are disjoint. This note points out a relation between these two conditions.

33. Mr. Hassler Whitney: *A peculiar function.*

The function considered is continuous at points of an everywhere dense set, and discontinuous in the complementary set. At points of continuity

it has no differential coefficient, right- or left-handed, while in the complementary set the differential coefficient is definitely infinite.

34. Dr. T. C. Benton: *A definition of an unknotted simple closed curve.*

A simple closed curve may be defined as a set of points $f(t)$ where t is a parameter ranging over the interval $t_1 \leqq t \leqq t_2$ and $f(t) \neq f(t')$ unless (i) $t=t'$, or (ii) $t=t_1$, $t'=t_2$, or (iii) $t=t_2$, $t'=t_1$. If there is a function $f(t, \lambda)$ (which is uniformly continuous) such that for the range $t_1 \leqq t \leqq t_2$, $0 \leqq \lambda \leqq 1$, we have the following properties: (i) $f(t, 1)=f(t)$, where $f(t)$ is the function defining the curve above; (ii) $f(t, 0)=f_0$, a constant; (iii) $f(t', \lambda')=f(t'', \lambda'')$ if and only if (a) $\lambda'=\lambda''=0$, or (b) $\lambda'=\lambda''\neq 0$ and one of the following holds: (1) $t'=t''$, (2) $t'=t_1$, $t''=t_2$, (3) $t'=t_2$, $t''=t_1$; then the simple closed curve $f(t)$ is an unknotted curve. It is shown that curves which satisfy this definition can be imbedded in a plane, and that those which are imbedded in a plane must be unknotted according to this definition.

35. Dr. W. L. Ayres (National Research Fellow): *Continuous curves homeomorphic with the boundary of a plane domain.*

In this note it is shown that in order that a continuous curve M be homeomorphic with a plane continuous curve which is the boundary of one of its complementary domains it is necessary and sufficient that every maximal cyclic curve of M be a simple closed curve.

36. Professor E. T. Browne: *On the signature of a quadratic form.*

In this paper a study is made of the signature of a (real) quadratic form of rank r in which $A_r \neq 0$ but which is not a regular form since we suppose that two or more consecutive terms vanish in the sequence of A's from which the signature is determined. The method employed is the rearrangement scheme of Frobenius in conjunction with Gundelfinger's rule and furnishes a very simple means of arriving at the results previously obtained in a more complicated manner by Frobenius when two consecutive A's vanish, and giving a result not given by Frobenius when three consecutive A's vanish. It is also shown how the same scheme of rearrangement may be employed in determining the signature of any real recurring form, whether regular or not.

37. Dr. T. H. Gronwall: *The number of arithmetic operations involved in the solution of a system of n linear equations.*

It is shown that, denoting by A_n, M_n, D_n the number of additions (or subtractions), multiplications, and divisions required for solving the system $\sum a_{ik}x_k=b_i$, $(i, k=1, \cdots, n)$, by the method of substitutions, then $A_n=M_n=n(n-1)(2n+5)/6$, $D_n=n(n+1)/2$, for the most advantageous

arrangement of the method. The reduction in these numbers when the matrix (a_{ik}) is symmetric is determined, and other methods of solution are discussed.

38. Professor Edward Kasner: *Biharmonic functions and certain generalizations.*

In an earlier paper the author showed that if a function $F(x, y, x_1, y_1)$ is converted by *every* conformal substitution $x_1=\alpha(x, y)$, $y_1=\beta(x, y)$ into a harmonic function of x, y, then F must be biharmonic in the sense of Poincaré (real part of an analytic function $f(z, z_1)$ of two complex variables). The same result holds if we use merely the similitude group. If, however, we use smaller groups, for example the translation or rotation groups, larger classes of functions F are obtained, which we may then regard as generalizations of the biharmonic class. The most interesting generalization arises for the group of rigid motions. The substitutions converting an arbitrary function F into a harmonic function do not always form a group, and examples of this sort are also studied. Finally an analogous question for polygenic (non-analytic) functions of two complex variables z, z_1 is discussed: when can such a *polygenic* function, by means of a set of *analytic* substitutions $z_1=\omega(z)$, be converted into an *analytic* function of z?

39. Professor Norbert Wiener: *Group theory and harmonic analysis.*

The author shows that his methods of generalized harmonic analysis may be extended to analysis in other Eigenfunktionen than the trigonometric functions by substituting for the transformation group $f(x)\rightarrow f(x+t)$ the cyclical transformation group T^{λ}, where T^{λ} is asymptotically permutable with the transformation which makes $f(x)$ vanish when $|x|>A$.

40. Professor J. L. Walsh: *Boundary values of analytic functions and the Tchebycheff method of approximation.*

Let $f(z)$ be a function of the complex variable z analytic on the unit circle C: $|z|=1$, but with singularities interior to this circle. This paper considers approximation to the function $f(z)$ by polynomials $p_n(z)$, when the measure of the closeness of the approximation is $\epsilon_n=\max |f(z)-p_n(z)|$, z on C. It is of course true that ϵ_n cannot approach zero with $1/n$, but ϵ_n nevertheless has (for all polynomials $p_n(z)$ without restriction) a lower limit ϵ. In the simplest cases, for instance if $f(z)$ is a polynomial in $1/z$, there exists a function $p(z)$, analytic for $|z|\leqq 1$, such that $\max |f(z)-p(z)|=\epsilon$, z on C, and it can be shown by the use of certain results due to Carathéodory that $\epsilon_n\rightarrow\epsilon$ implies $p_n(z)\rightarrow p(z)$ uniformly for $|z|\leqq r<1$. If $p_n(z)$ is the Tchebycheff polynomial of degree n, for approximation to $f(z)$ on C, then we have $p_n(z)\rightarrow p(z)$ uniformly in a region in whose interior C lies. There are extensions to more general curves C and to more general functions $p_n(z)$ and $f(z)$.

41. Dr. Lucien Féraud: *On Birkhoff's Pfaffian systems.*

The Birkhoff Pfaffian systems of differential equations, arising from a variational principle $\delta\int_{t_0}^{t_1}[\sum_{i=1}^{2m}X_i dx_i/dt+Z]dt=0$, include those of Hamiltonian type, and are invariant in form under an arbitrary point transformation. In the present paper most of the classical properties of the Hamiltonian systems are extended to the Pfaffian systems; particular attention is paid to those properties concerning the reduction of the order of the system in case a certain number of integrals are known. A "last multiplier" of simple form is obtained. The transformation of a Lagrangian system into a Pfaffian and also the reduction of the Pfaffian form to the Hamiltonian are studied. The fundamental fact that this last reduction is possible, not only in a *formal* sense but in an *actual* sense, is established for the case of two degrees of freedom.

42. Mr. C. I. Lubin: *A note on singular points of the differential equation* $dx/[X(x,y)]=dy/[Y(x,y)]$.

The real functions $X(x, y)$, $Y(x, y)$ appearing in the differential equation $dx/X=dy/Y$ are supposed to be analytic in the neighborhood of the origin and to vanish there. The determinant made up of the coefficients of the linear terms of the power series expansions of $X(x, y)$ and $Y(x, y)$ is assumed to be different from zero and to have characteristic roots λ_1, λ_2 of opposite signs. This case has not been covered at all in the classical theory, while the recent treatment by Birkhoff by means of divergent series supposes there is no commensurability relation between λ_1 and λ_2. In this note it is proposed to obtain a normal form for the equation when $p\lambda_1+q\lambda_2=0$, where p and q are positive integers without common factor. The author proves that there exists a function $g(x, y)=y+\cdots$ involving one arbitrary constant, analytic for $x\neq 0$ and $y\neq 0$, continuous together with all its partial derivatives for $x=0$ or $y=0$, analytic in x for $y=0$ and analytic in y for $x=0$, which, for $z=x$, $w=g(x, y)$, transforms the equation into the form $dz/(\lambda_1 z)=dw/(\lambda_2 w-Az^{mp}w^{mq+1}-Bz^{2mp}w^{2mq+1})$, where A and B are constants (either or both of which may be zero) depending on the functions $X(x, y)$, $Y(x, y)$.

43. Dr. Caroline E. Seely: *Note on kernels of positive type.*

In an earlier paper (Annals of Mathematics, (2), vol. 20, p. 174) the author showed that if a kernel $K(s, t)$ is such that an infinite number of its iterated kernels are of positive type with respect to all linear combinations of its fundamental functions $\Phi_i(s)$, $\Psi_i(s)$, then $\int_a^b\Phi_i(s)\Phi_j(s)ds=0$, $\int_a^b\Psi_i(s)\Psi_j(s)ds=0$, $\lambda_i\neq\lambda_j$. In the present note it is shown by an example that this condition is not necessary in order that $K(s, t)$ itself be of positive type with respect to every function developable in a Fourier series.

44. Professor H. S. Vandiver: *On the first case of Fermat's last theorem.*

In this paper the author examines the equation $x^l+y^l+z^l=0$, where

x, y, and z are rational integers prime to the odd prime l and none zero. The special case $x \equiv y \pmod{l}$ is discussed, and among other results the following theorem is obtained: If the above equation is satisfied under the conditions mentioned, and $x \equiv y \pmod{l}$, then there exists no prime integer in the set $1+2l, 1+4l, \cdots, 1+(l-1)l$. This paper will appear in the Annals of Mathematics.

45. Professor H. S. Vandiver: *A theorem of Kummer's concerning the second factor of the class number of a cyclotomic field.*

This paper has appeared in full in the May-June number of this Bulletin.

46. Professor H. S. Vandiver: *An algorithm for transforming the Kummer criteria in connection with Fermat's last theorem.*

In this paper the author applies the theory of power characters in the field $k(\zeta)$, $\zeta = e^{2i\pi/l}$, l an odd prime, to the equation $x^l+y^l+z^l=0$, where x, y, and z are rational integers prime to l and none zero. By means of this method various combinations of the well known Kummer criteria for the solution of this equation are obtained, including new derivations of the congruences $2^{l-1} \equiv 3^{l-1} \equiv 1 \pmod{l^2}$. This paper will appear in the Annals of Mathematics.

ARNOLD DRESDEN,
Associate Secretary

THE MARCH MEETING IN CHICAGO

The two hundred sixty-ninth regular meeting of the Society was held at the University of Chicago on Friday and Saturday, March 29–30, 1929. About one hundred fifty persons attended the meeting, among whom were the following one hundred fourteen members of the Society, the largest number of members ever attending a meeting in Chicago:

R. W. Babcock, Bardell, Barnard, Barnett, Bartky, Beenken, Bibb, G. A. Bliss, Boyce, Brand, Brixey, O. E. Brown, L. H. Bunyan, Butler, Cairns, G. N. Carmichael, R. D. Carmichael, E. H. Carus, Chittenden, Coble, H. H. Conwell, Copp, H. V. Craig, Curtiss, Dancer, H. A. Davis, Dickson, P. D. Edwards, Eiesland, Escott, H. P. Evans, Everett, Feltges, Georges, Germond, Gilmore, J. W. Glover, Gouwens, L. M. Graves, Griffiths, V. G. Grove, Hartung, Hawkins, E. R. Hedrick, Holgate, Louis Ingold, Ingraham, Dunham Jackson, Jonah, B. W. Jones, Kern, Ketchum, Knight, E. P. Lane, Langer, LaPaz, Larsen, Latimer, Longley, McCoy, McNair, MacDuffee, MacMillan, March, T. E. Mason, Merriman, E. H. Moore, Max Morris, D. C. Morrow, E. J. Moulton, Musselman, Oakley, Olson, Palmer, Parkinson, Pepper, Powell, Price, Rainich, Rashevsky, R. G. D. Richardson, H. L. Rietz, Ross, Roth, Rowland, Sanger, Schottenfels, Sheets, Shohat, R. C. Shook, W. G. Simon, Skinner, Slaught, E. R. Smith, Sokolnikoff, Speeker, Stafford, Stouffer, M. E. Taylor, Teach, J. S. Turner, Van Vleck, Vass, Wahlin, Wall, L. E. Ward, Warren Weaver, Weiss, Wiggin, R. L. Wilder, Wiley, K. P. Williams, F. E. Wood, F. L. Wren.

On Friday afternoon Professor E. B. Van Vleck gave a symposium address on *The location of roots of polynomials and entire functions*. A rising vote of thanks was given the speaker.

On Friday evening members and their guests attended a dinner in the Del Prado Hotel. Professor Holgate was toastmaster. He called upon Professor R. G. D. Richardson who spoke of the aspirations and future plans of the Society. Professor G. A. Bliss then spoke in appreciation of the scientific work and influence of Professor Van Vleck, and the toastmaster read a letter from Professor Birkhoff on the same subject. Professor Holgate then called on Professor Van

Vleck, and afterwards on Professor E. H. Moore. The last speaker was President Hedrick, who spoke of Professor Van Vleck's work and its lessons for the future. The dinner was attended by ninety-four persons. There was a telegraphic exchange of greetings between the groups meeting in Chicago and in New York.

Sessions of the Council were held both before and after the dinner, the President and Secretary being in attendance.

It was announced that the following twenty-eight persons were elected to membership:

Professor Edwin Brown Allen, Rensselaer Polytechnic Institute;
Miss Jean C. Hempstead, Drexel Institute;
Mr. Everett H. Johnson, Lehigh University;
Mr. Paul Allen Knedler, University of Pennsylvania;
Mr. Clarence Anding Lovell, Drexel Institute;
Mrs. Elizabeth Elliott Nixon (Mrs. J. C.), University of Alabama;
Miss Irene Price, Indiana University;
Mr. John Albert Roulton, engineer, Rainey-Wood Coke Company, Swedeland, Pa.;
Mr. Ronald G. Smith, Kansas University;
Professor Guy Stevenson, University of Louisville;
Mr. Alexander Tartler, Drexel Institute;
Mr. Charles Henry Vehse, Brown University;
Mr. Ralph Oaks Virts, Indiana University;
Professor Horatio Burt Williams, Columbia University;

Nominees of the Mathematical Association of America:

Professor I. F. Neff, Drake University, Des Moines, Iowa;
Mr. J. C. Nixon, University of Alabama, Tuscaloosa, Ala.;
Mr. R. E. Peterson, Pennsylvania State College, State College, Pa.;

Nominees of the University of Pennsylvania:

Mr. V. W. Adkisson; Mr. Leonard Carlitz; Mr. Alexander G. Makarov;

Nominees of the Metropolitan Life Insurance Company:

Mr. George V. Brady; Mr. J. Christman; Mr. Reinhard A. Hohaus;

Nominees of Members of the Department of Mathematics, University of Wisconsin:

Mr. R. Erickson; Mr. H. H. Germond; Mr. E. Hawkins; Mr. H. D. Larsen; Mr. J. S. McNair.

In pursuance of the action taken at the Annual Meeting of 1928, the San Francisco Section has been abandoned. Professor B. A. Bernstein was appointed as Associate Secre-

tary and Professors M. W. Haskell (chairman), E. T. Bell, H. F. Blichfeldt, and R. M. Winger on the Committee on Program to work with him, in organizing the future meetings of the Society on the Pacific slope.

It was decided that the meetings of the Southwestern Section be stopped and that the following working plan govern the allocation of meetings in the Middle West:

1. In case the Western Meeting of the Society in connection with the Association for the Advancement of Science in December is held in one of the two regions southeast of Chicago or southwest of Chicago, a meeting of the Society on the Saturday after Thanksgiving of the same year shall be held in the other region.

2. In case the above mentioned December meeting of the Society is not held in either region during a given year, but the Summer Meeting of the Society that year is held in one of the two regions, a meeting of the Society shall be held that year on the Saturday after Thanksgiving in the other region.

3. In case neither the December Meeting nor the Summer Meeting is held in one of the two regions during a given year, the Council shall authorize a meeting to be held on the Saturday after Thanksgiving that year in either the region southeast or the region southwest of Chicago, somewhat in rotation.

There are now no sections remaining and all meetings held will be meetings of the Society; hereafter all meetings are to be numbered seriatim and subsidiary numbering abandoned.

The 1929 Thanksgiving meeting of the Western group is to be held at Ann Arbor. Invitations from President Vinson of Western Reserve University and President Howe of Case School of Applied Science to hold a meeting in Cleveland in connection with the 1930–31 meeting of the American Association for the Advancement of Science were accepted.

The following appointments were made: as a committee to nominate officers and members of the Council: Professors O. D. Kellogg (chairman), C. R. Adams, R. D. Carmichael, L. E. Dickson, and Tomlinson Fort; as a committee for the

meeting in Ann Arbor: Professors J. W. Glover (chairman), J. W. Bradshaw, T. H. Hildebrandt, M. H. Ingraham, and W. G. Simon; as a committee for the meeting in Cleveland: Dean T. M. Focke (chairman), Professors W. D. Cairns, W. B. Ford, M. H. Ingraham, and W. G. Simon; to represent the Society on the Council of the American Association for the Advancement of Science for 1929: Professors Louis Ingold and M. H. Ingraham; to represent the Society on the American Year Book: Professor Tomlinson Fort.

Mr. Charles A. Cole has made a gift to the Society, in memory of his father, which more than doubles the Frank Nelson Cole Prize Fund. In thanking him, the Council expressed its conviction that this generous action will prove an important factor in the furtherance of those high ideals of the Society toward the establishing of which the father so notably contributed.

A memorandum of the financial needs of the Society as prepared at the request of the General Education Board was presented to the Council.

Minor changes in the by-laws were recommended to the Society for action at the Summer Meeting.

In view of the tremendous demands for space in the Bulletin, the editors of the Bulletin and Transactions were requested to study the problem of journal publication for the Society. The appointment also was authorized of a committee to investigate the demand and the probable financial support for a journal of applied mathematics.

The papers whose abstracts appear below were read in three sections: (1) Analysis—Friday morning; (2) Geometry and Point Sets—Friday morning; (3) General—Saturday morning. President Hedrick, Professors Van Vleck and Bliss presided at the first; Professors Lane and Ingold at the second; President Hedrick and Professor Rietz at the third. Papers 1–14 were read in the first section; papers 15–23 in the second; and papers 24–39 in the third. Papers 10, 11, 13, 14, 23, 35–38 were read by title.

1. Dr. H. S. Wall: *On the Padé table of approximants associated with a positive definite power series.*

This paper studies the diagonal files in the Padé table for series in which the Stieltjes inequalities $A_n>0$, $B_n>0$, are replaced by the single condition $A_n>0$. The results are somewhat analogous to those obtained by the writer for the less general Stieltjes series (Transactions of this Society, vol. 31 (1929), pp. 91–116). Let S_0, S_1, S_2, $\cdots$ denote the principal diagonal file and the successive parallel files to the right of it respectively, in the Padé table. Then all the files (1) S_0, S_2, S_4, $\cdots$ converge to limits which are analytic over any region not containing a part of the real axis. The files S_1, S_3, $\cdots$ need not converge. There are just three cases. Case i, all the files (1) have a common limit; Case ii, the files S_0, S_2, $\cdots$, S_{2k} have a common limit while no two of the subsequent files have equal limits; Case iii, no two of the files (1) have equal limits. Similar results are obtained for the files below the principal diagonal. In cases ii, iii the limits are all meromorphic functions of $1/z$.

2. Dr. H. S. Wall: *On the expansion of an integral of Stieltjes.*

This paper studies the formal expansion in descending powers of z of the integral $\int_0^\infty [(1-\lambda)/(e^{(1-\lambda)u}-\lambda)]e^{-zu}du$, a continued fraction development of which was given by Stieltjes, *Recherches sur les fractions continues* (Oeuvres, vol. 2, p. 547). The $(n+1)$st coefficient in this expansion is a reciprocal polynomial of degree $n-1$ in λ: $c_n(\lambda)=\sum_{i=0}^{n-1}a_{n,i}\lambda^i$, having only real, negative, simple roots, the roots of c_n being separated by those of c_{n-1}. Since $c_n(1)=n!$, the coefficients constitute a partition of $n!$, the number of elements in the symmetric group $G_{n!}$. Let t take k into i_k. Let $K_{n,p}$ denote the set of substitutions t in which exactly p inequalities $i_{k+1}>i_k$ obtain. Then $K_{n,p}$ contains $a_{n,p}$ elements and $T^{-1}K_{n,p}T=K_{n,p}$, where $T=(1,\ n)(2,\ n-1)\ (3,\ n-2)\cdots$. The sets $K_{n,p}$ and $K_{n,n-p-1}$ contain the same number of conjugate pairs e_p, $\bar{e}_p=T^{-1}e_pT$ and e_{n-p-1}, $\bar{e}_{n-p-1}=T^{-1}e_{n-p-1}T$ respectively, and also equal numbers of elements g_p, g_{n-p-1}, respectively, self-conjugate with respect to T. The set of all elements g forms a group of order $k!\,2^k$ when $n=2k$ or $2k+1$. Certain multiplication laws hold among the four elements e_p, $Te_p=e_{n-p-1}$, $\bar{e}_p$, and $T\bar{e}_p=\bar{e}_{n-p-1}$ and also among the elements g_p, $Tg_p=g_{n-p-1}$ and g_p', $Tg_p'=g_{n-p-1}'$.

3. Professor J. A. Shohat: *On a certain formula of mechanical quadratures with non-equidistant ordinates.*

The author investigates the following formula of mechanical quadratures:

$$\text{(A)}\qquad \int_{-n}^{n} u_x dx = a(u_{-n}+u_n)+\sum_{i=1}^{n}a_i(u_{-n_i}+u_{n_i}),\qquad (u_x\equiv u(x));$$

$(0<n_i<n)$, which has been discussed for $s=2$, 3 by G. F. Hardy, A. King and J. Glover. The existence of (A) is established for any positive integer s,

also its convergence for $s \to \infty$; the properties of the $n_i(s)$ and $a_i(s,a)$, also their numerical values for $s=3$, 4 are given. This paper will appear in full in the Transactions of this Society.

4. Professor G. A. Bliss: *An integral inequality.*

This paper contains a proof for $n>m>1$ of the inequality

$$\int_0^\infty y^n dx/x^{n-r} \leqq \left(\int_0^\infty f^m dx\right)^{n/m} \qquad (r=n/m-1),$$

when the integral in the second member exists in the sense of Lebesgue and

$$y=\int_0^x f\,dx, \quad k=(1/(n-r-1))\,[r\Gamma(n/r)/\Gamma(1/r)\Gamma(n-1/r)\,]^r.$$

The functions f for which the equality holds are specified. The method used constitutes a very interesting application of the calculus of variations to a case which is somewhat irregular. The inequality was suggested by Professor G. H. Hardy, and the paper is in answer to a query from him as to whether a proof by calculus of variations methods is possible. No other proof has as yet been given so far as is known to the writer.

5. Dr. Lincoln La Paz: *The Hamiltonian function and an inverse problem of the calculus of variations.*

The problem solved is the following: Given a function $H(x, y_1, \cdots, y_n, u_1, \cdots, u_n)$ arbitrary except for suitable continuity properties, under what conditions is this function the Hamiltonian function of a problem of the calculus of variations in $(x, y_1, \cdots, y_n)$-space? Necessary and sufficient conditions are found to be that H be of class C''' in a region S of $(x, y_1, \cdots, y_n, u_1, \cdots, u_n)$ values and that the determinant $|\partial^2 H/\partial u_k \partial u_m|$ be different from zero in S. The method used leads to an explicit expression for the integrand function of the variation problem associated with an admissible function H. As a corollary we obtain the conditions under which a prescribed first-order partial differential equation of the form $F(x, y_1, \cdots, y_n, W_x, W_{y_1}, \cdots, W_{y_n})=0$ can be the Hamilton-Jacobi partial differential equation of a problem of the calculus of variations in $(x, y_1, \cdots, y_n)$-space. Paulus has devoted a paper (Mathematische Zeitschrift, vol. 25 (1926), pp. 348–361) to the solution of this last problem in the special case $n=1$. His argument is based on properties of the Hilbert invariant integral rather than of the Hamiltonian function.

6. Dr. P. W. Ketchum: *Solution of partial differential equations by hypervariables.*

Consider the differential operator $P(p_1, \cdots, p_n)$, where $p_i=\partial/\partial w_i$, and P is a homogeneous polynomial of degree r, with constant coefficients. The necessary and sufficient conditions that the real part of all analytic functions $F(w)$, of the hypervariable $w=\sum w_i e_i$, be a general solution of the equation $P\cdot f(w_1, \cdots, w_n)=0$ are that $P(e_1, \cdots, e_n)=0$, and that the real part of αw^i contain a sufficient number of arbitrary constants. The number of such constants depends on n and on r, and, if n is greater than 2, the num-

ber increases indefinitely with t. It is thus seen why it is necessary to use an infinite hypervariable for Laplace's equation. By application of the above theorem, hypervariables were found which furnish general solutions of the wave equation and of Airy's equation.

7. Professor L. E. Ward: *On the region of convergence of a certain type of series.*

The series considered is that consisting of the characteristic functions of the irregular boundary value problem $d^3u/dx^3+\rho^3u=0$, $u(\pi)=u(-\omega_2\pi)=u(-\omega_3\pi)$. The author has shown (Transactions of this Society, Oct., 1927, pp. 741–745) that if such a series converges uniformly in any segment of the interval $(0, \pi)$, it will converge uniformly in and on any equilateral triangle T centered at $x=0$ and having one vertex at an interior point of the segment of assumed uniform convergence; and also that any function analytic in a region containing $x=0$ as an interior point can be expanded in such a series uniformly convergent in and on a certain triangle T. By examining the magnitude of the coefficients of the formal expansion of a function, the author shows in the present paper that if the function is analytic in a region of which $x=0$ is an interior point and also satisfies certain continuity conditions, the triangle of uniform convergence can be taken as large as one likes except that it shall not contain within or upon itself either a singularity of the function or the point π.

8. Professor P. D. Edwards: *Functions possessing an addition theorem from the standpoint of a functional equation given by Abel.*

The usual procedure in the study of functions having addition theorems has been to define a function $\theta(x)$ and then to determine whether or not there exists a function f such that $\theta(x+y)=f(\theta(x), \theta(y))$. This paper inverts the problem and looks at it from the standpoint of the function f. A necessary and sufficient condition that f represent an addition theorem is that it satisfy a functional equation given by Abel. This functional equation is used to find explicit formulas for f in the case where f is rational and in certain irrational cases. The function θ corresponding to a given f is found by the aid of a theorem due to Weierstrass, and thus the addition theorems of most of the well known functions are obtained.

9. Professor Louis Brand: *On the transformation of surface integrals to circuit integrals.*

The parent theorem of all transformation of this class is

$$\int n \times F d\sigma = \int tF ds, \tag{1}$$

where F is a scalar, vector or tensor function of position and ∇F is either a surface or space gradient. When F is a tensor function, many other theorems follow from this by equating invariants of both members which are distributive with respect to addition. One such theorem is

$$\int \{\nabla F \cdot n^{(2)} - n \operatorname{div} F\} d\sigma = \int t \times F ds, \tag{2}$$

where the index 2 means that the operation $\cdot n$ applies to the second vector of the tensor preceding and div F denotes a surface divergence if ∇F is a surface gradient. Now if f is a second function of the same generality as F and we put $F = nf$ in (2), we obtain the integral transformation

$$\int (\nabla f + J n f)\, d\sigma = \int t \times n f\, ds, \tag{3}$$

J being the mean curvature. This transformation is coextensive with (1); this may be shown by putting $f = nF$ in (3) and equating cross invariants. From (3) in turn other useful theorems may be deduced by the process of equating distributive invariants. One of these theorems finds an important application in the problem of minimizing a surface integral limited by a fixed closed curve.

10. Mr. W. T. Reid: *A boundary-value problem for a countable system of ordinary linear differential equations.*

The system considered is (1) $y'(x, \mu) = A(x, \mu)y(x, \mu)$ where $y(x, \mu)$ is an absolutely continuous vector in Hilbert space on X: $0 \leqq x \leqq 1$ and for values of the parameter μ on L: $L_1 < \mu < L_2$ the matrix A is summable on X and limited by the summable function $\phi(x)$ uniformly for μ on L; furthermore, for x almost everywhere on X and μ on L, $A(x; \mu)$ is a *vollstetig* matrix, as defined by Hilbert. This system with the following boundary conditions (2) $[E+G(\mu)]y(0; \mu)+[E+H(\mu)]y(1; \mu) = 0$, where E is the unit matrix, G and H are "vollstetig" matrices and $E+G$ and $E+H$ have unique reciprocals for each value of μ, has many of the properties of a finite system. When it is incompatible we may define uniquely a Green's matrix. It is proved that if for $\mu = \mu'$, system (1), (2) is compatible there exists a reduced system (1'), (2) which is incompatible, where (1'), $y'(x) = [A(x; \mu') + P(x)]y(x)$ and P is a matrix of non-negative continuous functions which is limited on X by an arbitrarily assigned ϵ.

11. Mr. W. T. Reid: *Boundary conditions for a countable system of linear differential equations.*

Two infinite matrices M and N that are limited, as defined by Hilbert, are said to belong to class $D_1[D_2]$ if there exists an infinite matrix (M', N') $[\binom{M'}{N'}]$ composed of columns [rows] of M and N which possesses a unique reciprocal, and furthermore there exist an infinite number of columns [rows] of M and N which do not belong to (M', N') $[\binom{M'}{N'}]$. We consider the infinite system $y'(x) = A(x)y(x)$, where $y(x)$ is an absolutely continuous vector in Hilbert space and the matrix $A(x)$ is summable and limited by a summable function $\phi(x)$ on X: $0 \leqq x \leqq 1$, with boundary conditions $My(0) + Ny(1) = 0$, where the matrices M and N are of class D_1. It is shown that for the adjoint system $z'(x) = -z(x)A(x)$ it is possible to determine boundary conditions $z(0)P + z(1)Q = 0$, where P and Q are matrices of class D_2 and $MP - NQ = 0$.

Such a definition of adjoint boundary conditions reduces for a finite system to that given by Bounitzky.

12. Professor H. W. March: *The field of a magnetic dipole in the presence of a conducting sphere.*

A plane coil through which an alternating current is flowing may be regarded as an alternating magnetic dipole at distances from the coil which are great in comparison with its dimensions. In this paper the general problem of the field of such a dipole in the presence of a conducting sphere is solved. From the general formulas, approximate expressions are obtained for the field components for the case in which the radius of the sphere is small in comparison with the wave length.

13. Dr. J. M. Earl: *The degree of convergence of Hermite and Laguerre series.*

In a previous paper dealing primarily with a more general problem (Bulletin of this Society, vol. 34, p. 416), the writer has discussed incidentally the degree of convergence of the Hermite development of a given function, the treatment being based directly on the fact that the partial sum $S_n(x)$ of order n is a polynomial of the nth degree which makes the integral of $e^{-x^2}[f(x)-S_n(x)]^2$ from $-\infty$ to $+\infty$ a minimum. By use of an asymptotic expression for the Hermite polynomial of degree n, it is now shown that on any finite interval, $S_n(x)$ does not exceed a constant multiple of $M \log n$, where M is the least upper bound of $|f(x)|e^{-hx^2}$ for all real values of x, and h is any given number less than $1/2$. This leads to improved results with regard to the degree of convergence of the Hermite series. Corresponding treatment is applicable to the Laguerre development on the simply infinite interval.

14. Mr. E. L. Mickelson: *On the approximate representation of a function of two variables.*

The first four sections of this paper extend known theorems on finite trigonometric and polynomial sums approximating a function of one variable which possesses a modulus of continuity to the case of a function of two variables which is similarly restricted. The first two consist of theorems on the existence of trigonometric and polynomial approximating double sums, the third treats double sums which are trigonometric in one variable and polynomial in the other, and the fourth applies the foregoing to discussion of the degree of convergence of certain double series—Fourier, Legendre, and a combination of these. In general terms, the fifth section proves that if a function possesses rth order directional derivatives of a certain type which have a modulus of continuity, $\omega(d)$, on the surface of a unit sphere, the sum of the first $m+1$ terms of Laplace's series for the function approximates it to the order of $\omega(2\pi/m)m^{1/2}(1/m)^r$, for all finite, positive, integral values of r, including $r=0$.

15. Professor J. R. Musselman: *A classification of planar six points.*

The author has made a classification of six points in the plane according to equalities existing between their irrational invariants. These invariants were developed by Coble, Transactions of this Society, vol. 16 (1915). In the present paper the author has divided six points into a certain number of types and developed canonical forms for them.

16. Dr. H. A. Davis: *Non-involutorial transformations belonging to a non-special linear complex.*

A brief synthetic discussion of non-involutorial transformations belonging to a non-special linear complex has been given by D. Montesano. The present paper extends the synthetic discussion and investigates the problem analytically. The equations of the general transformation are obtained and discussed.

17. Dr. H. A. Davis: *Non-involutorial transformations belonging to a special linear complex.*

The equations of the general non-involutorial transformation which belong to a special linear complex are here obtained and discussed. The analytic work reveals an error in the synthetic discussion of the problem by M. Pieri (Circolo Matematico di Palermo, vol. 6). This error causes a considerable part of Pieri's paper to be incorrect, especially the F- and P-systems of the transformation. In the present paper the equations of the components of the latter are obtained and discussed. It is also shown that the general transformation can be built up from the T_{3-3} in a plane through the directrix of the special linear complex. The study of this plane T_{3-3} reveals some interesting facts concerning the F- and P-systems of the space transformation.

18. Professor E. P. Lane: *On the contact of a quartic surface with an analytic surface.*

Since there are $(k+1)(k+2)/2$ conditions for an algebraic surface to have contact of order k at a point of an analytic surface, and since an algebraic surface of order h depends on $h(h^2+6h+11)/6$ parameters, there is a limitation on the order of contact possible for a given algebraic surface. A quartic surface depends on 34 parameters, so that there is a six-parameter family of quartics having contact of the sixth order at a point of an analytic surface. It is the primary purpose of this paper to initiate an investigation of the contact of a general quartic surface with an analytic surface. The cases in which the quartics have contact of the fourth, fifth, and sixth orders receive particular attention. In each case the possibilities as to composite quartic surfaces are considered, and emphasis is laid on the curve of intersection of a non-composite quartic and the tangent plane at the point of contact.

19. Professor May M. Beenken: *Surfaces in five-dimensional space.*

The method of Wilczynski is extended here to a systematic study of the projective differential geometry of surfaces in five-dimensional space S_5. A general surface in S_5 is determined, except for a projective transformation by a completely integrable system of four linear homogeneous partial differential equations of the third order. A local pyramid of reference is set up, and equations of linear osculants and of other linear spaces related to the surface are obtained in the local coordinate system. Quasi-asymptotic and principal curves, invented by the Italians as analogs of the asymptotic and conjugate curves on surfaces in ordinary space, are treated by a new method. Special surfaces in S_5, which sustain a conjugate net or one family of asymptotic curves, are investigated by means of other completely integrable systems of differential equations.

20. Professor John Eiesland: *On a class of ruled* $(n-1)$*-spreads in* S_n*, the* V_4^4 *in* S_5.

The locus of all the ∞^{n-2} lines that are incident with n given generic $(n-2)$-flats in S_n is a ruled $(n-1)$-spread. This spread has been studied by Segre for the case $n=4$. In a recent article in the Bulletin of this Society, vol. 34 (1928), B. C. Wong has constructed the generalized spread V_{n-1}^{n-1} synthetically, but owing to the limitation of this method in the case of n-dimensional geometry, only a few of the more obvious properties of the spread have been obtained. In the present paper a strictly analytic method has been adopted. The equation of the general V_{n-1}^{n-1} in S_n has been obtained and a more detailed discussion of the geometric properties of the V_4^4 in S_5 is given. The introduction of the Grassmannian coordinates of a 3-flat in S_5 is a special feature of this part of the work.

21. Professor R. L. Wilder: *On the imbedding of subsets of a metric space in Jordan continua.*

Let R be a metric space, and M a subset of R. Several conditions which are both necessary and sufficient in order that M should be a subset of a Jordan continuum ("continuous curve") J of R are obtained. Two of these follow: (1) If M is compact and closed, it is necessary and sufficient that M should be arcwise connected through R as well as arcwise connected im kleinen through R. In this case, J can be so chosen that $J-M$ is the sum of a denumerable set of non-overlapping open arcs $\{L_i\}$ such that $\lim_{i\to\infty} d(L_i)=0$, and such that if M has positive (Menger-Urysohn) dimension, J is of the same dimension. (2) If M is compact, it is necessary and sufficient that there should exist in R a denumerable set of points $\{P_i\}$ which is dense on M, and such that the set of points $\{P_i\}$ is arcwise connected through R as well as uniformly arcwise connected im kleinen through R. In this case, if the dimension of M is m, and the dimension of the outer shell of M is s, then J may be so chosen that its dimension is the greater

(or common) value of m and s, provided, of course, that either m or s is positive.

22. Professor R. L. Wilder: *On the relation of certain sets to their complements in euclidean spaces of three and higher dimensions.*

This paper is an attempt to establish some fundamental relations between certain basic sets and their complements in euclidean spaces of three and higher dimensions. In particular, a result of R. L. Moore (Proceedings of the National Academy, vol. 4 (1918), p. 369) to the effect that each of the domains complementary to a simple closed curve in the plane is uniformly connected im kleinen is extended to higher spaces as follows: In E_n (euclidean space of n dimensions) each of the domains complementary to a set M which is homeomorphic with an $(n-1)$-sphere, or, indeed, an $(n-1)$-manifold, is uniformly connected im kleinen. Furthermore, every point of M is regularly accessible by arcs from each of the complementary domains of M. A generalization of the theorem that every point of an arc in E_3 is accessible from its complement (see letter from C. Kuratowski to R. L. Moore, this Bulletin, vol. 31 (1925), p. 32) is obtained as follows: Every point of an i-cell $(0 \leqq i \leqq n-1)$ immersed in E_n is accessible by arcs from the complement of the cell.

23. Professor Arnold Emch: *On the geometry of symmetric functions.*

The geometric properties of symmetric substitution groups were discussed in a paper which appeared in the American Journal of Mathematics, vol. 45 (1923). In the present paper the author continues the study of symmetric functions from a geometric standpoint and investigates the properties of a number of algebraic curves which are invariant under finite collineation-groups, and also Cremona groups. The 8 cycles of order 3 on an elliptic cubic, for example, give rise to 4 elliptic class-sextics of order 12, which bear interesting relations to the cubic. Other examples for groups G_{12}, G_{24}, G_{48}, G_{108} are given.

24. Mr. O. E. Brown: *The equivalence of triples of bilinear forms.*

In the sense of Weierstrass, two triples of bilinear forms are equivalent if and only if their matrices are equivalent. We restrict the triples to those whose leading matrix is non-singular and which, thus, may be taken as the identity. Let the pairs (I, b) and (I, c) have the classical canonical forms (I, λ) and (I, μ). The triple (I, b, c) is equivalent to two classes of triples (I, λ, c_1) and (I, b_1, μ) called the first and second canonical neighborhoods. A transformation reducing a matrix of the first canonical neighborhood to the second is called a transposition. The set of all transpositions is the same for all triples equivalent to a given one.

25. Professor M. H. Ingraham: *An application of a theorem on matrices to the theory of equations.*

This paper calls attention to the use of the fact that if r_i, $i=1, \cdots, n$, are the roots of the characteristic equation of a matrix m, the roots of $|g(m, x)|=0$ can be divided into sets s_{ik} associated with the r_i in such a way that $g(r_i, s_{ik})=0(i, k)$, in finding an equation whose roots are connected by a polynomial equation with the roots of a given equation. All functions considered are polynomials. This is a generalization of results of A. Chatelet (Annales de l'Ecole Normale, (3), vol. 28 (1911), p. 124).

26. Professor L. E. Dickson: *All integral solutions of certain diophantine equations.*

Let $f(x, y)$ be a binary quadratic form whose discriminant d is not a square. Given one rational solution of $f=ez^2$, we can express all rational solutions as the products of ρ by certain quadratic functions X, Y, Z of parameters u and v, which are relatively prime integers. Express ρ as an irreducible fraction N/D. Then our solution is integral only when D divides X, Y, Z. It is proved that D then divides de. Hence all integral solutions are given by as many formulas (each involving only integral parameters) as there are positive divisors of de. For certain values of m and n all integral solutions are obtained of the problem to find two sets of n integers having the same sum, same sum of squares, $\cdots$, and same sum of mth powers.

27. Professor L. E. Dickson: *Binary quadratic forms with a single class in each genus.*

Let the form have a negative discriminant $-\Delta$. For Δ odd, $\Delta=4n-1$. For $ac=n$, $ax^2 \pm xy+cy^2$ are of disciminant $-\Delta$ and are both reduced for certain a, c, unless n is a prime or square of a prime. By using a form with the term $\pm 3xy$, when n is a prime, we conclude that $n+2$ must be a prime, a square of a prime, a triple of a prime, or 27 or 81. By means of such results we readily obtain the few answers as far as 23,000. The method is not quite so simple when Δ is even.

28. Professor C. C. MacDuffee: *On the independence of the first and second matrices of an algebra.*

This paper has appeared in full in the May-June issue of this Bulletin.

29. Professor G. A. Bliss: *Expansions for implicit functions near a very general type of singularity.*

In his Colloquium Lectures of 1909 the writer studied for real variables the inverse of a transformation of the form (1) $x=u+\phi_2+\cdots$, $y=\psi_2+\cdots$, where ϕ_2, ψ_2 are homogeneous of the second degree in u, v and the dots indicate terms of higher orders. The functional determinant of the second members of these equations with repect to u, v will be repre-

sented by $D(u, v)$. In a recent letter Professor J. L. Synge of the University of Dublin writes that W. R. Hamilton in his first published paper was led in 1828 to this same transformation in his effort to compare the intensity of light near a caustic surface in an optical field with the intensities in other portions of the field. By a lengthy and complicated argument, Hamilton finds that the solutions u, v of equations (1) can be expressed as power series in x, $[y-\phi(x)]^{1/2}$, where $y=\phi(x)$ is the locus of the points in the xy-plane corresponding to the zeros of $D(u, v)$. In this paper the writer shows that similar ones can be deduced for singular points of implicit functions defined by the much more general equations of the form $f_r(x_1, \cdots, x_n; y_1, \cdots, y_n) = 0$, $(r=1, \cdots, n)$. The singular points near which such expansions hold are of a very general type since their definition imposes only one inequality on the coefficients of the expansions of the functions f_r besides the vanishing of the functional determinant $|\partial f_r/\partial y_s|$. Analogous results hold in the domain of real variables.

30. Professor R. E. Langer: *The asymptotic location of the roots of a certain transcendental equation.*

The asymptotic location of the roots of the transcendental equation of the type $\sum_{i=1}^{n}\omega_i(\lambda)e^{\lambda B_i}=0$, in which the B_i are complex constants, has been studied by a number of authors under varying hypotheses on the coefficient functions $\omega_i(\lambda)$. In the most general case thus far treated—by Pólya and Schwengeler—these functions are taken to be polynomials. Wilder and Tamarkin had previously considered the case in which $\omega_i(\lambda)=\lambda^{r_1}\{a_i+E_i(\lambda)/\lambda\}$, the a_i being constants, the $E_i(\lambda)$ bounded, and the r_i being either all zero or else integers subject to special relations except when $n=2$. In this paper the equation is discussed for the general case $\omega_i(\lambda)=\lambda^{k_i}\{a_i+\epsilon_i(\lambda)\}$, where the k_i are any real constants and $\epsilon_i(\lambda)\to 0$, as $|\lambda|\to\infty$. The discussion is based on the papers of Tamarkin and Wilder.

31. Professor G. Y. Rainich: *A new proof of the theorem that the curvature field determines the electromagnetic field.*

Several years ago this theorem was proved by the author, using a vector method, for all cases with exception of the singular case where the electric and magnetic vectors are equal and at right angles (Transactions of this Society, vol. 27, p. 106). In the present paper the proof is given in coordinates and the theorem is shown to be true also in the case which was left out before.

32. Dr. H. H. Germond: *Solution of a non-linear equation of diffusion under specified boundary conditions.*

The problem of the moisture distribution in an initially soaked block of wood drying from one face involves the solution of a non-linear equation of diffusion. The substitution $z=b+x/(\alpha t)^{1/2}$ reduces the partial differential equation to an ordinary differential equation. A solution satisfying the boundary conditions is obtained as a power series in half-powers of z.

A simple approximate expression is obtained for the moisture distribution. The rate of moisture loss is deduced, and is compared with experimentally determined values.

33. Mr. N. H. McCoy: *Some commutation formulas in the algebra of quantum mechanics.*

For one pair of conjugate variables the special character of this algebra is due to the assumption $pq-qp=h/(2\pi i)=cI$, where q and p are matrices representing the coordinate and momentum respectively and I is the unit matrix. It has been shown by the author (Proceedings of the National Academy, March, 1929) that if f and ϕ are arbitrary polynomials in p and q,

$$f\phi - \phi f = \sum_{s=1}^{\infty}(c^s/s!)\left[\partial^s\phi/\partial q^s \cdot \partial^s f/\partial p^s - \partial^s f/\partial q^s \cdot \partial^s\phi/\partial p^s\right].$$

This formula is extended to polynomials in n pairs of variables. An application of the formula for one pair of variables to the problem of expanding functions such as $q^m p^{m'} q^n p^{n'}$, isobaric in p and q, in the form $\sum a_i p^i q^i$ gives rise to some interesting identities among binomial coefficients.

34. Mr. N. H. McCoy: *On vectors whose components are quantum variables.*

This paper deals with the rules of combination of vectors whose components are non-commutative quantum variables. Some analogs of the formula $pf-fp=c\partial f/\partial q$ are obtained for scalar and vector products. General commutation formulas for scalar and vector products of any two vectors are obtained by means of the results of the preceding paper.

35. Professor T. A. Pierce: *Symmetric functions of n-ic residues* (mod p).

This paper will appear in full in an early issue of this Bulletin.

36. Dr. T. W. Moore: *On the resultant of two binary forms.*

This paper contains a detailed discussion of the application of the method of elimination proposed by Morley and Coble (American Journal, vol. 49 (1927), pp. 463–488) to the case of two binary forms. The forms which the resultant determinant can take in the different cases are explained and a comparison drawn with the known methods of elimination. A non-symbolic method for writing down the resultant determinant by use of a multiple summation is furnished.

37. Professor E. G. Keller: *Periodic orbits about an anchor ring.*

This paper proves the existence of and constructs periodic orbits of the following types about a circular anchor ring, the radius of whose aperture and meridian sections are respectively $R-\epsilon$ and ϵ. Only orbits re-entrant

after one revolution are considered. Let (r, θ, z) and (ρ, θ, ϕ) be the cylindrical and polar coordinates of the particle. Orbits exist in the meridian plane of the form $r = \rho \cos \phi$, $z = \rho \sin \phi$, where

$$\rho = \sum_{i=0}^{\infty} \rho_i(t)\epsilon^i, \qquad \phi = \sum_{i=0}^{\infty} \phi_i(t)\epsilon^i$$

and where $\rho_i(t)$ and $\phi_i(t)$ are sums of sines and cosines of only even multiples of the independent variable t. These orbits are symmetrical with respect to both the equatorial plane and the axis of the ring. Orbits in the equatorial plane are expressible as power series in ϵ with trigonometric coefficients in t. Orbits inclined to the equatorial plane are of the form

$$r = \sum_{i=0}^{\infty} r_i(t)\epsilon^i; \qquad z = \sum_{i=1}^{\infty} z_i(t)\epsilon^i,$$

where $r_i(t)$ and $z_i(t)$ are either trigonometric or elliptic functions of t. The differential equations of motion of the particle admit one integral of areas and the vis viva integral, both of which are expressed in a finite number of terms.

38. Professor W. E. Milne: *On the degree of convergence of the Gram-Charlier Series.*

The author studies the series $\phi(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + \cdots$, where $\phi_0(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and where ϕ_n is the nth derivative of ϕ_0. It is shown that if $\phi(x)$ has a continuous kth derivative of bounded variation and if $x^m e^{x^2/4}\phi^{(k+1-m)}(x)$ $(m = 1, 2, \cdots, k+1)$ are of bounded variation in the interval $(-\infty, \infty)$, then the sum of the first n terms of the series will differ from $\phi(x)$ by a quantity which is $O(n^{-k/2})$. A similar theorem, but with lighter restrictions at infinity, is obtained for the series of orthogonal functions $f(x) = c_0w_0(x) + c_1w_1(x) + c_2w_2(x) + \cdots$ in which $w_n = e^{x^2/4}\phi_n$.

39. Professor C. G. Latimer: *On identities in theta-functions.*

In a former paper, the writer determined the sets of integral elements in the generalized quaternion algebras when both parameters are odd, the definition of *integral element* being due to Dickson. In this paper it is shown that Jacobi's fundamental identity in θ-functions may be proved by employing Hurwitz' integral quaternions, and the same method is applied to the above mentioned quaternion algebras to obtain new identities in θ-functions.

M. H. Ingraham,
Associate Secretary

THE USE OF CONTINUED FRACTIONS IN THE DESIGN OF ELECTRICAL NETWORKS*

BY T. C. FRY

1. *Introduction.* The literature on the subject of electrical circuit theory deals very largely with one type of problem: "Given an electrical system and the driving force which is imposed upon it, to find how it responds."

The converse problem, namely "Given a known driving force and the response which it is desired to produce, to find a system which will so respond," is much less frequently dealt with in the literature. It is this problem with which the present paper is to deal.†

About the electrical side of the problem we need only say that, whatever the language in which the engineer may phrase his requirements, they can always be satisfied provided we can produce one or more networks of preassigned *impedances*. It is, then, no serious restriction upon the prac-

* An address presented at the request of the program committee at the meeting of this Society at Cincinnati, December 1, 1928.

† Those who are interested in following the earlier literature will find the following references helpful. They will also give some idea of the technical problems to which the theory is applicable.

Fry, U. S. Patent #1,570,215, 1926, filed June 1921.

Zobel, *Theory and design of uniform and composite electric wave filters*, Bell System Technical Journal, vol. 2 (1923), p. 1.

Bartlett, *A note on the theory of artificial telephone and transmission lines*, Philosophical Magazine, vol. 48 (1924), p. 859.

Bartlett, *Properties of the generalized artificial line*, Philosophical Magazine, vol. 1 (1926), p. 553.

Cauer, *Die Verwirklichung von Wechselstromwiderständen vorgeschriebener Frequenzabhängigkeit*, Archiv für Elektrotechnik, vol. 17 (1926), pp. 355–388.

Bartlett, British Patent #290,701, 1928, filed January 1927.

Mead, *Phase distortion and phase distortion correction*, Bell System Technical Journal, vol. 7 (1928), p. 195.

Zobel, *Distortion correction in electrical networks with constant resistance recurrent networks*, Bell System Technical Journal, vol. 7 (1928), p. 438.

ticability of our theory from a technical standpoint to confine ourselves entirely to the design of networks with given impedance properties.

Our point of departure is a very simple observation regarding the ladder-type network shown in Fig. 1. The ordinary laws of circuit theory lead immediately to the formula*

$$F(p) = J_1 + \cfrac{1}{Y_2 + \cfrac{1}{J_3 + \cfrac{1}{Y_4 + \cdots}}} \tag{1}$$

If, then, the various J's and Y's are known, we can write down $F(p)$ at once. For example, if all the series elements

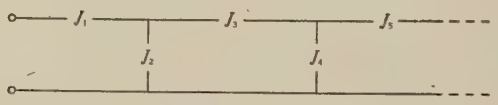

FIG. 1

are resistances $r_1, r_3, \cdots$, and all the shunt elements capacities $c_2, c_4, \cdots$, (1) will have the form of the Stieltjes fraction

$$F(p) = r_1 + \cfrac{1}{c_2 p + \cfrac{1}{r_3 + \cdots}} \tag{2}$$

Conversely, if we are given the problem of designing a network the impedance of which is to be some known function of frequency, and if we can expand this function $F(p)$ in a continued fraction of the Stieltjes type, we can certainly furnish the required network; for we need only cause the successive resistances in our ladder structure to be equal to the successive constant terms of the fraction, and the successive capacities to the coefficients of the corresponding terms in p.

* The admittances Y_i are reciprocals of the corresponding impedances J_i.

We can, then, apply the entire theory of Stieltjes fractions directly to the problem of network design. But there are many physical networks—even many networks of the ladder type—whose impedances cannot be expanded in the form (2), though they can be expressed in the form (1) if the J's and Y's are not restricted to be constants and multiples of p. For example, the simple function†

$$F^*(p) = \frac{2p + 3}{p + 2} \tag{3}$$

cannot be represented by a Stieltjes fraction, though it can be written in the form

$$F^*(p) = 1 + \cfrac{1}{1 + \cfrac{1}{p + 1}},$$

which is quite suitable for purposes of design. The Stieltjes fraction (2), in other words, is too special for our needs.

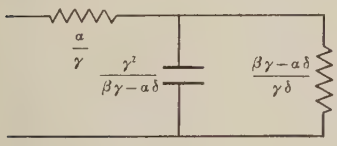

FIG. 2

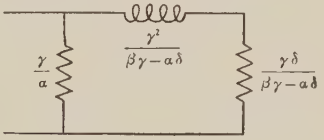

FIG. 3

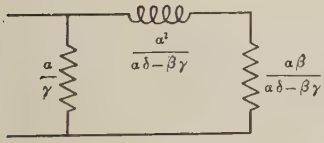

FIG. 4

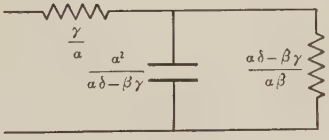

FIG. 5

In the present paper we shall obtain the necessary degree of generality by allowing the J's and Y's of (1) to be of the bilinear form

$$\frac{\alpha p + \beta}{\gamma p + \delta},$$

† The reason for the asterisk will appear later.

α, β, γ and δ being *positive* constants. Such a bilinear form can always be assigned a definite design significance. For if $\beta\gamma > \alpha\delta$, it can be interpreted as the impedance of the structure shown in Fig. 2 and the admittance of that shown in Fig. 3; while if $\beta\gamma < \alpha\delta$, it can be interpreted as the impedance of Fig. 4, or the admittance of Fig. 5.

2. *Resumé of Theorems about Stieltjes Fractions.* Obviously, since the Stieltjes fraction is so closely related to our problem, we must have the essential facts concerning it before us. They are as follows.

THEOREM 1. *A Stieltjes fraction*

$$f(z) = \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cdots}}} \tag{4}$$

*converges for every value of the complex variable z, except perhaps certain negative real values, provided the a's are positive reals, and provided $\sum a_j$ diverges.**

THEOREM 2. *The function $f(z)$ defined by a convergent Stieltjes fraction can always be expressed in the form*

$$f(z) = \int_0^\infty \frac{d\Phi(x)}{z + x}, \tag{5}$$

where $\Phi(x)$ is a monotonic non-decreasing real function of the real variable x, where $\Phi(0) = 0$ and $\Phi(\infty) = 1/a_1$. It is not necessary that $\Phi(x)$ be continuous.†

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* Stieltjes, Œuvres Complètes, vol. 2, p. 465. If Σa_j converges, the convergents of even and odd orders separately approach limits $f_1(z)$ and $f_2(z)$, but the limits are not equal. As we have no interest in this case we shall give no further attention to it.

The case of complex a's was afterward discussed by Van Vleck, Transactions of this Society, vol. 2 (1901), pp. 262, 344. Such coefficients have no meaning in the problem of network design, and will therefore not be considered here.

† Stieltjes, loc. cit., pp. 491, 493.

THEOREM 3. *The function $f(z)$ defined by a convergent Stieltjes fraction is therefore a regular analytic function, except at certain points of the negative real axis,* and is real for positive real values of z. The same is true of its reciprocal.†*

THEOREM 4. *The function $\Phi(x)$ in* (5) *is related to $f(z)$ by the law*

$$\Phi(x) = \frac{1}{2\pi i}\int f(z)dz, \tag{6}$$

the path of integration extending along a circle of radius x about the origin, beginning at $-x-i0$ and ending at $-x+i0$.‡

THEOREM 5. *Conversely, if $\Phi(x)$ is any monotonic non-decreasing real function of x in the "finite" interval $(0, b)$, and constant for $x \geqq b$, the function $f(z)$ defined by* (5) *possesses a convergent Stieltjes expansion with positive real coefficients.§*

THEOREM 6. *If $\Phi(x)$ is any monotonic non-decreasing function of x, such that the integrals*

$$\int_0^\infty x^n d\Phi(x) \tag{7}$$

all exist, the function $f(z)$ defined by (5) *possesses a "formal" Stieltjes expansion with positive real coefficients, which, if it converges at all, converges to the value $f(z)$ for all values of z except those on the negative real axis.‖*

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* Perron, *Die Lehre von den Kettenbrüchen*, p. 369.

† This is readily seen as follows: If (4) converges to $f(z)$, the fraction

$$\cfrac{1}{a_2 x + \cfrac{1}{a_3 + \cdots}}$$

must converge to $f_1(z) = 1/[zf(z)] - a_1$; for the convergents of this latter fraction are the corresponding convergents of the former one, divided by z and reduced by a_1. The new fraction, however, is a Stieltjes fraction, and therefore $f_1(z)$, and hence also $1/f(z)$, is regular everywhere except on the negative real z-axis.

‡ This is a slight modification of the theorem as given by Perron, loc. cit., p. 372.

§ Perron, loc. cit., p. 388.

‖ Stieltjes, loc. cit., p. 504.

Thus, if we are given a function $f^*(z)$ we can determine whether or not it possesses a Stieltjes expansion by the following systematic procedure:

(a) First we observe whether it is real for positive real values of z. If not, no usable expansion is possible.

(b) Next we investigate the singularities of $f^*(z)$ and its reciprocal. Unless they are all on the negative real axis no expansion is possible.

(c) If they are all on the negative real axis we compute a function $\Phi(x)$ from the formula

$$\Phi(x) = \frac{1}{2\pi i}\int f^*(z)dz, \tag{8}$$

the path of integration being that defined in Theorem 4. Unless it is monotonic non-decreasing, no expansion is possible. Even if it is monotonic non-decreasing no expansion is possible unless either $\Phi(x)$ is a finite constant for $x>b$, or else the integrals (7) all exist.

(d) If the possibility of expansion is still not disproved, we substitute our $\Phi(x)$ in (5) and evaluate the integral. The $f(z)$ which results may or may not be the function $f^*(z)$ with which we started. If it is not, $f^*(z)$ can neither be expressed in the form (5), nor expanded in a Stieltjes fraction. But if $f(z)$ is identical with $f^*(z)$, we know either that a convergent expansion *does* exist, or that it *may* exist, according as we are working under the conditions of Theorems 5 or 6.†

We may, then, fail to get a Stieltjes expansion for either of four reasons:

(a) Because $f^*(z)$ is not real when z is real and positive;

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† We say nothing about the processes by means of which the coefficients are obtained after we have shown that an expansion is possible, for the reason that these processes play no part in our study. They are tedious, but otherwise quite satisfactory. They will be found in almost any discussion of continued fractions, the most satisfactory references being Stieltjes, Œuvres Complètes, vol. 2, p. 184, for the more familiar method, and Thiele, *Interpolationsrechnung*, p. 138, for an interesting alternative.

(b) Because the singularities of $f^*(z)$ or its reciprocal are not properly located;

(c) Because $\Phi(x)$ is not of an allowable type;

(d) Because, though $f^*(z)$ gives rise to a $\Phi(x)$ of suitable type, it is still not true that

$$f(z) = \int \frac{d\Phi(x)}{z + x}$$

is equal to $f^*(z)$.

We shall consider these difficulties in order, seeking in each case to find a means of deriving a type of fraction which can be interpreted electrically.

We shall occasionally find it desirable to have separate names for the functions $f^*(z)$ and $f(z)$. The former of these—that is, the function which we wish to represent in fractional form, and from which we obtain $\Phi(x)$ by the use of (6)—we shall call the *generating function*. The other—that is, the function to which the fraction converges if we succeed in getting a convergent fraction at all, and which is obtained from $\Phi(x)$ by the use of (5)—will be said to *correspond* to $\Phi(x)$. The important distinction is, that while many functions may generate the same $\Phi(x)$, only the one which corresponds to it is capable of Stieltjes representation.

3. *Redistribution of Singularities by Change of Variable.* The first difficulty is inherent in the physical problem. For if a physical system had an impedance $Z(p)$ which was complex for a real positive $p = p_0$, a *real* driving force which varied with time according to the law $e^{p_0 t}$ would produce a *complex* current. This is absurd, since a complex current has no physical meaning.

Turning next to the second difficulty, we first note that physically possible systems cannot have singularities in the right-hand half of the plane. This is easily seen in the case of poles, for a pole would imply that once an oscillation of the proper frequency were established it would persist forever *with constantly increasing amplitude* even in the absence of any sustaining force, which is obviously absurd.

It is not so easy to exclude the possibility of essential singularities, but they are never met.

On the other hand, singularities in the left-hand half of the plane—or as a limiting case on the imaginary axis—are physically possible. In some simple cases they can be shifted to the negative real axis by a suitable change in variable, and an expansion thus obtained.

Thus, suppose we are given a function $F^*(p)$ which, together with its reciprocal, has only pure imaginary singular points. Then by the transformation

$$z = p^2, \tag{9}$$

we will arrive at a function $f^*(z) \equiv F^*(z^{1/2})$, whose singularities occur at negative real values of z. From this function we may be able to obtain a Stieltjes fraction of the form (4), and if so the equivalent fraction in p,

$$F(p) \equiv f(p^2) = \cfrac{1}{a_1 p^2 + \cfrac{1}{a_2 + \cdots}},$$

will converge for all values of p except possibly certain pure imaginaries.

Even though this $F(p)$ proved to be identical with $F^*(p)$, however, it would not be suitable for the purposes of circuit design; for we have no type of circuit element whose impedance or admittance varies as the square of the frequency. However, if we were to set $f^*(z) \equiv F^*(z^{1/2})/z^{1/2}$, we might arrive at a usable expansion.† For suppose we succeeded in finding a convergent fraction (4) the limit of which was identical with this $f^*(z)$. We would then have at once

$$F^*(p) = \cfrac{1}{a_1 p + \cfrac{1}{a_2 p + \cfrac{1}{a_3 p + \cdots}}}.$$

† We have probably introduced a new singularity at $z = 0$ by the introduction of the new factor, but as this is an end point of the negative real axis, it is not objectionable.

This form of development is altogether satisfactory. It corresponds to a ladder network all the series elements of which are inductances and all the shunt elements condensers.

Similarly, if the singularities of $F^*(p)$ and its reciprocal all lie on a line parallel to the imaginary axis and α units to the left of it, we may transfer them all to the negative real axis by either of the transformations†

$$z = (p + \alpha)^2, \tag{10}$$

or

$$z = p^2 + 2\alpha p. \tag{11}$$

It is easily seen that either leads to a usable type of fraction provided we identify $f^*(z)$, not with $F^*(p)$, but rather with $F^*(p)/(p+\beta)$, where $0 \leqq \beta \leqq 2\alpha$.‡

4. *Fractions which Represent Functions in Island Regions.* The function $f^*(z)$ obtained from $F^*(p)$ by such transformations as (9), (10) and (11) is ordinarily multiple-valued.

On the other hand, the function defined by any fraction of the form (4) must be single-valued. Hence if we succeed in obtaining such a fraction, it will represent one branch of $f^*(z)$ only. Consequently, when rewritten in terms of p, it will represent $F^*(p)$ over only that portion of the p-plane which corresponds to this branch of $f^*(z)$. In the remainder of the p-plane it will converge (except, perhaps, at certain points of the boundary which separates the two regions), but usually not to the value $F^*(p)$.

The fraction, in other words, will represent $F^*(p)$ only in an "island region" (espace lacunaire). For example, by using the transformation $z = p^2 + p$ we can expand the function (3) into the form

† The second of these transformations has a wider field of use than the first, for it carries to the negative real axis of z not only the p's on the vertical line mentioned, but p's in the *real* range $(-2\alpha, 0)$ as well. Singularities within this real interval are therefore also allowable.

‡ Since $p = -\beta$ corresponds to a negative real z, the new singularity at this point is not objectionable.

$$(12)\qquad F(p) = 1 + \cfrac{1}{\cfrac{2}{p+1} + \cfrac{1}{\cfrac{1}{p} + \cfrac{1}{\cfrac{1}{p+1} + \cdots}}},$$

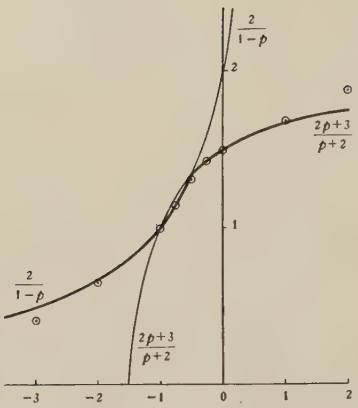

FIG. 6

the remaining terms being alternately $1/p$ and $1/(p+1)$. This fraction converges for every value of p, real or complex, except certain values with the real part $-1/2$. To the right of this dividing line the limit is actually equal to $F^*(p) = (2p+3)/(p+2)$. To the left, it is $2/(1-p)$. Fig. 6 shows the graphs of these functions for real values of p, together with certain points on the fifth convergent of the fraction (12). It is obvious that, in the region covered by the graph, convergence is quite rapid.

Now the outstanding thing about this expansion is, that its singular line bears no relation to $F^*(z)$ whatever, but is determined by the character of the transformation instead. Had we begun with any other function and used the same transformation, and had we succeeded in obtaining a convergent fraction for this new function, the new fraction also would have had this same line for a cut, and would have been regular everywhere else.

5. *Properties of a Particular Transformation.* These observations at once suggest two methods of dealing with functions the singularities of which are not on the negative real axis. One is, to search for a transformation which will carry the singularities to the negative real axis. The other is, to search for a transformation which will carry them all

into *one* branch of a multiple-valued function of z, thus leaving in the *other* branch only the singularities of the transformation itself. In either case, of course, we must require that the latter singularities shall lie on the negative real z-axis. The second of these methods is obviously the more powerful one, and we shall find that an especially suitable transformation is

$$z = \frac{(ap + b)(\alpha p + \beta)}{(cp + d)(\gamma p + \delta)}, \tag{13}$$

where the coefficients $a, b, c, d, \alpha, \beta, \gamma, \delta$ are all positive reals.

It requires only routine algebra to establish the following facts:

(1) In order that the singular points of the transformation (13) shall be on the negative real z-axis it is necessary and sufficient that both

$$\begin{cases} (b\gamma - a\delta)(c\beta - d\alpha) > 0, \\ (bc - ad)(\beta\gamma - \alpha\delta) > 0. \end{cases} \tag{14}$$

(2) The p-transformed of the negative real z-axis consists of part of the negative real p-axis together with a circle whose center is at

$$p_0 = -\frac{\beta\gamma(bc - ad) + ad(\beta\gamma - \alpha\delta)}{\alpha\gamma(bc - ad) + ac(\beta\gamma - \alpha\delta)},$$

and whose radius is

$$r^2 = p_0^2 - \frac{\beta\delta(bc - ad) + bd(\beta\gamma - \alpha\delta)}{\alpha\gamma(bc - ad) + ac(\beta\gamma - \alpha\delta)}.$$

We shall call this circle the "boundary of the transformation."

(3) So long as (14) is satisfied, p_0 is negative and $r^2 \leqq p_0^2$. Hence the boundary of the transformation never crosses the imaginary axis, though in a limiting case it may be tangent to it at the origin. It may also degenerate into a line parallel to, and to the left of, the imaginary axis; or into the imaginary axis itself. We may therefore state the following theorem.

THEOREM 7. *Any transformation of the type* (13) *which has its singularities located on the negative real z-axis, and for which all the constants a, b, c, d, α, β, γ, δ are positive or zero, maps a circular area in the left half of the p-plane upon one sheet of the z-plane, the remainder of the p-plane being mapped on the other sheet of the z-plane. The circular boundary may be tangent to the imaginary p-axis, or it may degenerate into a vertical straight line, but it cannot in any case enclose a portion of the right-hand half of the plane.*

In view of these conclusions we can rewrite (13) in one or the other of three convenient forms.* The first is

$$z = \frac{r^2 p^2}{\lambda^2 (p+s)^2 - p^2}, \qquad \begin{matrix} s \geqq 0, \\ \lambda \geqq 1, \end{matrix} \tag{15}$$

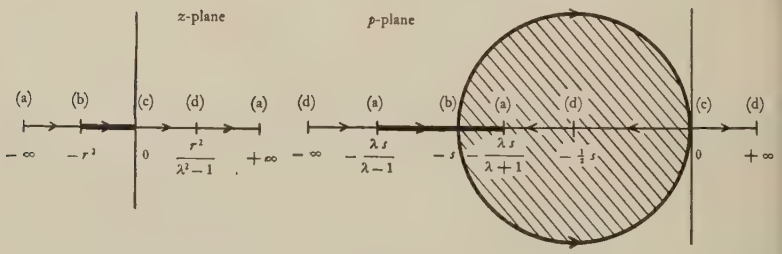

FIG. 7

which represents the configuration shown in Fig. 7. The boundary of the transformation in the p-plane, and the portion of the negative z-axis between the branch points, are drawn heavy. The letters (a), (b), $\cdots$, in the two drawings represent corresponding points, and the numbers written below them are their coordinates. Thus (a) is the point $z=\infty$, and transforms into $p=-\lambda s/(\lambda \pm 1)$. The singular points of the transformation are at (b) and (c).

* It will be noted that (16) is equivalent to an inversion in p, followed by (15); also that (17) is equivalent to a translation in p, followed by (15), and then by a translation in z. The set of transformations is closed under inversions in z.

By setting $\lambda=1$, $\mu=\alpha$, $r=s=\infty$, and $\nu=0$ or α in (17), we get (10) and (11).

The second form is

$$z = \frac{r^2}{\lambda^2(sp+1)^2 - 1}, \qquad \begin{matrix} s \geqq 0, \\ \lambda \geqq 1, \end{matrix} \tag{16}$$

which represents the configuration shown in Fig. 8. In this case the boundary of the transformation consists of a part of the negative p-axis, together with the vertical line through $p = -1/s$. The singular points of the transformation are at (b) and (c).

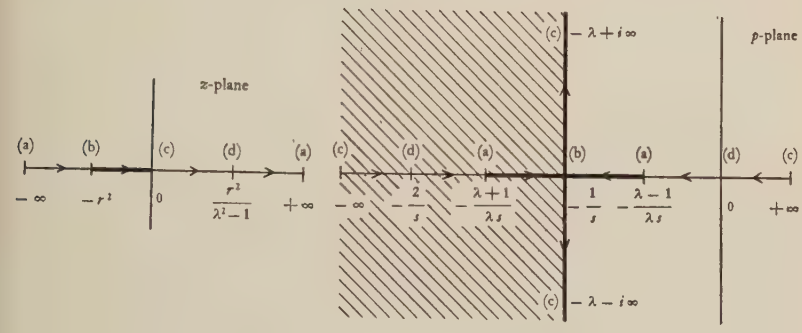

FIG. 8

The third form is

$$z + \nu = \frac{r^2(p+\mu)^2}{\lambda^2(p+\mu+s)^2 - (p+\mu)^2}, \qquad \begin{matrix} s \geqq 0, \\ \lambda \geqq 1, \end{matrix} \tag{17}$$

$$0 \leqq \nu \leqq \frac{r^2\mu^2}{\lambda^2(s+\mu)^2 - \mu^2},$$

for which the configurations are shown in Fig. 9. They are similar, except for the positions of the origins, to Fig. 7. The singular points are again at (b) and (c).

Finally, we notice that, if we had transformed $F^*(p)$ into a function of z by the use of (13), and if we had succeeded in obtaining a Stieltjes expansion for the latter, it would have been of the form (4), which becomes in terms of p

$$f(z) = \cfrac{1}{a_1\cfrac{(ap+b)(\alpha p+\beta)}{(cp+d)(\gamma p+\delta)} + \cfrac{1}{a_2 + \cdots}}.$$

This form in itself is not suitable for design purposes. But

$$(18)\quad \frac{(ap+b)}{(cp+d)}f(z) = \cfrac{1}{a_1\dfrac{\alpha p+\beta}{\gamma p+\delta} + \cfrac{1}{a_2\dfrac{ap+b}{cp+d} + \cfrac{1}{a_3\dfrac{\alpha p+\beta}{\gamma p+\delta} + \cdots}}}$$

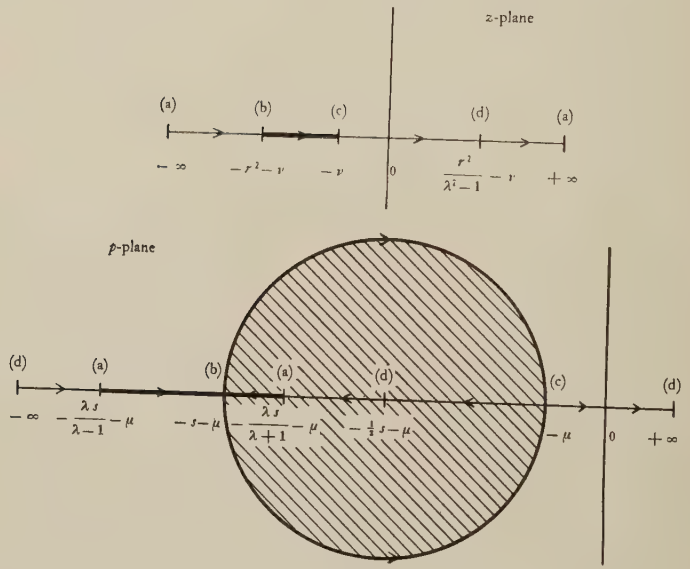

FIG. 9

is suitable for such uses. So also is

$$(19)\quad \frac{\alpha p+\beta}{\gamma p+\delta}f(z) = \cfrac{1}{a_1\dfrac{ap+b}{cp+d} + \cfrac{1}{a_2\dfrac{\alpha p+\beta}{\gamma p+\delta} + \cfrac{1}{a_3\dfrac{ap+b}{cp+d} + \cdots}}}$$

Hence, in using the transformation (13) for design purposes we would usually† identify $f^*(z)$ with the transformed of

† These are not the only *possible* bilinear multipliers which can be successfully used. However, we shall see that they serve our purpose remarkably well.

either $(cp+d)F^*(p)/(ap+b)$ or $(\gamma p+\delta)F^*(p)/(\alpha p+\beta)$ in order that $F^*(p)$ might itself appear in one of the forms (18) or (19). The additional factors introduce no objectionable singularities into $f^*(z)$, since the vanishing of either $ap+b$, $cp+d$, $\alpha p+\beta$ or $\gamma p+\delta$ leads to either $z=0$ or $z=\infty$, both of which are "negative real values of z." Inspection shows, moreover, that unless these points are essential singularities, the one at $z=0$ must be a pole, and the one at $z=\infty$ a zero.

6. *Theorems Relating to Redistribution of Singularities.* Suppose, now, that we are given a function $F^*(p)$ whose singularities, together with those of its reciprocal, are all more than $1/s$ units (s finite) to the left of the imaginary p-axis. Then by the use of the transformation (16), Fig. 8, we can obtain a function $f^*(z)$ with branch points at 0 and $-r^2$, *and no other singularities or zeros* in its one branch except possibly a simple zero at ∞. Moreover, the length r^2 of the cut in the z-plane may be finite.

If on the other hand we deal with Fig. 7 and suppose that either the given function $F^*(p)$ or its reciprocal is singular at $p=0$, but that all other singularities can be enclosed in the interior of a finite circle, it is a simple matter to show that all the essential facts of the last paragraph are again true. Hence we have the following theorem.

THEOREM 8. *From any $F^*(p)$ all the singularities of which, together with those of its reciprocal, lie at least $1/s$ units to the left of the imaginary p-axis, (or all the singularities of which, together with those of its reciprocal, except one at the origin, can be enclosed in the interior of a circle of finite radius), we can obtain a function $f^*(z)$ in at least two ways† which, together with its reciprocal, will be regular everywhere in the cut z-plane except at the two finite branch points 0 and $-r^2$, and except also for a simple zero at ∞. It has, moreover, at all points of the cut except the ends, a regular analytic continuation either from above or below.*

† Corresponding to the two methods of identification discussed in §3.

We shall, in this paper, have little to say about functions $F^*(p)$ which do not satisfy the conditions of Theorem 8, since they lead us into the ambiguities of Theorem 6. However, we state the following theorem without proof:

THEOREM 9. *If $F^*(p)$ and its reciprocal have no singular points to the right of the imaginary axis, but if the conditions of Theorem* 8 *can nevertheless not be met, the transformation* (13) *can still produce a function $f^*(z)$ the singularities of which, together with those of its reciprocal, are all located on the negative real axis. In this case, however, the entire negative z-axis is a cut, and the possibility of analytic continuation can in general not be asserted for every point on it.*

If $F^*(p)$ or its reciprocal has a singular point to the right of the imaginary p-axis no transformation of the form (13) *can carry it into an $f^*(z)$ the zeros and singularities of which all lie on the negative real axis.*

We have, then, found a way of overcoming the second of our difficulties; for we have found a method of creating, from any function $F^*(p)$ whose singularities are all to the left of, or on, the imaginary axis, another function $f^*(z)$ whose singularities are located as required by the Stieltjes theory; and this has been accomplished in such a way, moreover, that the existence of an expansion of the form (4) for $f^*(z)$ implies the existence of a usable expansion of the form (18) or (19) for $F^*(p)$ also. We now turn our attention to the third difficulty: the possibility that $f^*(z)$ may not generate a $\Phi(x)$ of suitable character.

7. *Existence Theorems concerning the Auxiliary Function* $\Phi(x)$. To begin with, we introduce a definition which will aid us in avoiding much circumlocution.

The function $f(w)$ is said to be "admissibly" singular at the point $w=w_0$, if it possesses there a simple pole and/or† *an essential singularity of such a character that*‡

† We do not wish to exclude such functions as $(w-w_0)^{1/2}+1/(w-w_0)$.

‡ It is obvious that this condition excludes isolated essential singularities. It includes many algebraic and logarithmic branch points, however.

$$(20) \qquad |(w-w_0)^k f(w)| < M, \quad 0 \leqq |w-w_0| < \delta,$$

where δ *and* M *are any positive numbers other than* 0 *and* ∞, *and where* $0 \leqq k < 1$.

We can then readily prove the following theorem.

THEOREM 10. *If the function* $F^*(p)$ *and its reciprocal have none but admissible singularities on the imaginary* p*-axis, the* $f^*(z)$ *obtained either from* $[(cp+d)/(ap+b)]F^*(p)$ *or from* $[\gamma p+\delta)/(\alpha p+\beta)]F^*(p)$ *by the use of* (13) *will have only admissible singularities on the negative real axis. The same is true of* $1/f^*(z)$.

At all but the branch points of the transformation this is immediately obvious, for the character of a singularity is never altered by a conformal transformation. At the singular points the transformation substantially replaces $p-p_0$ by $(z-z_0)^{1/2}$; whence even if the additional bilinear terms in p happen to be singular at such a point they cannot render $f^*(z)$ inadmissibly singular. We need hardly fill in the details of the proof.

Next we note that, since $F^*(p)$ is real for every real positive p, $f^*(z)$ is also real for every real positive z. This follows from a very simple study of analytic continuation in the p-planes of Figs. 7, 8 and 9. Then from an equally easy study of analytic continuation in the z-planes we find that $f^*(x+iy)$ and $f^*(x-iy)$ are conjugate complex. Hence, if we introduce the integral

$$(21) \qquad G(z) = \int_\alpha^z f^*(z)dz,$$

the integration being performed along any path which passes from the (arbitrary) positive real α to z without touching the negative real axis, we see at once that $G(x+iy)$ and $G(x-iy)$ are conjugate complex. But by (8)

$$(22) \qquad \Phi(x) = \lim_{y \to 0} \frac{G(-x+iy) - G(-x-iy)}{2\pi i}.$$

Hence if this limit exists at all it is real.

As for the existence of the limit, if $f^*(z)$ is regular and capable of analytic continuation across the axis, or if it possesses a singularity of the sort (20), $G(-x+iy)$ and $G(-x-iy)$ approach well-defined finite limits. If on the other hand $f^*(z)$ has a simple pole at $z=-x$, it may be written as $\alpha/(z+x)+f_1^*(z)$, where $f_1^*(z)$ has the same analytical character as $f^*(z)$ at all other points than $-x$, and at $-x$ satisfies (20). Then by actual integration

$$\Phi(x) = \frac{1}{2}\alpha + \lim_{y\to 0} \frac{G_1(-x+iy) - G_1(-x-iy)}{2\pi i},$$

where $G_1(z)$, being derived from $f_1(z)$, is known to have well-defined limits. Hence we have the following theorem.

THEOREM 11. *Any function $f^*(z)$ which is real for real positive z, regular for all but real negative z, and only admissibly singular for real negative z, generates a well-defined real function $\Phi(x)$.*

We are now prepared to state, as a consequence of Theorems 8, 9, 10 and 11, the following theorem, which has been the objective of the present section.

THEOREM 12. *From any function $F^*(p)$ which is real for positive real values of p, which has no singularities to the right of the imaginary axis, and none but admissible singularities on the imaginary axis, at least two functions $f^*(z)$ can be produced by the use of the transformation* (13), *each of which will generate a $\Phi(x)$ which is well-defined for every x.*

If, in addition, the conditions of Theorem 8 *are met by $F^*(p)$, $\Phi(x)$ will be of limited variation, and constant for $x>r^2$.*

Only the last paragraph of the theorem requires proof. That $\Phi(x)$ is constant for $x>r^2$ follows at once from (8) and from the fact, stated in Theorem 8, that $f^*(z)$ is regular for $|z|>r^2$.

To prove that $\Phi(x)$ is of limited variation we exclude for a moment the possibility of poles at the branch points, con-

sidering only singularities of the type (20). Then, since $f^*(z)$, and consequently $G(z)$ also, is regular as we approach the negative real axis from above and below, we have

$$\begin{aligned}\frac{d\Phi}{dx} &= \lim_{y\to 0} \frac{1}{2\pi i}\frac{d}{dx}[G(-x+iy) - G(-x-iy)] \\ &= \frac{1}{2\pi i}\lim_{y\to 0}[f^*(-x-iy) - f^*(-x+iy)]\end{aligned} \tag{23}$$

for all values of x except $x=0$ and $x=r^2$. Hence the variation of $\Phi(x)$ is

$$\text{Var } \Phi = \int_0^{r^2} \left|\frac{d\Phi}{dx}\right| dx \leqq \frac{1}{\pi}\int_0^{r^2} |f^*(-x)|\, dx.$$

We now consider the half of this interval nearest the origin. We know that, over the range $0<x\leqq\delta$, (20) is satisfied. We also know that $|f^*(z)|$ is bounded in the interval $\delta\leqq x\leqq r^2/2$. Call its upper bound m. Then if we take M_1 as the larger of the two quantities M and $(2/r^2)^k m$, it will be true that

$$|f^*(-x)| \leqq M_1/x^k$$

over the entire interval $0<x\leqq r^2/2$. A similar statement (with $M=M_2$) can be made for the other half of the interval, $r^2/2\leqq x<r^2$. Hence

$$\text{Var } \Phi \leqq (M_1+M_2)(r^2/2)^{1-k}/(1-k),$$

which proves our theorem, provided the singularities of $f^*(z)$ are of the type (20).

If, on the other hand, $f^*(z)$ possesses a pole at either $z=0$ or $z=-r^2$, or both, these poles will have no other effect then to introduce finite discontinuities into $\Phi(x)$ equal to their residues; and it is seen at once that the theorem will still be true.

Let us now take stock of our study from the standpoint of network design. We know that, from any $F^*(p)$ which satisfies the conditions of Theorem 8, we can obtain an $f^*(z)$ the singularities of which are on the negative real z-axis, and

which generates a $\Phi(x)$ the variation of which is bounded, and which is constant for $x > r^2$. We may, therefore, replace $\Phi(x)$ by the difference of two functions, $\Phi_1(x)$ and $\Phi_2(x)$, both of which are monotonic, non-decreasing, and both finite for $x > r^2$. Hence, by Theorem 5, the functions $f_1(z)$ and $f_2(z)$ which correspond to $\Phi_1(x)$ and $\Phi_2(x)$ possess Stieltjes fractions. That is, when the conditions of Theorem 8 are satisfied, $f^*(z)$ generates a $\Phi(x) = \Phi_1(x) - \Phi_2(x)$, to which corresponds an $f(z) = f_1(z) - f_2(z)$, which can be written as the *difference* of two Stieltjes fractions.

As it stands, this result has no practical utility, for we cannot connect two networks in such a way that their combined impedance is the difference of their separate impedances. If, however, we can find a way of obtaining a fraction with positive coefficients the $\Phi(x)$ corresponding to which is monotonic *non-increasing*, [say a fraction the limit of which is $f_2'(z)$], then $f'(z) = f_1(z) + f_2'(z)$ will also generate the same $\Phi(x)$ as $f^*(z)$, and may turn out to be identical with it. Such a result would be usable. This leads us, then, to the search for a form of fraction the $\Phi(x)$ of which is monotonic non-increasing.

8. *Functions for which $\Phi(x)$ is Monotonic Decreasing.* We can easily show that if $f^*(z)$ gives rise to a monotonic *non-increasing* $\Phi(x)$, certain related functions give rise to monotonic *non-decreasing* Φ's. Chief among these are $zf^*(z)$, $1/f^*(z)$ and $f^*(1/z)$.

We treat first the case of $zf^*(z)$. Wherever $f^*(z)$ is regular as it approaches the negative real axis from above and below, $f_1(z) = zf^*(z)$ is also, and therefore either $f^*(z)$ or $f_1^*(z)$ can be substituted in (23). Thus we obtain

$$\frac{d\Phi_1(x)}{dx} = \frac{1}{2\pi i} \lim_{y \to 0} [f_1^*(-x - iy) - f_1^*(-x + iy)]$$

$$= \frac{-x}{2\pi i} \lim_{y \to 0} [f^*(-x - iy) - f^*(-x + iy)]$$

$$- \frac{1}{2\pi} \lim_{y \to 0} y[f^*(-x - iy) + f^*(-x + iy)].$$

The last limit, however, vanishes owing to the finiteness of $*(-x \pm iy)$. Hence by (23) we have

$$\frac{d\Phi_1(x)}{dx} = -x\frac{d\Phi}{dx}$$

or

$$d\Phi_1(x) = -x\,d\Phi(x). \tag{24}$$

On the other hand, at a pole the residue of $f_1^*(z)$ is $-x$ times the residue of $f^*(z)$. That is, (24) is still true.

The extension to singularities of the type (20) is almost equally simple. It consists merely in showing that at such a point both $\Phi(x+\delta x)-\Phi(x-\delta x)$ and $\Phi_1(x+\delta x)-\Phi_1(x-\delta x)$ vanish with δx. There is, then, no discontinuity at the point, and (24) rules at every neighboring point. Hence we may state the following theorem.

THEOREM 13. *If the singularities of $f^*(z)$ are all on the negative real axis and are of admissible type, and if it generates a monotonic function $\Phi(x)$, then the function $f_1^*(z)=zf^*(z)$ generates a function $\Phi_1(x)$ which is monotonic in the opposite sense. More specifically, the two are related by equation* (24).

We could frame somewhat similar proofs to show that the Φ's generated by $f^*(1/z)$ and $1/f^*(z)$ are monotonic in the opposite sense to that generated by $f^*(z)$ itself. However we have no occasion to use them in the present paper.

The significance of Theorem 13 for our purposes lies in the fact that, if $f^*(z)$ possesses a Stieltjes expansion of the form (4), $f_1^*(z)=zf^*(z)$ will have the expansion

$$f_1^*(z) = \cfrac{1}{a_1 + \cfrac{1}{a_2 z + \cfrac{1}{a_3 + \cdots}}} \tag{25}$$

This form of expansion, therefore, is appropriate for functions which generate monotonic decreasing Φ's. Moreover, in view of the remarks made at the end of §7, if we were given a

function of bounded variation, we would naturally seek to expand it in the sum of two fractions, one of form (4) and one of form (25).

9. *The Difference Function,* $\delta(z)$. We now have methods for overcoming, to a large degree, our second and third difficulties. But while we are thus enabled to effect an expansion of

$$f(z) = \int_0^b \frac{d\Phi(x)}{z + x}$$

in a usable form, we still do not know that this $f(z)$ and $f^*(z)$ are identical. We must now study this fourth difficulty.

If we write

$$\delta(z) = f^*(z) - f(z),$$

we know that

$$\frac{1}{2\pi i} \int \delta(z) dz = 0, \tag{26}$$

the integral being taken about any (every) circle of radius x about the origin. We ask what sort of function $\delta(z)$ can be. We see at once that

(a) *It is regular except, perhaps, at negative real values of* z. This follows from the fact that it is the difference of two regular functions.

(b) *It may be an entire function if* $f^*(z)$ *is singular at infinity; or if* $f^*(z)$ *is regular at infinity* $\delta(z)$ *may be a constant.* That any entire function satisfies the requirement (26) is obvious. However, if $f^*(z)$ is regular at infinity, $\Phi(x)$ is constant for x greater than some finite value r^2; $f(z)$ is therefore also regular for $|z| > r^2$.† Hence the same will be true of $\delta(x)$ as well.

(c) *It may have any number of isolated singular points the contour integrals about which are zero. Such singularities, however, can only occur where* $f^*(z)$ *is itself singular.* For example,

† Perron. loc. cit., p. 369.

$$\delta(x) = \frac{1}{(z + x)^2},$$

which has residue zero, is possible. To establish that $\delta(z)$ is regular wherever $f^*(z)$ is regular, we need only show that this is true of $f(z)$. Let us suppose, then, that $f^*(z)$ is represented on a Riemann surface of any number of sheets, connected along certain segments of the negative real axis, and that it has no singularities in the interval $(-x_2, -x_1)$. Elsewhere on the negative real axis it may be of any analytic character whatever consistent with the fact that it generates $\Phi(x)$. Then the $G(z)$ defined by (21) is regular for every z not on the negative real axis, and also for values of z between $(-x_2, -x_1)$; though in the case of this interval it may take different values $F_1(z)$ and $F_2(z)$ as we approach the axis from above and below. It follows, then, that

$$\Phi(z) = \frac{F_1(-z) - F_2(-z)}{2\pi i}$$

is a regular analytic function of z in the vicinity of (x_1, x_2). Along the real axis this function is, by definition, identical with $\Phi(x)$. Then by a well known theorem†, it is also true that

$$\int_{x_1}^{x_2} \frac{d\Phi(x)}{z + x} = \int_{x_1}^{x_2} \frac{d\Phi}{dx} \frac{dx}{z + x}$$

is regular—though perhaps multiple valued—between $-x_2$ and $-x_1$. On the other hand

$$\int_0^{x_1} \frac{d\Phi(x)}{z + x}$$

and

$$\int_{x_2}^{\infty} \frac{d\Phi(x)}{z + x}$$

are both analytic in this same region.‡ Since $f(z)$ is the sum of these three integrals, it too must be regular in the interval (x_1, x_2).

† Goursat, *Cours d'Analyse*, vol. 2, p. 254.

‡ Perron, loc. cit., p. 369.

(d) $\delta(z)$ *does not admit of non-essential cuts.* For suppose $(-x_2, -x_1)$ were such a cut: then it would be possible to effect analytic continuation not only of $\delta(z)$, but of the integral

$$\Delta(z) = \int_{\alpha}^{z} \delta(z)dz$$

also, across this boundary. Let us call the value taken by $\Delta(z)$ as reached through the upper half of the plane $\Delta_1(z)$, and that reached through the lower half $\Delta_2(z)$. Then $\Delta_1 - \Delta_2$ possesses a well-defined set of derivatives, which we can obtain by choosing dz in any manner we like. For example, we may choose $dz = dx$ if we wish. But by (26) $\Delta_1 - \Delta_2$ is identically zero along the negative real axis; hence all the derivatives vanish. That is $\Delta_1(z) = \Delta_2(z)$, and the function $\delta(z)$ is its own analytic continuation across $(-x_2, -x_1)$.

As a final result of all these considerations we may enunciate the following theorem.

THEOREM 14. *Two functions $f_1(z)$ and $f(z)$ which are regular everywhere except on the negative real axis and generate the same $\Phi(x)$ can differ at most by a function $\delta(z)$ which is single-valued and regular, except perhaps at certain points where one or both of the functions f_1 and f are singular.*† *At such points $\delta(z)$ may possess isolated singularities of residue zero; or, in case the singularities of f_1 or f are everywhere dense in an interval $(-x_2, -x_1)$, the same may be true of δ also; but in this latter case the relation* (26) *will still be satisfied.*

A special case of peculiar interest is the type of function derived through the use of the transformation (13), when $F^*(p)$ satisfies the conditions of Theorem 8. In such cases, $f^*(z)$ has singularities at only $-r^2$ and 0; hence $\delta(z)$ can have singularities nowhere else. Usually it is found to be a constant.

† Our proof has assumed *one* function to *correspond* to $\Phi(x)$. But the difference between any two functions which generate the same $\Phi(x)$ is the sum of two δ's of the type with which our proof deals, whence the theorem is true as stated.

10. *An Example.* As an example of this process of expansion, let us consider

$$f^*(z) = (z + \tfrac{1}{2})(z^2 + z)^{1/2} - z^2 - z, \tag{27}$$

a function which is everywhere regular (including ∞) except at the winding points $z=0$ and $z=-1$. We wish to represent that branch of it for which the radical is positive for positive real z. We readily find that it generates

$$\begin{cases} \Phi(x) = -\dfrac{1}{3\pi}(x - x^2)^{3/2}, & 0 < x < 1, \\ \Phi(x) = 0, & 1 < x\,; \end{cases} \tag{28}$$

or, since it is the derivative† of Φ rather than Φ itself which is of interest,

$$\frac{d\Phi}{dx} = \frac{x}{\pi}(x - x^2)^{1/2} - \frac{1}{2\pi}(x - x^2)^{1/2}.$$

We can, then, divide Φ into a monotonic non-decreasing term Φ' and a monotonic non-increasing term Φ_1'' by choosing

$$\frac{d\Phi'}{dx} = \frac{x}{\pi}(x - x^2)^{1/2}$$

and

$$\frac{d\Phi_1''}{dx} = -\frac{1}{2\pi}(x - x^2)^{1/2}.$$

The function $f'(z)$ which corresponds to Φ' is found directly from (5) to be

$$\begin{aligned} f'(z) &= \int_0^1 \frac{d\Phi'}{dx}\,\frac{dx}{z + x} \\ &= z(z^2 + z)^{1/2} - z^2 - \frac{z}{2} + \frac{1}{8}. \end{aligned} \tag{29}$$

We know, then, that this function possesses a Stieltjes expansion.

† Since $\Phi(x)$ is differentiable, the Stieltjes integral $\int d\Phi/(z+x)$ reduces simply to $\int (d\Phi/dx)\,dx/(z+x)$.

To find the residual function $f_1''(z)$ which corresponds to Φ_1'', we adopt the process suggested by Theorem 13 and compute the auxiliary function

$$f''(z) = \frac{f_1''(z)}{z}$$

which corresponds to

$$\frac{d\Phi''}{dx} = -\frac{1}{x}\frac{d\Phi_1''}{dx} = \frac{1}{2\pi}\frac{(x^2 - x)^{1/2}}{x}.$$

Again, we find from (5) that

$$f''(z) = \frac{1}{2}\left(\frac{(z^2 + z)^{1/2}}{z} - 1\right).$$

This also possesses a Stieltjes expansion. Then

$$f_1''(z) = \tfrac{1}{2}\left[(z^2 + z)^{1/2} - z\right] \tag{30}$$

possesses an expansion of the type

$$f''(z) = \cfrac{1}{a_1 + \cfrac{1}{a_2 z + \cdots}}.$$

By adding together (29) and (30) we get the function which corresponds to (28). It is

$$f(z) = (z + \tfrac{1}{2})(z^2 + z)^{1/2} - z^2 - z + \tfrac{1}{8}. \tag{31}$$

Now, by comparing this with (27), we find that the two differ by only a constant, which is what we would expect in the light of Theorem 14.

We could also have proceeded as follows: Since $\Phi(x)$, as given by (28), decreases steadily up to $x = 1/2$, when it reaches the value $-1/(24\pi)$, and then increases, we may choose

$$\begin{cases} \Phi' = 0, & 0 < x < \tfrac{1}{2}, \\ \Phi' = \dfrac{1}{24\pi} - \dfrac{1}{3\pi}(x - x^2)^{3/2}, & \tfrac{1}{2} < x < 1, \\ \Phi' = \dfrac{1}{24\pi}, & 1 < x ; \end{cases}$$

$$(32) \qquad \begin{cases} \Phi_1'' = -\dfrac{1}{3\pi}(x - x^2)^{3/2}, & 0 < x < \tfrac{1}{2}, \\ \Phi_1'' = -\dfrac{1}{24\pi}, & \tfrac{1}{2} < x. \end{cases}$$

In other words,

$$\frac{d\Phi'}{dx} = \frac{1}{\pi}\left(x - \frac{1}{2}\right)(x - x^2)^{1/2}, \quad \tfrac{1}{2} < x < 1,$$

and

$$\frac{d\Phi_1''}{dx} = -\frac{1}{\pi}\left(\frac{1}{2} - x\right)(x - x^2)^{1/2}, \quad 0 < x < \tfrac{1}{2},$$

the values of the derivatives being zero outside the indicated ranges. Then by direct computation we find for $f'(z)$

$$f'(z) = \frac{1}{4\pi(2z+1)} - \frac{1}{16} - \frac{z^2+z}{2} + \frac{2z+1}{4}(z^2+z)^{1/2} - \frac{1}{\pi}\frac{z^2+z}{2z+1}\left[F\left(1,1,\frac{1}{2};\frac{1}{(2z+1)^2}\right) - 1\right],$$

where $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function

$$F(\alpha, \beta, \gamma; x) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha\cdot\alpha + 1\cdot\beta\cdot\beta + 1}{1\cdot 2\cdot\gamma\cdot\gamma + 1}x^2 + \cdots.$$

Similarly, we set $d\Phi''/dx = -(1/x)(d\Phi_1''/dx)$, compute $f''(z)$, and then make $f_1''(z) = zf''(z)$. The result is

$$f_1''(z) = \frac{z}{2\pi(2z+1)} - \frac{z^2+z}{2} + \frac{2z+1}{4}(z^2+z)^{1/2} + \frac{1}{\pi}\frac{z^2+z}{2z+1}\left[F\left(1,1,\frac{1}{2};\frac{1}{(2z+1)^2}\right) - 1\right].$$

Adding f' and f_1'' together we get

$$(33)\quad f(z) = \left(\frac{1}{4\pi} - \frac{1}{16}\right) - z^2 - z + \left(z + \frac{1}{2}\right)(z^2 + z)^{1/2}.$$

This result again differs from (27) only to the extent of an additive constant. This time, however, the constant is smaller than before, being 0.017 instead of 0.125.

This value $(4-\pi)/(16\pi)$ is, in fact, the smallest difference which can exist between the sum of two fractions of the form

$$(34)\quad \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cdots}}} \qquad + \cfrac{1}{b_1 + \cfrac{1}{b_2 z + \cfrac{1}{b_3 + \cdots}}}$$

and the function (27), as we can readily show by the following argument.

It is evident that, at $z = \infty$, such a pair of fractions takes the value $1/b_1$. However, it is known (Theorem 2) that $1/b_1$ is equal to $\Phi''(\infty)$. Hence the fraction will have its least value at ∞ when $\Phi''(\infty)$ is smallest. We see at once from the relation between Φ'' and Φ_1'' that $\Phi''(\infty)$ will be smallest when $\Phi_1''(x)$ is given *for every* x the smallest value which is consistent with the requirement that Φ' shall nowhere decrease. This, however, is obviously the function $\Phi_1''(x)$ which we have adopted in (32). From it we find by direct computation that $\Phi''(\infty) = (4+\pi)/(16\pi)$. This, then, is the smallest value which $f(\infty)$ can have. We easily see from (27), however, that $f^*(\infty) = 1/8$. Hence the smallest *difference* that can exist between the two is $(4-\pi)/(16\pi)$, which is, indeed, the constant term in (33).

11. *The Final Process for Obtaining a Usable Expansion.* In the discussions of §§7–10 we have tacitly assumed that an expansion of the form (34) is satisfactory for purposes of design; and so it is if z is synonymous with our frequency variable p, for as we have seen in §1 the two fractions then correspond to constructible networks. But when the func-

tion $f^*(z)$ has been identified with $(cp+d)F^*(p)/(ap+b)$ or with $(\gamma p+\delta)F^*(p)/(\alpha p+\beta)$ by use of the transformation (13), the situation is not quite so simple, and something still remains to be accomplished.

To see why this is true, let us abbreviate the expressions $(ap+b)/(cp+d)$ and $(\alpha p+\beta)/(\gamma p+\delta)$ to q_1 and q_2, so that $z=q_1q_2$. Then from (34) we may get a pair of fractions representing $F(p)=q_1f(z)$. The terms of one of these are of the form aq_1 and aq_2 and are therefore constructible. But in the other, terms of the form $bq_1^2q_2$ appear, and these cannot be interpreted in electrical terms.† Hence the result, in exactly this form, is useless.

This difficulty can be overcome by separating $F^*(p)$ itself into two components each of which generates an expansion of usable form, instead of dealing with the transformed function $f^*(z)$.

In order to make our study somewhat more readable, we shall think entirely in terms of the transformation (15) and its accompanying Fig. 7. This is indeed quite general, for all distributions of singularities which can be enclosed by the boundary of (17) can also be enclosed by (15), and, as we have already observed, (16) can be obtained from (15) by a simple inversion on p. Then we have

$$(35) \qquad q_1 = \frac{rp}{(\lambda+1)p+s}, \qquad q_2 = \frac{rp}{(\lambda-1)p+s}.$$

We shall also assume $F^*(p)$ to be broken up into the sum of two components $G^*(p)$ and $H^*(p)$ by a procedure which will become clear as we proceed, and shall indicate the three functions of z into which they transform by $f^*(z)$, $g^*(z)$ and $h^*(z)$. That is, $f^*(z)\equiv F^*(p)$, $\cdots$.

We shall be interested in the values of all these various functions along the upper and lower edges of the cut in the

† We assume that $f^*(z)$ has been identified with $F^*(p)/q_1$. If it has been identified with $F^*(p)/q_2$ the fractions are slightly different, but still unsatisfactory.

z-plane. In the case of q_1 and q_2 we readily find them to be

$$(36) \qquad \begin{cases} rq_1 = x \pm i(r^2x - x^2)^{1/2}, \\ rq_2 = -x \pm i(r^2x - x^2)^{1/2}, \end{cases}$$

where the upper and lower signs apply to the upper and lower borders of the cut, respectively. About the values of f^*, g^* and h^* we can say nothing definite, but we introduce the notation

$$(37) \qquad \begin{cases} f^*(-x) = u_f \pm iv_f, \\ g^*(-x) = u_g \pm iv_g, \\ h^*(-x) = u_h \pm iv_h. \end{cases}$$

Then in terms of this notation, we readily find from (23) that the Φ's generated by $F^*(p)$, $G^*(p)/q_2$ and $H^*(p)/q_1$, are, respectively

$$(38) \qquad \begin{cases} \pi \dfrac{d\Phi_f(x)}{dx} = -v_f, \\ r\pi \dfrac{d\Phi_g(x)}{dx} = u_g\left(\dfrac{r^2}{x} - 1\right)^{1/2} + v_g, \\ r\pi \dfrac{d\Phi_h(x)}{dx} = u_h\left(\dfrac{r^2}{x} - 1\right)^{1/2} - v_h. \end{cases}$$

Now, what we purpose to show is, that so long as $F^*(p)$ is bounded along the entire boundary of the transformation,† $G^*(p)$ and $H^*(p)$ can always be so chosen that the left hand members of the second and third of equations (38) are everywhere positive. $\Phi_g(x)$ and $\Phi_h(x)$ then satisfy the conditions of Theorem 5. From them we can therefore obtain convergent Stieltjes fractions in the variable $z = q_1q_2$; whence, calling these fractions $G(p)/q_2$ and $H(p)/q_1$, respectively, we have

† This condition is sufficient, but not necessary, as we shall see in §12.

$$(39)\qquad \begin{cases} G(p) = \cfrac{1}{a_1q_1 + \cfrac{1}{a_2q_2 + \cfrac{1}{a_3q_1 + \cdots}}}, \\ H(p) = \cfrac{1}{b_1q_2 + \cfrac{1}{b_2q_1 + \cfrac{1}{b_3q_2 + \cdots}}}, \end{cases}$$

both of which are constructible.

To show that this is so, we start with the first of equations (38) and set $v_g' = v_f$ wherever v_f is positive, and $v_g' = 0$ elsewhere. Similarly we put $v_h' = v_f$ wherever v_f is negative, and zero elsewhere.† Thus we have

$$(40)\qquad v_g' + v_h' = v_f.$$

Next we define two new functions of z by the formulas

$$(41)\qquad \begin{cases} g'(z) = -\frac{1}{\pi}\int_0^{r^2} \frac{v_g'\,dx}{z+x}, \\ h'(z) = -\frac{1}{\pi}\int_0^{r^2} \frac{v_h'\,dx}{z+x}. \end{cases}$$

From (5), (40) and the first of equations (38) we see that $f'(z) = g'(z) + h'(z)$ is either identical with $f^*(z)$, or else differs from it only to the extent of a function $\delta(z)$ the Φ of which is everywhere zero. Further, by an argument essentially similar to that in §10 we may readily show that $g'(z)$ and $h'(z)$ are regular analytic functions as we approach the negative real axis from above and below, except perhaps at 0 and $-r^2$, and are bounded at these points. By analogy with (37) we shall call their values along the cut

$$(42)\qquad \begin{cases} g'(-x) = u_g' \pm iv_g', \\ h'(-x) = u_h' \pm iv_h', \end{cases}$$

† Any other method of division for which v_g' is everywhere positive and v_h' everywhere negative is satisfactory.

the v_g' being by definition everywhere *positive*, the v_h' everywhere *negative*, while u_g' and u_h' are only known to be *bounded*.

Finally, we take the arbitrary step of defining $g^*(z)$ and $h^*(z)$—and therefore also the $G^*(p)$ and $H^*(p)$ into which $F^*(p)$ is divided—by the equations

$$g^*(z) = g'(z) - M\left(\frac{r^2}{z} + 1\right)^{1/2},$$

$$h^*(z) = h'(z) + M\left(\frac{r^2}{z} + 1\right)^{1/2} + \delta(z),$$

M being a positive constant the value of which will be assigned later. Then we see that along the cut

$$g^*(-x) = u_g' \pm iv_g' \pm iM\left(\frac{r^2}{x} - 1\right)^{1/2},$$

$$h^*(-x) = u_h' \pm iv_h' \pm iM\left(\frac{r^2}{x} - 1\right)^{1/2};$$

and upon comparing these values with (37) we can identify u_g, v_g, u_h and v_h. Substituting the values thus found in the second and third of equations (38), we arrive at

$$(43) \qquad \begin{cases} r\pi \dfrac{d\Phi_g(x)}{dx} = (u_g' + M)\left(\dfrac{r^2}{x} - 1\right)^{1/2} + v_g', \\ r\pi \dfrac{d\Phi_h(x)}{dx} = (u_h' + M)\left(\dfrac{r^2}{x} - 1\right)^{1/2} - v_h'. \end{cases}$$

Now, since u_g' and u_h' are bounded, M may be taken so large that the first terms are positive, while the terms v_g' and $-v_h'$ are positive by definition. Hence both $\Phi_g(x)$ and $\Phi_h(x)$ are monotonic increasing and lead to Stieltjes fractions in z.

Thus we are assured that expansions of the form (39) really exist, where $G(p)$ and $H(p)$ are related to $F^*(p)$ by the equation

$$F^*(p) = G(p) + H(p) + \Delta(p),$$

$\Delta(p)$ being some function which generates a Φ that is everywhere zero. Hence we may state the following theorem.

THEOREM 15. *If $F^*(p)$ satisfies the conditions of Theorem 8 and is finite at the origin, it can be expressed as the sum of three functions $G(p)$, $H(p)$ and $\Delta(p)$, the first two of which possess constructible expansions of the form* (39), *while the third generates a Φ which is everywhere zero.*

We have thus avoided the difficulty to which attention was called at the beginning of the section.

12. *An Example.* We can make the material of §11 somewhat more intelligible by means of an illustration.

Let us consider the function†

$$F^*(p) = \left(p^2 + p + \frac{1}{2}\right) \log \frac{p+1}{p}.$$

By using (15) in the form

$$z = \frac{p^2}{(p+1)^2 - p^2}, \tag{44}$$

(which means, of course, that $r = s = \lambda = 1$), we readily convert $F^*(p)$ into

$$f^*(z) = \left(z + \frac{1}{2}\right)\left(z + \frac{1}{2} + (z^2 + z)^{1/2}\right) \log \frac{z+1}{z}.$$

Along the negative real axis this takes the value

$$\begin{aligned} f^*(-x) = {} & \left(x - \frac{1}{2}\right)^2 \log \frac{1-x}{x} + \pi\left(\frac{1}{2} - x\right)(x - x^2)^{1/2} \\ & \pm i\left[\left(\frac{1}{2} - x\right)(x - x^2)^{1/2} \log \frac{1-x}{x} - \pi\left(x - \frac{1}{2}\right)^2\right] \\ & \equiv u_f \pm i v_f. \end{aligned}$$

† This function has a pole at ∞, and is not bounded at the points 0 and -1. Thus it violates the conditions laid down in Theorem 15. The choice of such a function is deliberate, as it serves to emphasize that the conditons, while sufficient, are not always necessary.

This is the analog of the first of equations (37).

We now define

$$(45)\qquad \begin{cases} v_g' = \left(\frac{1}{2} - x\right)(x - x^2)^{1/2} \log \frac{1-x}{x}, \\ v_h' = -\pi\left(x - \frac{1}{2}\right)^2, \end{cases}$$

this definition serving the purpose of making v_g' everywhere positive and v_h' everywhere negative. Then we find from (41)

$$g'(z) = \left(z + \frac{1}{2}\right)\left(-1 + (z^2 + z)^{1/2} \log \frac{z+1}{z}\right),$$

$$h'(z) = \left(z + \frac{1}{2}\right)\left[-1 + \left(z + \frac{1}{2}\right) \log \frac{z+1}{z}\right].$$

We must next find the values (42) which these functions take along the negative real axis. Of course the imaginary components v_g' and v_h' are given by (45). The real components are readily found to be

$$(46)\qquad \begin{cases} u_g' = \left(\frac{1}{2} - x\right)\left[(-1 + \pi(x - x^2)^{1/2}\right], \\ u_h' = \left(\frac{1}{2} - x\right)\left[-1 + \left(\frac{1}{2} - x\right) \log \frac{1-x}{x}\right]. \end{cases}$$

Substituting (45) and (46) in (43), and remembering that $r = 1$, we obtain

$$\pi \frac{d\Phi_g(x)}{dx} = \pi\left(\frac{1}{2} - x\right)(1 - x)$$

$$+ \left[M - \left(\frac{1}{2} - x\right) + x\left(\frac{1}{2} - x\right) \log \frac{1-x}{x}\right]\left(\frac{1-x}{x}\right)^{1/2};$$

$$\pi \frac{d\Phi_h(x)}{dx} = \pi\left(\frac{1}{2} - x\right)^2$$

$$+ \left[M - \left(\frac{1}{2} - x\right) + \left(\frac{1}{2} - x\right)^2 \log \frac{1-x}{x}\right]\left(\frac{1-x}{x}\right)^{1/2}.$$

Our problem now is to choose M so that these Φ's will be monotonic non-decreasing in the interval $0 \leqq x \leqq 1$. Inspection shows that this can be done by setting $M = 1/2$.

Using this value of M we readily find from (5) that the corresponding functions, which we will call $g(z)/q_2$ and $h(z)/q_1$, are

$$\frac{g(z)}{q_2} = \left(z + \frac{1}{2}\right)\left(z + \frac{1}{2} - (z^2 + z)^{1/2}\right) \log \frac{z+1}{z} + z - (z^2 + z)^{1/2},$$

$$\frac{h(z)}{q_1} = \left(z + \frac{1}{2}\right)^2 \left[1 + \left(\frac{z+1}{z}\right)^{1/2}\right] \log \frac{z+1}{z} - z - 1 - (z^2 + z)^{1/2}.$$

Both of these functions, then, have Stieltjes expansions.

But by (44) and (35),

$$z = \frac{p^2}{2p+1}, \quad q_1 = \frac{p}{2p+1}, \quad q_2 = p.$$

Hence

$$G(p) \equiv g(z) = \frac{2p^2 + 2p}{(2p+1)^2}\left(p^2 + p + \frac{1}{2}\right) \log \frac{p+1}{p} - \frac{p^2}{2p+1}$$

and

$$H(p) \equiv h(z) = \frac{2p^2 + 2p + 1}{(2p+1)^2}\left(p^2 + p + \frac{1}{2}\right) \log \frac{p+1}{p} - \frac{p^2 + p}{2p+1}$$

must have expansions in the form (39).

We see at once that

$$F^*(p) = G(p) + H(p) + p,$$

the extra term p being an entire function which generates a zero $\Phi(x)$. It occurs because $F^*(p)$ has a pole at ∞.

This result is an entirely satisfactory one from the standpoint of design, for not only $G(p)$ and $H(p)$, but $\Delta(p) \equiv p$ as well, are constructible, the resultant structure consisting of an inductance and two ladder networks in series.

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A CORRECTION

By E. T. Bell

Dr. M. A. Basoco has kindly pointed out to me that the isomorphism established in my paper in the last issue of this Bulletin,* is not, as there stated, partial, but complete. In the second sentence of §3, page 323, the word *not* should be deleted. Obviously the relation stated is identical with (B). On page 324 (8), the words *but not* should be replaced by the word *and*. The error arose from replacing a certain intermediary function of my first draft by $\lambda(z)$; the discarded function did not satisfy (B), but it failed to give a satisfactory analogy in some other respects.

As the matter now stands, it is clear that *more* is proved than was stated in my paper: *my revised Lucas functions are identical with the stated Weierstrassian normal forms having the given invariants.* Complete isomorphism *is* possible, provided we attend in Halphen's theorem to the degenerate case of one period infinite.

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* *A partial isomorphism between the functions of Lucas and Weierstrass*, this Bulletin, vol. 35 (1929), pp. 321–325.

THE APPROXIMATION OF HARMONIC FUNCTIONS BY HARMONIC POLYNOMIALS AND BY HARMONIC RATIONAL FUNCTIONS*

BY J. L. WALSH

1. *Introduction.* The following theorem of Weierstrass is classical.

Let the function $f(\theta)$ be continuous for all values of the argument and periodic with period 2π. Then if an arbitrary positive ϵ be given, there exists a trigonometric polynomial which differs from $f(\theta)$ at most by ϵ; that is, the inequality

$$(1) \qquad \left| f(\theta) - \sum_{k=0}^{n} (a_k \cos k\theta + b_k \sin k\theta) \right| \leqq \epsilon$$

holds for all values of θ.

An equivalent statement of the conclusion of this theorem is that *$f(\theta)$ can be expanded in a series of the form*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_{nk} \cos k\theta + b_{nk} \sin k\theta),$$

which converges uniformly for all values of θ. This is of course a general fact, that if a given function can be uniformly approximated as closely as desired by a linear combination of other functions, then that function can be expanded in a uniformly convergent series of which each term is a linear combination of those other functions, and conversely. This fact is easy to prove and will be frequently used in the sequel.

If in the (x, y)-plane we introduce polar coordinates (r, θ) and consider the function $f(\theta)$ defined on the unit circle C, Weierstrass's theorem refers to the approximation of a function $f(\theta)$ continuous on C by trigonometric polynomials, or what is the same thing, by polynomials of the form

* An address delivered by invitation of the program committee at the meeting of the Society in New York, February 23, 1929.

$$\sum_{k=0}^{n} r^k(a_k \cos k\theta + b_k \sin k\theta). \tag{2}$$

A polynomial of the form (2) is of course harmonic, for the expressions $r^k \cos k\theta$ and $r^k \sin k\theta$ are respectively the real parts of the analytic functions z^k, $-iz^k$, and can be written as harmonic polynomials in x and y.

If we interpret Weierstrass's theorem as yielding a uniform *expansion* rather than uniform approximation of $f(\theta)$, we have a sequence $\{p_n(r, \theta)\}$ of functions each of type (2) which converges uniformly to the function $f(\theta)$ on C and likewise converges uniformly in the *closed* interior of C. For a necessary and sufficient condition for uniform convergence on C is that if an arbitrary ϵ be given, there exist M so that

$$| p_m(r,\theta) - p_n(r,\theta) | \leqq \epsilon, \quad m,n > M,$$

holds on C. But this inequality holding for $r=1$ holds also for $r \leqq 1$, since the functions $p_n(r, \theta)$ are harmonic for $r \leqq 1$ and a harmonic function has its maximum and minimum values on the boundary of the region considered. The limit $f(r, \theta)$ of this sequence $\{p_n(r, \theta)\}$ is thus harmonic* interior to C, continuous in the corresponding closed region, and on C coincides with the given continuous function $f(\theta)$. That is, by Weierstrass's theorem we have established the existence of the function $f(r, \theta)$, the solution of the Dirichlet problem for the region interior to C and for arbitrary continuous boundary values $f(\theta)$; the solution of the Dirichlet problem is known to be unique.

The problem of the uniform approximation to $f(\theta)$ *on* C by harmonic polynomials is precisely equivalent to the problem of the uniform approximation to $f(r, \theta)$ *in the closed interior of* C, for a function harmonic in a region has no maximum or minimum interior to that region. Inequality (1) is equivalent to the inequality

* A function $u(x, y)$ is harmonic in a region if it is continuous there, together with its first and second partial derivatives, and if $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$. A function is harmonic at a point if it is harmonic throughout a neighborhood of that point.

$$\left| f(r,\theta) - \sum_{k=0}^{n} r^k (a_k \cos k\theta + b_k \sin k\theta) \right| \leqq \epsilon, \qquad r \leqq 1 ;$$

each of these inequalities implies the other.

The discussion just given suggests the general problem of studying the approximation to a given continuous function on an arbitrary point set C by means of harmonic polynomials or more generally by harmonic rational functions, particularly in the more interesting cases that the given function is harmonic in the interior points of C, or that C is itself the boundary of a region. The present paper is devoted to results on this general problem, chiefly a *report* so far as concerns approximation by polynomials, but a detailed exposition of results on approximation by more general rational functions, for these results are in the main new. As in the special case already considered to some extent, namely the theory of approximation to functions of a single real variable by trigonometric polynomials, there are particularly three special problems to be taken up here: first, the general problem of the possibility of approximation, corresponding to Weierstrass's theorem as already mentioned; second, the problem of approximation by *special types* of polynomials corresponding for example to expansion in Fourier's series; and third, the study of the *degree* of approximation, the study of the relation between the continuity properties of the functions approximated and the least maximum error for approximation by a polynomial of degree n. We shall examine these problems in some detail in the order given, first the problems for approximation by polynomials and later the corresponding problems for approximation by more general rational functions.

In the theory of functions of the complex variable $z=x+iy$, similar problems arise in connection with the approximating on a given point set of a given continuous or analytic function by means of polynomials in z or rational functions of z. This analogous theory frequently suggests results in the present theory of harmonic functions, and

may even furnish methods of proving those results; but the connection between the two theories is not always simple, and some situations in each theory have no analogs in the other. The two theories both have interesting applications to and connections with harmonic and analytic continuation, conformal mapping, the study of the Dirichlet problem, and topology.

2. *General Approximation by Harmonic Polynomials.* The fundamental theorem in the approximation of analytic functions by polynomials is due to Runge [1]*:

Let $f(z)$ be a function of z, analytic in a simply-connected region† C of the z-plane which does not contain the point at infinity in its interior. Then $f(z)$ can be developed in C in a series of polynomials in z, which converges uniformly on any closed point set interior to C.

We shall use Runge's theorem to prove the following analog for harmonic functions.‡

Let $u(x, y)$ be a function of (x, y), harmonic in a simply-connected region C of the (x, y)-plane which does not contain the point at infinity in its interior. Then $u(x, y)$ can be developed in C in a series of harmonic polynomials in (x, y), which converges uniformly on any closed point set interior to C.

Let $v(x, y)$ be a function conjugate to $u(x, y)$ in C; the function $v(x, y)$ is single-valued in C if one considers only an element of $v(x, y)$ and its harmonic extensions along curves interior to C. Runge's theorem as applied to the function $f(z) = u(x, y) + iv(x, y)$ yields a development of $f(z)$ in polynomials in z. The series whose terms are the real parts of the respective terms of that development represents the function $u(x, y)$ and has the required properties with respect to convergence.

* The numbers in square brackets refer to the works of the authors indicated, in the bibliography at the close of this paper.

† A region is a connected set of interior points. A region plus its boundary points forms the corresponding *closed* region.

‡ Compare Walsh [1, p. 198]. Bergmann [1] uses similar reasoning, but does not bring out clearly the uniformity of the convergence of the series of harmonic polynomials.

The essential content for our present purposes of the theorem just proved may also be stated as follows.

If the function $u(x, y)$ is harmonic in a closed Jordan region, then $u(x, y)$ can be developed in that closed region in a uniformly convergent series of harmonic polynomials in (x, y).

The present writer has proved, however, results much more general than this [1], regarding the uniform development of arbitrary harmonic functions. We state a number of these results, simply mentioning at this time the fact that the proof is based on Lebesgue's important work [1] on harmonic functions. We shall go into more detail regarding methods in §5, when we take up the more general problem of development in terms of harmonic rational functions.

Let C be an arbitrary limited simply-connected region of the (x,y)-plane. A necessary and sufficient condition that an arbitrary function harmonic in C, continuous in the corresponding closed region, be uniformly developable in the closed region by harmonic polynomials in (x, y), is that the boundary of C be also the boundary of an infinite region.

Let C be an arbitrary limited closed point set. A necessary and sufficient condition that an arbitrary function continuous on C be uniformly developable on C by harmonic polynomials in (x, y), is that C be the boundary of an infinite region.

An arbitrary limited closed point set C divides the plane in general into a number of regions; in particular one of these, D, is infinite. A continuous function $u(x, y)$ defined on C is in particular defined on the boundary B of D, and on this boundary can, by the theorem just stated, be uniformly developed in harmonic polynomials. This expansion of $u(x, y)$ on B converges uniformly on B, hence uniformly on the entire point set E which is complementary (with respect to the entire plane) to D, and therefore defines a function $U(x, y)$ on the entire set E. The set C is a subset of E. The functions $u(x, y)$ and $U(x, y)$ coincide on B. *A necessary and sufficient condition that $u(x, y)$ be uniformly developable on C in harmonic polynomials is that $u(x, y)$ and $U(x, y)$ should coincide on C.*

These theorems obviously include Weierstrass's theorem on the approximation to an arbitrary continuous function by trigonometric polynomials. These theorems include also Weierstrass's classical theorem that an arbitrary function continuous on a closed interval of the axis of reals can be uniformly approximated on that closed interval as closely as desired by a polynomial in the real variable. For if the point set C is chosen as such an interval, the only harmonic polynomials $1, r\cos\theta, r\sin\theta, \cdots, r^n\cos n\theta, r^n\sin n\theta, \cdots$ not vanishing identically on C can be written on C in the form $1, x, x^2, \cdots$, so that on C a harmonic polynomial is a polynomial in x.

With these results, the most important questions regarding the expansion of arbitrary functions on *limited* point sets are answered. The corresponding questions for unlimited point sets have apparently not been treated previously. On some unlimited point sets C only the polynomial 1 of the set $\{r^n\cos n\theta, r^n\sin n\theta\}$ is uniformly bounded, so the only functions continuous on C which can be uniformly approximated on C by harmonic polynomials are constants. On certain other unlimited point sets C only a finite number of the harmonic polynomials $\{r^n\cos n\theta, r^n\sin n\theta\}$ are continuous, and a necessary and sufficient condition that a function continuous on such a point set C be uniformly developable on C by harmonic polynomials is that the function should be a linear combination of those particular harmonic polynomials. On other unlimited point sets C an infinity of the harmonic polynomials $\{r^n\cos n\theta, r^n\sin n\theta\}$ are continuous, so necessary and sufficient conditions for the expansion of arbitrary functions on such point sets C are not obvious, nor can they be obtained without modification of the methods hitherto employed.

It is illuminating to contrast the present problem with the problem of approximating given analytic or continuous functions by means of polynomials in the complex variable. It is not yet known what is the most general region C such that an

arbitrary function analytic interior to C and continuous in the corresponding closed region can be approximated uniformly in the closed region as closely as desired by polynomials in the complex variable. It can be shown by an example [Walsh, 1, p. 203] however, that this most general region C is not the most general region such that an arbitrary function harmonic interior to C and continuous in the corresponding closed region can be approximated uniformly in the closed region as closely as desired by harmonic polynomials. *Unlimited* point sets C are easy to treat for approximation by polynomials in the complex variable. Every such polynomial not a constant becomes infinite on C, so the only continuous functions which can be uniformly approximated on C are constants, and every such function can be so approximated.

3. *Harmonic Polynomials Belonging to a Region.* It is natural to inquire whether more precise results than those just considered are obtainable regarding *expansions in a given region in terms of a particular set of harmonic polynomials belonging to that region.* For instance, if the region is chosen as the interior of the unit circle C, then an arbitrary function $f(x, y)$ harmonic in the closed interior of C can be expanded on the circumference in a series of Fourier:

$$(3)\quad f(x,y) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta),$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f \cos k\theta \, d\theta, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f \sin k\theta \, d\theta,$$

where the integrals are computed over C. The series converges uniformly on C. The series

$$(4)\qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k(a_k \cos k\theta + b_k \sin k\theta)$$

is a series of harmonic polynomials which converges uni-

formly on C and hence converges uniformly throughout the closed interior of C. The sum of the series (4) coincides with $f(x, y)$ on C and hence throughout the interior of C as well.

If we start not with the unit circle but a more general Jordan curve C as the boundary of our region, can there be found harmonic polynomials $\{p_k(x, y)\}$ such that an arbitrary function harmonic on and within C can be expanded in a series

$$\sum_{k=0}^{\infty} a_k p_k(x, y) \tag{5}$$

which converges uniformly in the closed interior of C? The answer here is affirmative, if the curve C is analytic. In fact, Faber [1], Fejér [1], Szegö [1], Carleman [1], Bergmann [1], Bochner [1], and others have studied polynomials $\{s_k(z)\}$ in the complex variable z belonging to such a region and have shown that an arbitrary function $f(z)$ analytic in the closed region can be expanded in a series

$$\sum_{k=0}^{\infty} c_k s_k(z)$$

which converges uniformly in the closed region. A suitable set of harmonic polynomials of the kind we desire can be found by separating $s_k(z)$ into its real and pure imaginary parts. Thus, let the function $u(x, y)$ be given harmonic in the closed interior of C; there exists a function $v(x, y)$ conjugate to it, likewise harmonic in this closed region. Then by the properties of the polynomials $s_k(z)$, there exists an expansion

$$u(x, y) + iv(x, y) = \sum_{k=0}^{\infty} (c_k' + ic_k'')(s_k'(x, y) + is_k''(x, y)),$$

where $c_k = c_k' + ic_k''$, $s_k(z) = s_k'(x, y) + s_k''(x, y)$. From this we derive the series

$$u(x, y) = \sum_{k=0}^{\infty} [c_k' s_k'(x, y) - c_k'' s_k''(x, y)], \tag{6}$$

so we have an expansion of type (5)* by setting

$$p_{2k}(x,y) = s_k'(x,y), \qquad p_{2k+1}(x,y) = s_k''(x,y).$$

In particular if the derivatives of the polynomials $s_k(z)$ are orthogonal with respect to the area interior to C, the formulas for the coefficients in (6) are remarkably simple in form, as Bergmann has shown [1].

There are, however, various other ways of determining sets of harmonic polynomials belonging to a given region. Let us consider the plane of the auxiliary variables (x', y'), and map conformally the interior of C onto the interior of the unit circle Γ of the (x', y')-plane. A function harmonic in the closed interior of C corresponds to a function harmonic in the closed interior of Γ, and can be uniformly expanded in the closed interior of Γ in a series (4) of harmonic polynomials in (x', y'). The harmonic polynomials $r^n \cos n\theta$, $r^n \sin n\theta$ in the (x', y')-plane correspond in the (x, y)-plane to functions harmonic in the closed interior of C; moreover these functions in the (x, y)-plane can be replaced by harmonic *polynomials* in (x, y) which differ only slightly from them, without essentially altering the convergence properties of series expansions of arbitrary functions in terms of them. More explicitly, we state the following theorem [Walsh, 2].

Let C be a simple finite analytic curve in the (x, y)-plane. Then there exist harmonic polynomials $\{p_k(x, y)\}$ such that if $f(x, y)$ is defined and continuous on C and on C is of bounded variation, then $f(x, y)$ can be developed into a series

$$f(x,y) = \sum_{k=0}^{\infty} a_k p_k(x,y), \tag{7}$$

which converges uniformly in the closed interior of C. Series (7) thus represents a function harmonic interior to C, continuous in the corresponding closed region, and having the value $f(x, y)$ on C. There exist continuous functions $\{q_k(x, y)\}$ on C

* Removal of brackets in series (6) can be justified in the cases to which reference has been made.

with which the polynomials $\{p_k(x, y)\}$ *form a biorthogonal set:*

$$\int_C p_k(x, y)q_m(x, y)ds = \begin{cases} 0, & \text{if } k \neq m, \\ 1, & \text{if } k = m. \end{cases}$$

The coefficients of (7) *are given by the formulas*

$$a_k = \int_C f(x, y)q_k(x, y)ds \; ; \tag{8}$$

the functions $q_k(x, y)$ *depend on* C *but not on* $f(x, y)$.

If the function $f(x, y)$ *is known merely to be continuous on* C, *then the series* (7), *where the* a_k *are given by* (8), *converges throughout the interior of* C, *uniformly on any closed point set interior to* C; *if summed by the method of Cesàro this series converges uniformly on and within* C *and thus represents a solution of the Dirichlet problem for the region interior to* C *and the boundary values* $f(x, y)$.

If the function $f(x, y)$ *is of bounded variation on* C, *but not necessarily continuous, then the series* (7), *where the* a_k *are given by* (8), *converges at every point of the closed region, uniformly on any closed point set interior to* C. *The function represented is bounded in the closed interior of* C *and approaches the boundary values* $f(x,y)$ *continuously at every point of continuity of* $f(x, y)$.

This theorem as stated applies only to an analytic curve C, but the writer has some further results, as yet unpublished, which apply to much more general curves.

Still another method of defining a set of harmonic polynomials belonging to a given region bounded by a rectifiable Jordan curve is that of orthogonalization. We begin with the harmonic polynomials

$$1, \; r\cos\theta, \; r\sin\theta, \; r^2\cos 2\theta, \; r^2\sin 2\theta, \; \cdots$$

and orthogonalize them with respect to the given curve C. This yields a set of polynomials $\{p_k(x, y)\}$ such that

$$\int_C p_k(x, y)p_m(x, y)ds = \begin{cases} 0, & k \neq m, \\ 1, & k = m. \end{cases}$$

An arbitrary function $f(x, y)$ defined on C can be expanded formally in terms of these polynomials:

$$(9) \qquad f(x,y) \sim \sum_{k=0}^{\infty} a_k p_k(x,y), \qquad a_k = \int_C f(x,y)p_k(x,y)ds.$$

This orthogonalization method was used by Szegö [1] in the corresponding case of polynomials in the complex variable. The results in the present case have been established by Merriman [1], who determines the asymptotic formulas for the polynomials $\{p_k(x, y)\}$, and shows, under suitable restrictions, that the series (9) converges interior to C or even uniformly in the closed interior of C. The sum of the series is of course the solution of the Dirichlet problem for the continuous boundary values $f(x, y)$.

It would be of interest to investigate the corresponding problems where the harmonic polynomials are orthogonalized not with respect to arc length on the boundary, but with respect to the area of the region interior to C [compare Bergmann 1, Carleman 1]. The former method has the theoretical advantage, however, of not requiring for expansion the knowledge of the values of the harmonic function except on the boundary of the region. Another general problem which deserves treatment is the detailed study of the convergence on C of expansions in terms of the special polynomials of Faber and others, either for the case of analytic or of harmonic functions, where the given function to be expanded is not known to be analytic or harmonic in the *closed* region.

4. *Degree of Approximation.* We have considered thus far the *possibility* of approximation by harmonic polynomials and the expansion in terms of a *particular set* of harmonic polynomials. We turn now to consideration of the *degree* of approximation, that is, the relation between the properties of the function approximated, with regard to existence of derivatives etc. on the one hand, and the asymptotic properties of the maximum error in the best approximation

by a harmonic polynomial of degree n,* on the other hand. A simple result [Walsh, 3] here is the analog and consequence of the corresponding result in approximation of analytic functions by means of polynomials in the complex variable.

Let C be an arbitrary closed Jordan region of the (x, y)-plane, and let $w=\phi(z)$, $z=x+iy$, be a function which maps conformally the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve $|\phi(z)|=R$, $R>1$, that is, the transform in the z-plane of the circle $|w|=R$.

A necessary and sufficient condition that an arbitrary function $u(x, y)$, defined in C, be harmonic in the (closed) region C is that there should exist harmonic polynomials $p_n(x, y)$ of degree n, $n=0, 1, 2, \cdots$, and numbers M, $R>1$, such that the inequalities

$$|u(x,y)-p_n(x,y)| \leqq \frac{M}{R^n}, \tag{10}$$

where M and R are independent of n and of (x, y), should be valid for every point (x, y) of C.

If the polynomials $p_n(x, y)$ are given so that (10) is satisfied for every (x, y) of C, the sequence $\{p_n(x, y)\}$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, so the function $u(x, y)$ is harmonic throughout the interior of C_R.†

If $u(x, y)$ is given harmonic in the closed region interior to C_ρ, the polynomials $p_n(x, y)$ can be chosen to satisfy (10) with $R=\rho$, for (x, y) in C.

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* The polynomial $a_0x^n+a_1x^{n-1}y+\cdots+a_ny^n+b_0x^{n-1}+b_1x^{n-2}y+\cdots$ is considered to be of degree n, but if the term degree is used *in the restricted sense*, is of degree n if and only if at least one of the coefficients a_i is different from zero.

† Here and below we tacitly assume that if $u(x, y)$ is not originally supposed to be defined on the entire point set considered, then the definition in the new points is to be made by harmonic extension—or what amounts to the same thing—by means of the convergent series of harmonic polynomials.

This theorem seems to be the only one in the literature concerning degree of approximation by harmonic polynomials, except in the case that C itself is a circle, when approximation by harmonic polynomials reduces to approximation on the circumference by trigonometric polynomials, for which well known results have been obtained by Bernstein, Jackson, Montel, de la Vallée-Poussin, and others [see de la Vallée-Poussin, 1]. There is obviously occasion here for further investigation. If $u(x, y)$ is given harmonic within, continuous on and within the Jordan curve C, what is the relation between the maximum error for the best approximation of $u(x, y)$ on C by a harmonic polynomial of degree n on the one hand, and the continuity properties of the curve C and of the function $u(x, y)$ on C on the other hand? If $u(x, y)$ is given harmonic within, continuous on and within C_R, what is the relation between the maximum error for the best approximation on C by a harmonic polynomial of degree n on the one hand, and the continuity properties of the curve C and of the function $u(x, y)$ on the curve C_R on the other hand?

Let us stop for a moment to consider the Tchebycheff harmonic polynomial for the function $u(x, y)$ on a point set C, that is, the harmonic polynomial $p_n(x, y)$ of degree n for which the maximum $|u(x, y) - p_n(x, y)|$, (x, y) on C, is least. It is convenient here to refer to a general theorem due to Haar [1], which deals with approximation on a given point set C of a given function $u(x, y)$ by linear combinations of given functions $\{u_i(x, y)\}$; the coefficients a_i are to be determined so that the maximum

$$(10') \qquad |u(x,y) - a_1u_1(x,y) - \cdots - a_mu_m(x,y)|$$

is least for (x, y) on C. If C is closed and the u's are continuous, such a determination of the coefficients is possible [Haar, loc. cit.] and the corresponding linear combination of the $u_i(x, y)$ may be called a Tchebycheff polynomial of order m. Haar's theorem asserts—except for incidental restrictions on the continuity of the functions and the

closure of the point set—that *a necessary and sufficient condition for the uniqueness of the Tchebycheff polynomial of order m for approximation on C to an arbitrary function continuous on C, is that no function*

$$A_1u_1(x,y) + \cdots + A_mu_m(x,y), \qquad |A_1| + |A_2| + \cdots + |A_m| \neq 0, \tag{10''}$$

shall vanish at more than $m-1$ *points of C.* It is immaterial whether one studies on the one hand the approximation in a closed Jordan region C by harmonic polynomials of a function harmonic interior to the region, continuous in the closed region, or on the other hand the approximation on the boundary of the region of the boundary values of the harmonic function. For the maximum in C of the function in (10′) must occur on the boundary of C. Fréchet has proved that the Tchebycheff trigonometric polynomial is unique for approximation to an arbitrary continuous function $f(\theta)$ with period 2π on the interval $0 \leqq \theta \leqq 2\pi$. That is, from our present standpoint (compare §1), *the Tchebycheff harmonic polynomial is unique for approximation to a continuous function on a circumference C, or for approximation on and within a circle to a function continuous there, harmonic in the interior.* This statement is true, however, only with a restriction. The Tchebycheff polynomial which is a linear combination of the functions $1, r\cos\theta, r\sin\theta, \cdots, r^n\cos n\theta, r^n\sin n\theta$ is indeed unique, for an arbitrary linear combination (10″) of these functions vanishes on an algebraic curve of degree n, and this curve has at most $2n$ points in common with the circumference C.* In fact, this method proves that *the corresponding Tchebycheff harmonic polynomial is unique for approximation on an ellipse C to a function continuous on C, or for approximation on and within C of a function harmonic interior to C, continuous in the corresponding closed region.*

The Tchebycheff polynomial, which is a linear combination

* The circumference cannot be a branch of this curve, for the function (10″) cannot vanish everywhere on any (limited) Jordan curve.

of the functions $1, r\cos\theta, r\sin\theta, \cdots, r^{n-1}\cos(n-1)\theta, r^{n-1}\sin(n-1)\theta, r^n\cos n\theta$, is not necessarily unique either when C is a circle or an ellipse. Indeed, if $n=1$, the function (10″) is A_1+A_2x, which obviously for suitable choice of A_1 and A_2 vanishes at more than a single point of the circle or ellipse C.

It would be an interesting investigation, to determine what algebraic curves C have this property, that the Tchebycheff harmonic polynomial for approximation to an arbitrary function continuous on C is unique. This is a subject dealing with the real intersections of plane algebraic curves, in which a single point of intersection counts merely as a single point, in spite of singularities and multiple points (in the usual sense of the term) of either curve.

Whether or not the Tchebycheff harmonic polynomial $t_n(x, y)$ of degree n is unique, a Tchebycheff polynomial $t_n(x, y)$ of degree n exists, and we can derive certain properties of the sequence $\{t_n(x, y)\}$. In the notation of the previous theorem, we shall prove the following facts.

Let the function $u(x, y)$ be harmonic in the closed interior of C, and have at least one singularity on the curve C_ρ but no singularity interior to C_ρ. Then a sequence $\{t_n(x, y)\}$ of Tchebycheff harmonic polynomials for the function $u(x, y)$ considered in the closed interior of C (or what is essentially the same, considered on C itself), whether or not the Tchebycheff polynomial is unique, converges throughout the interior of C_ρ, and uniformly on any closed point set interior to C_ρ. The limit of the sequence throughout the interior of C_ρ is $u(x, y)$. The sequence can converge uniformly in no region C_ρ, with $\rho'>\rho$.

The proof is immediate. There exists, by the theorem already stated, some sequence $\{p_n(x, y)\}$, where $p_n(x, y)$ is of degree n, such that we have

$$|u(x,y)-p_n(x,y)| \leqq \frac{M}{R^n}, \qquad (x,y) \text{ in } C, \tag{10}$$

provided merely that $R<\rho$. If (10) is valid for the poly-

nomial $p_n(x, y)$, it is also valid if $p_n(x, y)$ is replaced by $t_n(x, y)$:

$$| u(x,y) - t_n(x,y) | \leqq \frac{M}{R^n}, \qquad (x,y) \text{ in } C,$$

provided still that $R<\rho$. Again by virtue of the theorem stated, the sequence $\{t_n(x, y)\}$ must converge interior to every C_R for which $R<\rho$, hence throughout the interior of C_ρ, uniformly on any closed point set interior to C_ρ. In particular, if the function $u(x, y)$ has no singularity except at infinity, the sequence converges at every point of the plane, uniformly on any limited closed point set. In the general case, the sequence $\{t_n(x, y)\}$ cannot converge uniformly on any curve $C_{\rho'}$ for which $\rho'>\rho$, for then it would likewise converge uniformly in a region containing C_ρ, which has on it a singularity of the function $u(x, y)$.

The theorem just established is the analog of a theorem due to Faber [2, p. 105] for the case of approximation to an analytic function by polynomials in the complex variable. The proofs are, however, different, and the present proof applies without essential change in that other situation, even if C is a Jordan arc or a certain more general point set, instead of a Jordan region.

It will be noticed that the sequence $\{t_n(x, y)\}$ may converge for certain points (x, y) exterior to C_ρ. Let us suppose for instance that the region C is symmetric on the x-axis and that the function $u(x, y)$ satisfies the equation $u(x, y) = -u(x, -y)$. Then a given $t_n(x, y)$ may be replaced by another Tchebycheff polynomial $t_n'(x, y)$ of degree n which satisfies the equation $t_n'(x, y) = -t_n'(x, -y)$ and which approximates $u(x, y)$ on C as closely as does $t_n(x, y)$. If we have

$$| u(x,y) - t_n(x,y) | \leqq \epsilon, \qquad (x,y) \text{ on } C,$$

we have likewise by symmetry

$$| u(x, -y) - t_n(x, -y) | \leqq \epsilon, \qquad (x,y) \text{ on } C,$$

that is,

$$| u(x,y) + t_n(x,-y) | \leqq \epsilon, \qquad (x,y) \text{ on } C.$$

If we set

$$t_n'(x,y) = \frac{t_n(x,y) - t_n(x,-y)}{2},$$

the harmonic polynomial $t_n'(x, y)$ obviously satisfies the functional equation considered, and we have the inequality

$$| u(x,y) - t_n'(x,y) | \leqq \epsilon, \qquad (x,y) \text{ on } C;$$

that is to say, the polynomial $t_n'(x, y)$ is as good an approximation to $u(x, y)$ on C as is $t_n(x, y)$.* Interpret $t_n'(x, y)$ as a linear combination of the functions $r^n \cos n\theta, r^n \sin n\theta$, hence as a linear combination of the functions $r^n \sin n\theta$. The sequence $\{t_n'(x, y)\}$ converges for $\theta = 0$ for *all* values of r, no matter how small C may be or where the singularities of $u(x, y)$ may lie.

We have thus far restricted our entire discussion of the degree of approximation by harmonic polynomials to the consideration of point sets C which are regions. That is probably the simplest case; the study of the same problems where C is, let us say, a Jordan arc, is more complicated. Complication arises because a harmonic polynomial may vanish identically on C, so that convergence on C of a sequence of harmonic polynomials, even so that (10) is satisfied on C, does not imply convergence of the sequence for points not on C. Let us treat here in detail the simplest pos-

* We have essentially proved here that if the function $u(x, y)$ can be uniformly expanded by harmonic polynomials on a point set C which is symmetric in the x-axis and if we have $u(x, y) = -u(x, -y)$, then $u(x, y)$ can likewise be uniformly expanded on C by harmonic polynomials $p_n(x, y)$ which satisfy the equation $p_n(x, y) = -p_n(x, -y)$.

Another theorem of the same general nature easily proved by methods already used elsewhere for functions of a complex variable is that if an arbitrary function $u(x, y)$ can be uniformly expanded by harmonic polynomials on a point set C, then $u(x, y)$ is the uniform limit of a sequence of harmonic polynomials each of which is equal to $u(x, y)$ in n arbitrary preassigned points of C.

sible case, namely that C is an interval of the axis of reals; the result is also valid if C is any line segment.

Suppose that $u(x, y)$ is defined on C and that a set of polynomials $\{p_n(x, y)\}$ exists so that (10) holds on C for $R>1$. On C a harmonic polynomial of degree n is a polynomial in x of degree n. Inequality (10), valid for a polynomial $p_n(x, y)$ in x of degree n, implies the convergence of this sequence of polynomials *considered as polynomials in the complex variable* x throughout the interior of C_R (which is defined here as already indicated for a *region* C), uniformly on any closed point set interior to C_R, and hence represents interior to C_R an analytic function of the complex variable x. This is indeed a theorem due to Bernstein, and in the present case the curve C_R is a certain ellipse whose foci are the extremities of the interval C. It follows that *there exists a function* $U(x, y)$, *harmonic interior to* C_R, *coinciding on* C *with the given function* $u(x, y)$. There are two important differences between this result and the result established for the case that C is a Jordan region. First, we have not shown in the present case, nor is it necessarily true, that the original given sequence of harmonic polynomials $\{p_n(x, y)\}$ converges everywhere interior to C_R. Second, the function $U(x, y)$ is not uniquely determined by the requirements of being harmonic interior to C_R and coinciding on C with the given function $u(x, y)$; if any such function $U(x, y)$ is at hand, we may find another by adding to it an arbitrary function harmonic interior to C_R and vanishing on C.

Reciprocally, if it is desired to establish (10) when the function $u(x, y)$ is given harmonic on and within C_R, that can always be done if C is an arbitrary Jordan arc, or indeed a much more general point set. The result follows from the corresponding result for the development of analytic functions in terms of polynomials in the complex variable [compare Walsh, 3].

Before we leave the subject of approximation by harmonic polynomials and turn to harmonic rational functions, we mention another topic which seems not to have been treated

in the literature and yet which deserves to be investigated, namely, interpolation by means of harmonic polynomials. When does a harmonic polynomial of degree n exist which takes on preassigned values at $2n+1$ points? When is it unique? What of the asymptotic character of the polynomial as the number of points becomes infinite? What application is there to approximation and expansion, if these points are chosen on the boundary of a region? These questions have been answered in special cases, corresponding to trigonometric interpolation [see Faber, 3, Jackson, 1, de la Vallée-Poussin, 1], and also for the analogous problem of interpolation by polynomials in the complex variable [Fejér, 1]. The former case yields satisfactory results (compare §1) for interpolation by harmonic polynomials either on a circumference or on and within a circle, but for more general situations the questions seem still to be untouched.*

5. *General Approximation by Harmonic Rational Functions.* The results thus far established for approximation by harmonic polynomials have precise analogs for approximation of arbitrary harmonic functions by harmonic rational functions; these analogs will now be treated in the same order, and the proofs follow, in the main, the proofs for the simpler case. The results in the present case, however, are mostly new to the literature and must therefore be treated here in some detail. This newness explains the apparent lack of balance between the treatment of the two cases of approximation, by harmonic polynomials and by more general harmonic rational functions.

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* These results of Fejér do yield simple theorems immediately. Thus we may make the following statement.

Let C be an arbitrary Jordan curve, and let the function $u(x, y)$ be harmonic on and within C. Then there exists a sequence of harmonic polynomials $\{p_n(x, y)\}$ of respective degrees n coinciding with $u(x, y)$ at n points P_i of C, which converge uniformly on and within C to the function $u(x, y)$. These n points P_i can be chosen as points of C which correspond, under conformal mapping of the exterior of C on the exterior of a circle K so that the points at infinity correspond to the vertices of a regular polygon of n sides inscribed in K.

Let the function $u(x, y)$ be harmonic in the closed region C bounded by two analytic Jordan curves C_1 and C_2, with C_2 interior to C_1 and the origin interior to C_2. Green's formula

$$u(x,y) = \frac{1}{2\pi}\int_{C_1+C_2}\left[u\frac{\partial \log r}{\partial n} - \log r\frac{\partial u}{\partial n}\right]ds,$$

where the integrals are to be taken in the positive sense with respect to the region, n being the inner normal, breaks up the function $u(x, y)$ into a function harmonic on and interior to C_1, plus a function harmonic on and exterior to C_2, plus a multiple of $\log r$, where $r^2 = x^2 + y^2$ [see Osgood, 1, pp. 642–644, Walsh, 1, p. 206]. Consequently the function $u(x, y)$ can be approximated in the closed region C as closely as desired by a harmonic polynomial in (x, y) plus a harmonic polynomial in

$$\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

plus a multiple of $\log r$. A similar situation obtains if C is bounded not by two Jordan curves but by n Jordan curves, and is not materially altered if these bounding curves are not analytic, nor if $u(x, y)$ is not harmonic in the closed region, but is continuous in the closed region and harmonic in the interior.

We are in a position however to establish a more general* theorem.

Let C be a closed point set which does not contain the point at infinity and which contains no region of infinite connectivity not included in a larger region of finite connectivity belonging to C. Let $f(x, y)$ be an arbitrary function continuous on C and

* More general, that is, so far as concerns the point sets and functions considered; actually less specific in one regard, for the older theorem splits up the given function into $2n-1$ functions, of which $n-1$ are logarithms and left unchanged in the approximation. The other n functions are harmonic in simply-connected regions in whose interiors C lies, and *each* of these n functions can be uniformly approximated or expanded in terms of harmonic rational functions in the corresponding simply-connected region.

harmonic in the interior points of C. Then if an arbitrary positive ϵ be given, there exists a harmonic function $\phi(x, y)$, namely, a polynomial in (x, y) and in

$$\left(\frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2}, \frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2}\right),$$

such that we have

$$(11) \quad \left| f(x,y) - \phi(x,y) - a_1 \log r_1 - a_2 \log r_2 - \cdots - a_m \log r_m \right| < \epsilon, \quad (x,y) \text{ in } C,$$

where the a_i are suitable constants and $r_i^2 = (x-x_i)^2+(y-y_i)^2$. Here the points (x_i, y_i), $i = 1, 2, \cdots$, are exterior to C, and can be preassigned, one in each of the regions R into which C separates the plane, although all of such preassigned points do not necessarily appear in (11).

In particular if C has no interior points, an arbitrary function $f(x, y)$ continuous on C can be so approximated.

Let us outline the proof; the method is essentially due to Lebesgue. Let K be a circle which contains C. There exists a function $F(x, y)$ continuous on and within K and which coincides with $f(x, y)$ on C. There exists a polynomial $p(x, y)$ in (x, y) which is not necessarily harmonic but which throughout K differs from $F(x, y)$ by less than $\epsilon/3$. Divide the plane into squares and continue subdivision indefinitely, by halving the sides of the squares already constructed, so that we determine a sequence of closed point sets S_i, each of which consists of a finite number of regions each bounded by a finite number of non-intersecting Jordan curves, of such a nature that each S_i contains C in its interior, so that each S_i contains its successors, but so that every point exterior to C is exterior to some S_i. By suitable modification of these point sets S_i if necessary, we can make sure that no two of the closed regions composing any particular S_i have a point in common. Let $h_i(x, y)$ be the function harmonic throughout the interior of S_i, continuous on S_i, and coinciding on the boundary of S_i with the function $p(x, y)$.

Then we have [Walsh, 1, p. 199] $\lim_{i\to\infty} h_i(x, y) = p(x, y)$ uniformly on the boundary points of C. That is, in particular we can choose k so that $|h_k(x, y) - p(x, y)| < \epsilon/3$ uniformly on the boundary of C. But on C the function $h_k(x, y)$ can be uniformly approximated by a function of the sort considered in the theorem:

$$(12) \quad \left| h_k(x,y) - \phi(x,y) - a_1 \log r_1 - \cdots - a_m \log r_m \right| < \frac{\epsilon}{3}, \quad (x,y) \text{ on } C,$$

and the points (x_i, y_i) lie exterior to C. In fact, if the points (x_i, y_i) are not preassigned, we can [loc. cit., p. 208] satisfy (12) on the entire point set S_k. If the points (x_i, y_i) are preassigned, we can make the approximation (12) not on the point set S_k but on a point set S_k' of the same connectivity as S_k, bounded by a finite number of non-intersecting Jordan curves, which is contained in S_k, which contains C, but which contains precisely those of the preassigned points (x_i, y_i) which lie in regions R lying entirely in S_k. In this new approximation (12), which holds in S_k' and hence in C, we use only the preassigned points (x_i, y_i).

Combination of the inequalities obtained yields (11) uniformly for all points on the boundary of C. But all the functions in the left-hand member of (11) are harmonic in the interior points of C, continuous on the corresponding closed point set, and such a function has no maximum or minimum in an interior point. Hence inequality (11) holds for *all* points of C, and the theorem is established.

We mention explicitly that it is not true that an arbitrary function continuous in a limited closed region and harmonic interior to the region can be uniformly approximated in that region as closely as desired by a harmonic rational function, either with or without logarithmic terms as considered in (11). Consider for example the region C formed from the circle $x^2+y^2<1$ by cutting out the line segment $y=0$, $-1/2 \leqq x \leqq 1/2$; let $f(x, y)$ be the function harmonic interior

to this region, continuous in the closed region, zero on the circumference, and unity on the line segment. The function $f(x, y)$ cannot be uniformly approximated in the closed region. For the approximating functions have only isolated singularities [Osgood 1, p. 680], are continuous without exception in the closed region (else are not uniformly bounded), hence are harmonic in the closed region. These approximating functions can be uniformly approximated as closely as desired in the closed region C by harmonic polynomials in (x, y), so $f(x, y)$ can also be uniformly approximated in C as closely as desired by a harmonic polynomial in (x, y). But if such a polynomial differs from $f(x, y)$ by less than ϵ for points on $x^2+y^2=1$, that polynomial differs from zero by less than ϵ for all points $x^2+y^2<1$, which is a contradiction for $x=y=0$ if $\epsilon<1/2$.

The general theorem we have proved, culminating in inequality (11), is obviously not an exhaustive discussion of its subject-matter. If C is a closed region of infinite connectivity, is it true that an arbitrary function $f(x, y)$ harmonic interior to C and continuous in the closed region, can in the closed region be uniformly approximated as closely as desired, as in (11)? Is our general theorem true without any restriction as to regions of infinite connectivity? Is it true that if C is an arbitrary closed point set without interior points, then an arbitrary function $f(x, y)$ continuous on C can be uniformly approximated on C as closely as desired by a rational harmonic function *without logarithmic terms*?

Even though we are not in a position to answer this last question, there are specific closed point sets C on which an arbitrary continuous function can certainly be approximated as closely as desired by a harmonic rational function. In fact, *an arbitrary closed point set C which consists of a finite number of Jordan arcs which do not divide the plane into an infinite number of regions has this property.* More generally, [compare Walsh, 5] *if C is such a point set that an arbitrary function continuous on C can be approximated on C as closely as desired by a rational function of the complex variable, then*

C also has the property considered. For let $f(x, y)$ be the given (real) function; a rational function $p(z)$ of z exists so that we have $|f(x, y)-p(z)|<\epsilon$, (x, y) on C, where ϵ is preassigned. Hence we have also $|f(x, y)-r(x, y)|<\epsilon$, (x, y) on C, where $r(x, y)$ is a rational harmonic function of (x, y), the real part of $f(z)$.

We do not attempt to approximate the most general function harmonic in a multiply-connected region C by harmonic rational functions without the use of logarithmic terms, for that is impossible, as we shall now prove, if the closed region cannot be considered as a closed simply-connected region.* We choose C limited, so there exists an analytic Jordan curve J interior to C, whose interior contains points not belonging to the closed region C; let us assume the origin to be such a point, so that the function $\log r$, where $r^2=x^2+y^2$, is harmonic in the closed region C. Assume the approximation possible, so that $\log r$ can be expanded in C in a uniformly convergent series of harmonic rational functions. These rational functions have only isolated singularities and are continuous in C, hence harmonic throughout the interior of C. Differentiate this series term by term in the direction of the normal to J, and integrate the resulting series over J term by term [Osgood 1, pp. 652–653]. For each term of the series, the result of this process is zero [Osgood 1, p. 680], but for the function $\log r$ the result is 2π. We thus have the contradiction $2\pi=0$, and the statement is established.

The entire discussion we have given enables us to answer the question as to whether a function given on an arbitrary closed point set C can be uniformly approximated on that point set by harmonic rational functions plus logarithmic terms, with singularities in assigned points not belonging to C, provided that no region of infinite connectivity is involved. The facts and proofs are so similar to the corresponding facts and proofs for the case of approximation by harmonic polynomials [Walsh, 1] that they are omitted.

* Already proved [Walsh 1, p. 206] in a special case.

It is not our purpose to study non-uniform expansion of harmonic functions, but one result is now interesting and yet so obvious that we mention it here; it is still an open problem to determine in general what functions can be expanded not necessarily uniformly in terms of harmonic rational functions either with or without logarithmic terms.

Let C_0 be an arbitrary open point set, which may be empty. Let $C_1, C_2, \cdots$ be closed point sets (any or all of which may be empty) mutually exclusive and having no common point with C_0, and on each of the point sets $C_0, C_1, \cdots$ let the function $u(x, y)$ be expansible by harmonic rational functions plus logarithmic terms. Then the function $u(x, y)$ can be expanded on $C_0+C_1+C_2+\cdots$ in a series of rational harmonic functions plus logarithmic terms. The series converges uniformly on each of the point sets $C_1, C_2, \cdots$ on which $u(x, y)$ is uniformly expansible, and if $u(x, y)$ is harmonic on C_0, except possibly for logarithmic singularities or singularities corresponding to rational harmonic functions, then the series converges uniformly on every closed point set contained in C_0.

We use $\psi(x, y)$ generically to denote a rational harmonic function with logarithmic terms, such as occurs in (11). Let $S_1, S_2, \cdots$ be closed point sets, each consisting of a finite number of mutually exclusive regions, each bounded by a finite number of non-intersecting Jordan curves, and such that S_k lies interior to C_0, that S_k lies interior to S_{k+1}, and that every point of C_0 lies in some S_k. Let us suppose

$$u(x,y) = \lim_{n\to\infty} \psi_{kn}(x,y), \qquad [(x,y) \text{ on } C_k,\ k = 0,1,2, \cdots] ,$$

where this limit is approached uniformly if $u(x, y)$ is uniformly expansible on C_k, $k \geqq 1$, and uniformly on every closed point set in C_0 for $k=0$ if that is possible. Then we can determine a closed point set B_{11}, consisting of a finite number of mutually exclusive closed regions each bounded by a finite number of non-intersecting Jordan curves, which contains C_1 but has no point in common with S_1. We can also determine $\psi_1(x, y)$ (see the lemma below) such that

$$|\psi_1(x,y) - \psi_{02}(x,y)| < \tfrac{1}{2}, \qquad (x,y) \text{ on } S_1,$$
$$|\psi_1(x,y) - \psi_{11}(x,y)| < \tfrac{1}{2}, \qquad (x,y) \text{ on } B_{11}.$$

Let B_{12} be a closed point set of the topological simplicity of B_{11}, which contains C_1 but has no point in common with S_2 or C_2. Let B_{21} be a closed point set likewise of the topological simplicity of B_{11}, which contains C_2 but has no point in common with S_2 or B_{12}. Choose $\psi_2(x, y)$ so that we have

$$|\psi_2(x,y) - \psi_{03}(x,y)| < \tfrac{1}{3}, \qquad (x,y) \text{ on } S_2,$$
$$|\psi_2(x,y) - \psi_{12}(x,y)| < \tfrac{1}{3}, \qquad (x,y) \text{ on } B_{12},$$
$$|\psi_2(x,y) - \psi_{21}(x,y)| < \tfrac{1}{3}, \qquad (x,y) \text{ on } B_{21}.$$

We continue this process; in general B_{1n} shall contain C_1 but shall have no point in common with S_n or C_2, C_3, $\cdots$, C_n. The point set $B_{2,n-1}$ shall contain C_2 but shall have no point in common with C_3, $\cdots$, C_n, S_n or B_{1n}. The point set B_{n1} shall contain C_n but shall have no point in common with S_n or B_{1n}, $B_{2,n-1}$, $\cdots$, $B_{n-1,2}$. The function $\psi_n(x, y)$ is then to be chosen so that we have

$$|\psi_n(x,y) - \psi_{0,n+1}(x,y)| < \frac{1}{n+1}, \quad (x,y) \text{ on } S_n,$$
$$|\psi_n(x,y) - \psi_{1n}(x,y)| < \frac{1}{n+1}, \quad (x,y) \text{ on } B_{1n},$$
$$\cdots \qquad \cdots$$
$$|\psi_n(x,y) - \psi_{n1}(x,y)| < \frac{1}{n+1}, \quad (x,y) \text{ on } B_{n1},$$

and the sequence $\{\psi_n(x, y)\}$ has the property required in the theorem. Uniform convergence of the sequence $\{\psi_n(x, y)\}$ on a closed point set belonging to $C_0+C_1+\cdots$ but not necessarily a C_k depends merely on the uniform convergence of the corresponding sequence (or sequences) $\{\psi_{kn}(x, y)\}$ on that point set. The function $u(x, y)$, if harmonic on C_0 except possibly for logarithmic singularities or singularities corresponding to rational harmonic functions, can be chosen arbitrarily on C_0, and the sequence $\{\psi_n(x, y)\}$ converges to the

value $u(x, y)$, uniformly on any closed point set contained in C_0.

If none of the point sets $C_0, C_1, C_2, \cdots$ separates any point of another of those point sets from the point at infinity the theorem is true if the words *harmonic rational functions plus logarithmic terms* are replaced by *harmonic polynomials*. For uniform convergence on any closed point set contained in C_0, however, we require that $u(x, y)$ should be harmonic in C_0 and that C_0 should be composed of mutually exclusive simply-connected regions. In the discussion just given we have had occasion to apply the following lemma.

LEMMA. *If S_1 and S_2 are mutually exclusive closed point sets each consisting of a finite number of mutually exclusive regions each bounded by a finite number of non-intersecting Jordan curves, and if the functions $\psi_1(x, y)$ and $\psi_2(x, y)$ are rational harmonic functions with logarithmic terms, then if $\epsilon > 0$ be given there exists a rational harmonic function $\psi(x, y)$ with logarithmic terms so that we have*

$$|\psi(x,y) - \psi_1(x,y)| < \epsilon, \qquad (x,y) \text{ in } S_1,$$
$$|\psi(x,y) - \psi_2(x,y)| < \epsilon, \qquad (x,y) \text{ in } S_2.$$

For simplicity in the proof we assume that both S_1 and S_2 are limited. We write $\psi_1(x, y) = \psi_1'(x, y) + \psi_1''(x, y)$, where $\psi_1'(x, y)$ has no singularities in S_1 and $\psi_1''(x, y)$ has none in S_2. Similarly set $\psi_2(x, y) = \psi_2'(x, y) + \psi_2''(x, y)$, where $\psi_2'(x, y)$ has no singularities in S_2, and $\psi_2''(x, y)$ none in S_1; this splitting up is possible in the present case, but a slight modification may be necessary, due to the presence of logarithmic terms, if the point at infinity belongs to S_1 or S_2. It is possible [Walsh, 1, p. 208] to determine $\psi'(x, y)$ so that the inequalities

$$|\psi'(x,y) - \psi_1'(x,y) + \psi_2''(x,y)| < \epsilon, \qquad (x,y) \text{ in } S_1,$$
$$|\psi'(x,y) - \psi_2'(x,y) + \psi_1''(x,y)| < \epsilon, \qquad (x,y) \text{ in } S_2,$$

are satisfied. These inequalities, if we set

$$\psi(x,y) = \psi'(x,y) + \psi_1''(x,y) + \psi_2''(x,y),$$

are the inequalities it is desired to establish.

We mention explicitly the general question of the possibility of the expansion of given functions in terms of harmonic polynomials or in terms of harmonic rational functions when the restriction of uniformity of convergence is not made. This question seems not to be completely answered in the literature, although Lavrentieff [1] has recently announced some results without proof, and Hartogs and Rosenthal [1] have published an important paper on the corresponding subject for expansion in terms of polynomials in the complex variable. It is worth noting that Osgood's classical theorem [2] in this corresponding subject has the following analog in the present one.

If there converges in a region R a sequence of functions $\{u_n(x, y)\}$ harmonic in R, then this sequence converges uniformly in some sub-region of R.

From this theorem follows directly Osgood's better known result, that *if there converges in a region R a sequence of functions $\{f_n(z)\}$ analytic in that region, then this sequence converges uniformly in some sub-region of R.* For since the sequence $\{f_n(z)\}$ converges, the sequence $\{u_n(x, y)\}$ of the real parts of these functions converges in R, hence uniformly in some sub-region R' of R. Convergence of the sequence $\{iv_n(x, y)\}$ of the pure imaginary parts of the functions $f_n(z)$ at a single point of R' is now sufficient to ensure convergence of the sequence $\{iv_n\}$ and hence of the sequence $\{f_n(z)\}$ uniformly in any simply-connected closed proper sub-region of R'.

Let us indicate briefly the proof of the theorem for harmonic functions. We prove first, after Montel [1, p. 109], but this is only a modification of Osgood's proof, that the sequence $\{u_n(x, y)\}$ is uniformly bounded in some sub-region of R. Otherwise we should have some $|u_{n_1}(x, y)| > 1$ at some point of R, hence in some sub-region R_1 of R. If the sequence $\{u_n(x, y)\}$ is not uniformly bounded in R_1, we must have $|u_{n_2}(x, y)| > 2$ at some point of R_1, and therefore in some sub-region R_2 of R_1. Proceeding in this way we arrive at

a sequence $\{u_{n_k}(x, y)\}$ greater in absolute value than $\{k\}$ in R_k. There is at least one point common to all the R_k, and at this point the original sequence cannot converge.

From the boundedness of the original sequence in some sub-region R' it follows [compare Osgood, 2] that the first partial derivatives of these functions are uniformly bounded in an arbitrary closed sub-region of R', hence that the functions $\{u_n(x, y)\}$ are equicontinuous in this sub-region and therefore converge uniformly there.

We shall not go into great detail on the subject of rational harmonic functions *belonging to a region*, for the discussion can be made to depend upon the discussion for harmonic polynomials belonging to a region. Let a region C be bounded by the Jordan curves $C_0, C_1, \cdots, C_k$, of which no two have a common point, and so that the curves $C_1, C_2, \cdots, C_k$ lie interior to C_0. Let $u(x, y)$ be an arbitrary function harmonic interior to C and continuous in the closed region. As has already been suggested, Green's formula applied to $u(x, y)$ not for the region C but for a neighboring region C' of the same connectivity interior to C splits up the function $u(x, y)$ into k logarithmic terms with singularities at infinity and at points interior to $C_1, C_2, \cdots, C_k$ respectively, which may be preassigned, plus k functions harmonic exterior respectively to $C_1, C_2, \cdots, C_k$, including the point at infinity, and continuous in the corresponding closed regions, plus a function harmonic interior to C_0 and continuous in the corresponding closed region. If the curves $C_0, C_1, \cdots, C_k$ are suitably restricted, the regions interior to C_0 and exterior to $C_i (i = 1, 2, \cdots, k)$ respectively have associated with them (compare §3) sets of rational functions, in fact polynomials in (x, y) and in

$$\left(\frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2}, \frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2} \right)$$

respectively, and the last mentioned $k+1$ functions can be respectively expanded in terms of these functions.

It would be an interesting problem, however, to construct

for the region C *a single set* of normal orthogonal harmonic rational functions (with also k logarithmic functions) in terms of which an arbitrary harmonic function could be expanded. Here orthogonal may mean (1) with respect to area (that is, surface integral over C) or (2) with respect to length (that is, line integral over $C_0, C_1, \cdots, C_k$). Such a set of functions can easily be constructed by the process of orthogonalization, starting with the logarithmic and rational functions we have been using. If interpretation (2) is used, it is true that *if the curves* $C_0, C_1, \cdots, C_k$ *bounding the region* C *are analytic, and if the function* $u(x, y)$ *is harmonic interior to* C, *continuous in the corresponding closed region, then the formal expansion of* $u(x, y)$ *in terms of these normal orthogonal functions belonging to* C *converges to the value* $u(x, y)$ *throughout the interior of* C, *uniformly on any closed point set interior to* C. This follows from the reasoning as given by Merriman [1]; but questions of asymptotic character of the normal orthogonal functions and of uniform convergence of the formal expansion in the closed region C, are there left unanswered under the present general hypothesis on the region C.

6. *Degree of Approximation by Rational Harmonic Functions.* Our results on the degree of approximation to a harmonic function by rational harmonic functions are to be obtained with the help of the corresponding results on the degree of approximation to an analytic function by rational functions of the complex variable, so as a preliminary study we need to consider the relation between the degree of a rational harmonic function and the degree of the corresponding analytic function. In this discussion we use the word *degree* to indicate degree *in the restricted sense.**

If $u(x, y)$ is a harmonic polynomal of degree n, then its conjugate function $v(x, y)$ is likewise a harmonic polynomial of degree n and hence the function $f(z) = u(x, y) + iv(x, y)$ is a

* Compare §4. The degree of a rational function is the greater of the degrees of numerator and denominator, or the common degree if the two are of the same degree.

polynomial in z of degree n. In fact, we may define $v(x, y)$ by

$$v(x,y) = \int_{(0,0)}^{(x,y)} \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \right) + C,$$

where C is an arbitrary constant. The function $f(z) = u(x, y) + iv(x, y)$ can have no singular points other than those of $u(x, y)$, hence is an entire function. In the Taylor's development of $f(z)$ about the origin, the coefficients of powers of z higher than the nth vanish; otherwise $u(x, y)$ is not a polynomial of degree n.

The situation is somewhat more complicated if $u(x, y)$ is a harmonic rational function not a polynomial. Osgood [1, p. 680] has shown that if $u(x, y)$ is harmonic and rational, then $f(z) = u(x, y) + iv(x, y)$, where $v(x, y)$ is defined by the equation just given, is a rational function of z. If $f(z)$ is given, with no factor containing z common to numerator and denominator,

$$f(z) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}, \qquad a_0 b_0 \neq 0,$$

the real part is found by multiplying numerator and denominator by the conjugate complex quantity of the original denominator,

$$f(z) = \frac{a_0 \bar{b}_0 z^m \bar{z}^n + \cdots + a_m \bar{b}_n}{b_0 \bar{b}_0 z^n \bar{z}^n + \cdots + b_n \bar{b}_n},$$

and then separating into real and pure imaginary parts. The function $u(x, y)$ is then a rational function whose numerator is of total degree $m+n$ or less, whose denominator is of degree $2n$, and with no factor common to numerator and denominator; the actual degree of $u(x, y)$ is $m+n$ or less, or $2n$ according as $m \geqq n$ or $m < n$. Reciprocally, let $u(x, y)$ be given in its lowest terms; denote the degrees of numerator and denominator by p and $2q$ respectively, and those of the numerator and denominator of the corresponding analytic function $f(z)$, which is determined only to within an additive constant, by m and n. A necessary and sufficient condition

that $u(x, y)$ be harmonic at infinity is that $f(z)$ be analytic at infinity, that is, $m \leqq n$; in this case we have $n = q$, $m \geqq p - q$; the actual degree of $u(x, y)$ is $2n$ and that of $f(z)$ is n. A necessary and sufficient condition that $u(x, y)$ be singular at infinity is that $f(z)$ be singular at infinity, that is, $m > n$. Here we may write $f(z)$ as a rational function whose numerator is of degree $n-1$ or less and denominator of degree n plus a polynomial of degree $m-n$. Then $u(x, y)$ is a rational function whose numerator is of degree $2n-1$ or less and denominator of degree $2n$ plus a polynomial of degree $m-n$, or a rational fraction whose numerator is of degree $p = m+n$ and denominator of degree $2q = 2n$; the actual degree of $u(x, y)$ is $m+n$ and that of $f(z)$ is m.

We can prove the following result concerning approximation by harmonic rational functions; henceforth we use the word *degree* in its inclusive sense.

Let S be an arbitrary closed Jordan region of the (x, y)-plane. If there exist rational harmonic functions $r_n(x, y)$, real parts of rational functions of $z = x+iy$ of respective degrees n such that we have*

$$| u(x,y) - r_n(x,y) | \leqq \frac{M}{R^n}, \qquad R > 1,$$

for all points (x, y) of S and for all sufficiently large n, and if the singular points of the functions $r_n(x, y) - r_{n-1}(x, y)$ have no limit point on S, then $u(x, y)$ is the real part of a function $f(z)$ meromorphic on S. If in addition the functions $r_n(x, y)$ have no singularities on S, then $u(x, y)$ is harmonic on S.

Let $w = \Phi(z)$ denote a function which maps the complement of S onto the exterior of the unit circle in the w-plane, so that the points at infinity correspond to each other. Let S_R denote the Jordan curve $|\Phi(z)| = R$, where $R > 1$. If the singularities of the functions $r_n(x, y) - r_{n-1}(x, y)$ have no limit point interior to

* The rational function of z is not determined uniquely by $r_n(x, y)$, but merely to within an additive constant. Nevertheless the degree, and nature and location of the singularities of this rational function are determined by $r_n(x, y)$ uniquely.

S_ρ, then the sequence $\{r_n(x, y)\}$ converges interior to S_ν, where $\nu=(1+\rho R^{1/2})/(\rho+R^{1/2})$, and the convergence is uniform on any closed point set interior to S_ν. Hence $f(z)$ is meromorphic interior to S_ν and if $r_n(x, y)$ has no singularities interior to S_ν, the function $u(x, y)$ is harmonic interior to S_ν.

If the function $r_n(x, y)-r_{n-1}(x, y)$ is the real part of a function which has at most n poles, for n sufficiently large, we may set $\nu=(1+\rho R)/(\rho+R)$, and in particular if the only singularities of $r_n(x, y)$ lie at infinity, we may set $\rho=\infty$, $\nu=R$, so that the sequence $\{r_n(x, y)\}$ converges interior to S_R.

The proof follows the proof of the corresponding theorem [Walsh 3, Theorem I] for approximation by means of polynomials. If the functions $f(z)$ and $t_n(z)$, of which $u(x, y)$ and $r_n(x, y)$ are respectively the real parts, are properly chosen, and if S' is an arbitrary closed region interior to S, then the inequality $|f(z)-t_n(z)|\leqq M'/R^n$ is valid for z in S'. If the region S' is a region bounded by a Jordan curve uniformly near the boundary of S, the function $w=\Phi'(z)$ which maps the complement of S' onto the exterior of the unit circle in the w-plane differs little from the function $w=\Phi(z)$, and the curve $S_R'\colon|\Phi'(z)|=R$ lies uniformly near the curve $S_R\colon|\Phi(z)|=R$. The theorem follows by virtue of the corresponding theorem for approximation to analytic functions by means of rational functions [Walsh 4, Theorem IV].

It is a positive simplification here not to mention the degree of $r_n(x, y)$ except in connection with $t_n(z)$, for such widely differing functions as

$$x,\ \frac{x}{x^2+y^2},\ \frac{3x^2+3y^2+x}{x^2+y^2}$$

belong in the same category.

We turn now to the study of the approximation of functions by rational harmonic functions in multiply connected regions. Here logarithmic terms are essential for the approximation of the most general harmonic functions, as we have seen, so we prove the following theorem.

Let C be a closed region bounded by Jordan curves C_0, C_1, $\cdots$, C_k, such that no two of these curves have a common point and so that C_1, C_2, $\cdots$, C_k lie interior to C_0. If the sequence

$$\begin{aligned} &r_n(x,y) + A_{n1}\log\left[(x-x_{n1})^2+(y-y_{n1})^2\right] \\ &\quad + A_{n2}\log\left[(x-x_{n2})^2+(y-y_{n2})^2\right] \\ &\quad + \cdots + A_{nk}\log\left[(x-x_{nk})^2+(y-y_{nk})^2\right], \\ &\qquad\qquad (n=1,2,\cdots), \end{aligned} \tag{13}$$

converges uniformly in C, where $r_n(x, y)$ is a harmonic rational function of (x, y) with no singularities interior to C, and where the point (x_{ni}, y_{ni}) lies in or on C_i and approaches a limit (x_i, y_i) as n becomes infinite, then $\lim_{n\to\infty} A_{ni}$ exists, $i=1,2,\cdots, k$, and each of the sequences*

$$r_n(x,y),\ A_{ni}\log\left[(x-x_{ni})^2+(y-y_{ni})^2\right],\ (i=1,2,\cdots,k), \tag{14}$$

converges uniformly in any closed region interior to C; if the functions in (14) *have no singularity in C and no limit point of singularities in C, this convergence is uniform in the closed region C.*

The proof is simple; choose an analytic Jordan curve C_i' interior to C and enclosing C_i but none of the curves C_1, C_2, $\cdots$, C_{i-1}, C_{i+1}, $\cdots$, C_k in its interior. The sequence of the derivatives of (13) in the direction of the normal ν to C_i' converges uniformly on C_i', and can be integrated on C_i' term by term [Osgood 1, pp. 652–653, Walsh 1, p. 206]. It follows from the theorem of Osgood already quoted that

$$\int_{C_i'} \frac{\partial r_n(x,y)}{\partial \nu} ds = 0\ ;$$

for we have

$$\int_{C_i'} \frac{\partial r_n(x,y)}{\partial \nu} ds = \int_{C_i'} \frac{\partial s_n(x,y)}{\partial s} ds = 0,$$

* It is of course sufficient if the point (x_i, y_i) lies on or within C_i without the assumption that (x_{ni}, y_{ni}) lies interior to or on C_i; we integrate (as below) over C_i' when n is chosen so large that all points (x_{ni}, y_{ni}) but no points (x_{nj}, y_{nj}), $i \neq j$, lie interior to C_i'.

where $s_n(x, y)$ is a function conjugate to $r_n(x, y)$. The total result of the integration of the nth term of the sequence (13) is then precisely $4\pi A_{ni}$, so $\lim_{n\to\infty} A_{ni} = A_i$ exists. It follows that

$$\lim_{n\to\infty} A_{ni} \log [(x - x_{ni})^2 + (y - y_{ni})^2] = A_i \log [(x - x_i)^2 + (y - y_i)^2],$$

uniformly in any closed region interior to C, and hence it follows that the sequence $\{r_n(x, y)\}$ converges uniformly as asserted.

In the next theorem we shall not trouble to consider sequences so general as (13).

Let C be a closed region bounded by Jordan curves C_0, C_1, $\cdots$, C_k, such that no two of these curves have a common point and such that C_1, C_2, $\cdots$, C_k lie interior to C_0. A necessary and sufficient condition that the function $u(x, y)$ defined in C should be harmonic in (the closed region) C, is that there should exist rational harmonic functions $r_n(x, y)$, real parts of rational functions of z of degrees $(k+1)n$, $n = 0, 1, 2, \cdots$, so that we have

$$(15) \quad \left| u(x, y) - \sum_{i=1}^{k} A_{in} \log [(x - x_i)^2 + (y - y_i)^2] - r_n(x, y) \right| \leqq \frac{M}{R^n}, \quad R > 1,$$

for all points (x, y) of C; here the A_{in} are constants and the points (x_i, y_i) are supposed to lie interior to C_i respectively. It follows that $\lim_{n\to\infty} A_{in} = A_i$ exists if (15) is satisfied.

The sufficiency of this condition follows, as in the proof already given, from Osgood's theorem and from the corresponding result for approximation of analytic functions of the complex variable [Walsh 4, Theorem V]. It is found by the integration that

$$|A_{in} - A_i| \leqq M'/R_n;$$

hence (15) holds if A_{in} is replaced by A_i, provided that M is replaced by a suitable M''. The necessity of the condition is likewise easy to establish. The points (x_i, y_i) may be chosen arbitrarily interior to the curves C_i respectively. Then [com-

pare Osgood 1, pp. 642–644; Walsh 1, p. 206] in C the function $u(x, y)$ may be expressed as the sum of the k logarithmic terms which appear in (15) plus k functions harmonic respectively on and exterior to C_i, and at infinity, for $i=1, 2, \cdots, k$, plus a function harmonic in the closed interior to C_0. The results of §4 can now be applied, and yield the desired rational functions $r_n(x, y)$.

This theorem can be extended by considering (1) variable logarithmic terms, (2) more explicit regions or point sets for the location of the singularities of the functions $r_n(x, y)$, (3) more explicit regions (that is, regions containing C in their interiors) for the harmonic character of $u(x, y)$ and for the convergence of the sequence $\{r_n(x, y)\}$, (4) more general boundaries for regions than Jordan curves. There is comparatively little difficulty involved in making any of these generalizations however, and the essential reasoning involved already appears in the literature (either here or elsewhere), so these generalizations are left to the reader.

It is also possible to study sequences of harmonic rational functions which satisfy such a relation as (15) not in a region but on a Jordan arc or curve. Here the situation is not so simple, as we have indicated in the analogous situation in §4. If (15) holds merely on the unit circle C, for instance, the sequence of harmonic rational functions need not converge elsewhere; for the function $A_n \log r$ vanishes on C and a change in A_n does not alter (15), yet may alter the convergence off of C of the sequence in (15). The rational harmonic functions are not uniquely determined, moreover, by their values on C; the two functions $r^n \cos n\theta$ and $r^{-n} \cos n\theta$ are equal on C. Knowledge even of the *location* of the singularities of a harmonic rational function (such as $ar^n \cos n\theta + br^{-n} \cos n\theta$) and of the value on C does not determine the rational function uniquely. Nor is a function $u(x, y)$ harmonic in a region containing C in its interior determined by its values on C.

If (15) holds on the unit circle C for rational functions $\{r_n(x, y)\}$ with singularities only at the origin and at infinity

and where (x_i, y_i), $i=1$, is the origin, there exists a function $U(x, y)$ harmonic on C, in fact harmonic for $r<R$, and coinciding on C with the values $u(x, y)$. Indeed, the logarithmic term of (15) vanishes on C and hence may be entirely omitted. The rational function $r_n(x, y)$ is a polynomial in

$$r^{-n}\cos n\theta, r^{-n}\sin n\theta, \cdots, 1, \cos\theta, \sin\theta, \cdots, r^n\cos n\theta, r^n\sin n\theta,$$

but the negative powers of r may be changed into positive powers without altering $r_n(x, y)$ or (15) on C, so we are dealing with a sequence of harmonic polynomials, converging like a geometric series on C. This situation has already been treated in §4, and the existence of the required function $U(x, y)$ is established there.

7. *Expansions in Three Dimensions.* The theory of the expansion of harmonic functions in three dimensions is not nearly so far developed as the corresponding theory in two dimensions, but we shall discuss a few results and a few open problems.

It is classic that Poisson's integral yields directly an expansion of functions which are given harmonic interior to the sphere, continuous in the corresponding closed region.

If the function $u(x, y, z)$ is harmonic interior to a sphere and continuous in the corresponding closed region, then interior to the sphere $u(x, y, z)$ can be expanded in a series of harmonic polynomials in (x, y, z), the series converging uniformly in any closed region interior to the sphere.

This theorem will be used in proving the analog of the corollary (§2) to Runge's theorem. This analog has already been proved by Bergmann [1], but only for the case of convex regions. The method we shall use is closely related to that of Bergmann, and both methods are intimately related to that of Runge.

If the function $u(x, y, z)$ is harmonic in a closed region S bounded by a simple closed (limited) surface, then in that closed region the function $u(x, y, z)$ can be approximated uni-

formly as closely as desired by a harmonic polynomial in (x, y, z).

We shall find it convenient to have for reference the following lemma.

LEMMA. *Let the two points* (x_0, y_0, z_0) *and* (x', y', z') *and the simple polygonal curve* C *joining them lie exterior to the closed point set* S. *If the function* $u(x, y, z)$ *can be uniformly approximated in* S *as closely as desired by a harmonic function whose only singularity lies in* (x_0, y_0, z_0), *then this function* $u(x, y, z)$ *can likewise be uniformly approximated in* S *as closely as desired by a harmonic function whose only singularity lies in* (x', y', z') *and which is rational except for the factor* $[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}$.

We give the proof of this lemma for the case that both (x_0, y_0, z_0) and (x', y', z') are finite points, but that is simply a matter of convenience. The proof holds with only obvious changes even if (x', y', z') is the point at infinity,* and it is in this latter form that the lemma will be applied later. When (x', y', z') is the point at infinity, the approximating functions are polynomials in (x, y, z), and the factor $[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}$ does not enter.

Construct a finite sequence of spheres $S_0, S_1, \cdots, S_n$ extending from (x_0, y_0, z_0) to(x', y', z'), so that no sphere contains in its interior a point of S, but so that successive spheres of the sequence have a region common to them. The point (x_0, y_0, z_0) shall lie interior to S_0 and so shall another particular point (x_1, y_1, z_1) of C. The sphere S_1 shall contain (x_1, y_1, z_1) in its interior and likewise another particular point (x_2, y_2, z_2) which lies on C between (x_1, y_1, z_1) and (x', y', z'). We proceed in this way to construct $S_2, S_3, \cdots, S_{n-1}$, finally arriving at the sphere S_n which contains in its interior (x_n, y_n, z_n) and (x', y', z'). The construction of these spheres is surely possible, for it is no loss of generality to assume that C is composed of merely a finite number of line

* It is to be remembered that vanishing at infinity is a necessary condition that a function should be harmonic there.

segments and has the length l. Assume likewise that the nearest distance from C to a point of S is δ. Choose the sequence of points (x_0, y_0, z_0), (x_1, y_1, z_1), $\cdots$, (x_n, y_n, z_n), (x', y', z') on C such that the greatest distance between two successive points is less than $\delta/2$. The sphere S_i may be taken as the sphere with center (x_i, y_i, z_i) and diameter δ.

Let an arbitrary positive ϵ be given. We choose the harmonic function $r_0(x, y, z)$, whose only singularity lies in (x_0, y_0, z_0), such that we have

$$\left| u(x,y,z) - r_0(x,y,z) \right| < \frac{\epsilon}{n+2}, \qquad (x,y,z) \text{ in } S.$$

This choice is possible by hypothesis. We next choose the harmonic function $r_1(x, y, z)$, whose only singularity lies in (x_1, y_1, z_1), rational except for the factor $[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2]^{-1/2}$, such that we have

$$\left| r_0(x,y,z) - r_1(x,y,z) \right| < \frac{\epsilon}{n+2}, \qquad (x,y,z) \text{ in } S. \tag{16}$$

We prove by means of an inversion in the unit sphere whose center is (x_1, y_1, z_1) that such choice is possible:

$$\xi = \frac{x - x_1}{r^2}, \quad \eta = \frac{y - y_1}{r^2}, \quad \zeta = \frac{z - z_1}{r^2}, \tag{17}$$

$$r^2 = \frac{1}{\rho^2} = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

The function $\rho_0(\xi, \eta, \zeta) = (1/\rho)\, r_0(x, y, z)$, where ξ, η, ζ, x, y, z are connected by the relations (17), is a harmonic function of (ξ, η, ζ) on and within S_0', the transform of the sphere S_0 under the inversion. Then by the theorem already stated, the function $r'(\xi, \eta, \zeta)$, a polynomial in (ξ, η, ζ), can be determined so that we have

$$\left| \frac{1}{\rho} r_0(x,y,z) - r'(\xi,\eta,\zeta) \right| < \frac{\epsilon}{(n+2)d}, \ (\xi,\eta,\zeta) \text{ interior to } S_0',$$

where d is the distance from (x_1, y_1, z_1) to the farthest point of S_0'. This inequality implies the inequality

$$| r_0(x,y,z) - \rho r'(\xi,\eta,\zeta) | < \frac{\epsilon}{n+2}, \quad (x,y,z) \text{ exterior to } S_0.$$

But $(1/r)r'(\xi, \eta, \zeta)$ is a harmonic function of (x, y, z), rational except for the factor $1/r$ and with its only singularity in the point (x_1, y_1, z_1); hence its equal $\rho r'(\xi, \eta, \zeta)$ may be identified with the desired function $r_1(x, y, z)$ and yields (16).

This same process can now be continued, and yields the additional inequalities

$$| r_1(x,y,z) - r_2(x,y,z) | < \frac{\epsilon}{n+2}, \qquad (x,y,z) \text{ in } S,$$

. . . . . .

$$| r_n(x,y,z) - r_{n+1}(x,y,z) | < \frac{\epsilon}{n+2}, \qquad (x,y,z) \text{ in } S.$$

Hence we have finally

$$| u(x,y,z) - r_{n+1}(x,y,z) | < \epsilon, \qquad (x,y,z) \text{ in } S,$$

where $r_{n+1}(x, y, z)$ represents a harmonic function whose only singularity lies in (x', y', z') and is, except for the factor $[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}$, rational in (x, y, z). That is, the lemma is established.

We need a second lemma in the proof. For the sake of simplicity we state and prove the lemma for integrals of a function of a single variable, but the proof holds without essential change for multiple integrals, and to double integrals we shall apply the lemma. Likewise in the proof we consider merely a single parameter α, but no difficulty is introduced by the appearance of several, and several parameters appear in the application we shall make.

LEMMA. *Let $f(x, \alpha)$ be a continuous real function of the arguments for $a \leqq x \leqq b$, $\alpha_1 \leqq \alpha \leqq \alpha_2$. If an arbitrary positive δ be given, then there exists δ' so that the inequality*

$$(18) \qquad \left| \int_a^b f(x,\alpha)dx - \sum_{i=1}^{n} f(\xi_i,\alpha)(x_i - x_{i-1}) \right| \leqq \delta,$$

$$a = x_0 \leqq x_1 \leqq \cdots \leqq x_n = b, \qquad x_{i-1} \leqq \xi_i \leqq x_i,$$

holds uniformly for all α, $\alpha_1 \leqq \alpha \leqq \alpha_2$, *provided merely that we have* $|x_i - x_{i-1}| \leqq \delta'$.

By the mean value theorem for integrals, we can write

$$\int_a^b f(x,\alpha)dx = \sum_{i=1}^{n} f(\eta_i,\alpha)(x_i - x_{i-1}),$$

where $x_{i-1} \leqq \eta_i \leqq x_i$, and where η_i depends on α. Let us choose δ' so small that the inequality $|\zeta_i - \zeta_i'| \leqq \delta'$ implies

$$|f(\zeta_i,\alpha) - f(\zeta_i',\alpha)| \leqq \delta/(b-a)$$

uniformly for all α, $\alpha_1 \leqq \alpha \leqq \alpha_2$; such a δ' exists by the uniform continuity of $f(x, \alpha)$. Then under the hypothesis $|x_i - x_{i-1}| \leqq \delta'$, we have also $|\xi_i - \eta_i| \leqq \delta'$, from which follows the inequality

$$\left| \sum_{i=1}^{n} f(\eta_i,\alpha)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(\xi_i,\alpha)(x_i - x_{i-1}) \right| \leqq \delta,$$

which is precisely (18).

We are now in a position to prove the theorem. Denote by σ a simple closed surface consisting entirely of a finite number of portions of planes parallel to the coordinate planes, which contains S in its interior, and on and within which the given function $u(x, y, z)$ is harmonic. The value of $u(x, y, z)$ in S is given by Green's integral

$$(19) \qquad u(x,y,z) = \frac{1}{4\pi} \int_\sigma \int \left[\frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \left(\frac{1}{r} \right)}{\partial n} \right] d\sigma,$$

where n represents the exterior normal. Let an arbitrary positive ϵ be given. We divide σ into a finite number n of pieces σ_i, so that each piece consists only of a portion of a plane parallel to one of the coordinate planes, so that each piece σ_i lies interior to a sphere S_i which contains on or within it no point of S, and so that we have

$$(20)\qquad \left|\frac{1}{4\pi}\int_\sigma\int \frac{1}{r}\frac{\partial u}{\partial n}d\sigma - \frac{1}{4\pi}\sum_{i=1}^{n}\frac{1}{r_i}\left(\frac{\partial u}{\partial n}\right)_i\Delta_i\sigma\right| \leqq \frac{\epsilon}{4}, \quad (x,y,z) \text{ in } S,$$

$$\left|\frac{1}{4\pi}\int_\sigma\int u\frac{\partial(1/r)}{\partial n}d\sigma - \frac{1}{4\pi}\sum_{i=1}^{n}u_i\left[\frac{\partial(1/r)}{\partial n}\right]_i\Delta_i\sigma\right| \leqq \frac{\epsilon}{4}, \quad (x,y,z) \text{ in } S,$$

where r_i is the distance from (x, y, z) to a particular but arbitrary point P_i of σ_i, where $(\partial u/\partial n)_i$ is the normal derivative of $u(x, y, z)$ at P_i, where $\Delta_i\sigma$ is the area of σ_i, where u_i is the value of $u(x, y, z)$ at P_i, and where $[\partial(1/r)/\partial n]_i$ is the value of $\partial(1/r)/\partial n$ at P_i. The last expression may also be written as the value at P_i of a partial derivative of $1/r$ with respect not to (x, y, z) but to running coordinates on σ, and this latter form serves better to indicate that we are dealing with a harmonic function of (x, y, z).

Each term of the sums in (20) is a function of (x, y, z) harmonic not merely in the closed region S but throughout the exterior of the corresponding sphere S_i, even at infinity. It is immediately seen by inversion with P_i as center of inversion that in the (x, y, z)-space each of these terms can be approximated by a harmonic function of (x, y, z) whose only singularity lies in P_i and which is rational except for the factor $1/r_i$, where r_i indicates distance measured from P_i. The approximation can be made uniformly as closely as desired in the exterior of the sphere S_i, and hence uniformly as closely as desired in the closed region S. But each point P_i can be joined to the point at infinity by a Jordan curve which does not meet S. Then by the first lemma we can approximate each term of the sums in (20) as closely as desired uniformly in the closed region S by a harmonic polynomial in (x, y, z). That is, we can determine harmonic polynomials $p_1(x, y, z)$ and $p_2(x, y, z)$ such that we have

$$(21)\qquad \left|\frac{1}{4\pi}\sum_{i=1}^{n}\frac{1}{r_i}\left(\frac{\partial u}{\partial n}\right)_i\Delta_i\sigma - p_1(x,y,z)\right| \leqq \frac{\epsilon}{4}, \quad (x,y,z) \text{ in } S,$$

$$\left|\frac{1}{4\pi}\sum_{i=1}^{n}u_i\left(\frac{\partial(1/r)}{\partial n}\right)_i\Delta_i\sigma - p_2(x,y,z)\right| \leqq \frac{\epsilon}{4}, (x,y,z) \text{ in } S.$$

It follows immediately that we have

$$(22)\quad \left| u(x,y,z) - [p_1(x,y,z) - p_2(x,y,z)] \right| \leqq \epsilon,\ (x,y,z) \text{ in } S,$$

and the theorem is proved.

The present proof has the advantage over the proof of §2 of holding (with only minor modifications) in two, three, or n dimensions, and since it does not involve the theory of functions of a complex variable, has the additional advantage of being more satisfactory from the standpoint of the purist. This remark applies also to the results which immediately follow.

Another remark is of interest. If the point (x, y, z) is exterior to the closed surface σ, the integral (19) has the value zero, where r still indicates distance from (x, y, z) to an arbitrary point of σ. Indeed it is a general theorem that if the functions u and v are both harmonic on and within σ, then we have

$$\int_\sigma \int \left[v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right] d\sigma = 0.$$

Thus if the function $u(x, y, z)$ is given harmonic on a point set S composed of the closed interiors of two non-intersecting simple closed surfaces, two new non-intersecting simple closed surfaces σ_1 and σ_2 can be constructed which consist entirely of a finite number of portions of planes parallel to the coordinate planes, such that each point of S lies interior either to σ_1 or σ_2, and such that $u(x, y, z)$ is harmonic on and within both σ_1 and σ_2. Then formulas (19), (20), (21), (22) are valid without any change whatever, if we set $\sigma = \sigma_1 + \sigma_2$. A similar fact holds if S falls into n distinct parts. In fact we can prove the following theorem.

Let S be an arbitrary closed limited point set whose complementary set (with respect to the entire space) is connected. Then if $u(x, y, z)$ is harmonic on S, the function $u(x, y, z)$ can be uniformly approximated on S as closely as desired by a harmonic polynomial in (x, y, z).

There exists a point set σ consisting of a finite number of mutually exclusive closed limited simply-connected regions

$\sigma_1, \sigma_2, \cdots, \sigma_n$ containing all points of S in their interiors, the individual regions bounded entirely by a finite number of portions of planes parallel to the coordinate planes, and such that $u(x, y, z)$ is harmonic on the entire set $\sigma_1+\sigma_2+\cdots+\sigma_n$. The previous method applies without change and the theorem follows.

The following application corresponds to the theorem of Runge; the proof is omitted.

Let C be a point set composed of a finite or infinite number of mutually exclusive simply-connected regions, none of which contains the point at infinity in its interior. If the function $u(x, y, z)$ is harmonic interior to each of these regions, then $u(x, y, z)$ can be expanded on C in a series of harmonic polynomials in (x, y, z), and the series converges uniformly on any closed point set contained in C.

We add one further result on general approximation. A closed region S is said to be *convex with respect to the interior point P* if S is bounded by a surface which is cut in a single point by each ray through P. A region which is convex in the usual sense of the word is convex with respect to each of its interior points.

If the region S is a limited closed region convex with respect to some interior point P, then an arbitrary function $u(x, y, z)$, harmonic interior to S and continuous in the closed region, can be approximated as closely as desired uniformly in the closed region S by a harmonic polynomial in (x, y, z).

Choose P as origin of coordinates, which involves no loss of generality, and consider the transformation

$$x' = \rho x, \qquad y' = \rho y, \qquad z' = \rho z, \qquad \rho > 1,$$

which transforms S into a region S' which contains S in its interior. The function $u(x, y, z)$ is transformed into a new function

$$u'(x, y, z) = u\left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right)$$

defined throughout S'. It follows from the uniform contin-

uity of $u(x, y, z)$ in S that if $\epsilon > 0$ be given, then ρ can be chosen so near to unity that we have

$$| u'(x,y,z) - u(x,y,z) | < \frac{\epsilon}{2}, \qquad (x,y,z) \text{ in } S.$$

For $u'(x, y, z) - u(x, y, z)$ is the same as $u(x/\rho, y/\rho, z/\rho) - u(x, y, z)$, which becomes uniformly small with $\rho - 1$. The function $u'(x, y, z)$ is harmonic in the closed region S, so there exists a harmonic polynomial $p(x, y, z)$ such that we have

$$| u'(x,y,z) - p(x,y,z) | < \frac{\epsilon}{2}, \qquad (x,y,z) \text{ in } S,$$

whence follows

$$| u(x,y,z) - p(x,y,z) | < \epsilon, \qquad (x,y,z) \text{ in } S,$$

and the theorem is established.*

It will be noticed that the study of the approximation to an arbitrary function in the plane is fairly well developed, while for space the study is only begun. In particular the general question as to when a function harmonic interior to a three-dimensional region and continuous in the closed region (or more generally, defined on a given closed point set) can be uniformly approximated in that closed region (or on that point set) as closely as desired by a harmonic polynomial, seems to be well worth investigating. There is an essential difference in methods and results for two and for three dimensions, for in the latter case the Dirichlet problem does not always have a solution, even for simply-connected regions. Methods of solution of the problem of approximation in three dimensions would presumably depend on the study of the solution of the Dirichlet problem for variable regions, and the result would presumably depend to some extent on whether the region is normal [compare Kellogg, 1].

* The corresponding theorem for analytic functions of a complex variable was given independently by Hilb and Szász, Encyklopädie der mathematischen Wissenschaften, vol. 2, C, II, p. 1276, and by Walsh, Transactions of this Society, vol. 26 (1924), pp. 155–170; p. 168, footnote.

We mention other problems for three or more dimensions: the study of harmonic polynomials belonging to a general region*, the degree of approximation to functions by harmonic polynomials, the question of interpolation by harmonic polynomials, approximation by harmonic rational functions,—the solution of all of these problems still lies in the future.

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* For convex regions a treatment is given by Bergmann, 2.

IRREDUNDANT SETS OF POSTULATES FOR THE LOGIC OF PROPOSITIONS*

BY B. A. BERNSTEIN

1. *Introduction.* Alonzo Church† has introduced the interesting notion of "irredundance" in connection with a set of postulates. A set of postulates is irredundant "if the postulates are independent and no one of them can be weakened with respect to the set." I give in this paper a number of irredundant sets of postulates for the logic of propositions.‡

Church gives a "mechanical" method by which any set of independent postulates can be made irredundant: if

(I) $$A, B, C$$

is a set of independent postulates, the set

(II) $$A,\ \textit{if } A \textit{ then } B,\ \textit{if } A \textit{ and } B \textit{ then } C$$

is equivalent to (I) and is irredundant. My postulate sets below have the form (II). But they are free from the irrelevances that (II) usually has when obtained mechanically by Church's rule. The postulates are simple, and in every case the hypothesis is necessary to the conclusion.

There should, of course, be no objection to an irredundant set of postulates simply because it is in form (II).§ For, as Church has shown,¶ *any* irredundant postulate set is equivalent (postulate for postulate) to an irredundant set of form (II). In fact, it can easily be shown that if postulate

* Presented to the Society, San Francisco Section, June 2, 1928.

† Alonzo Church, *On irredundant sets of postulates*, Transactions of this Society, vol. 27 (1925), p. 318.

‡ For the nature of the logic of propositions see B. A. Bernstein, *Sets of postulates for the logic of propositions*, Transactions of this Society, vol. 28 (1926), p. 472. I shall refer to this paper in succeeding footnotes as Paper I.

§ Compare H. M. Gehman, this Bulletin, vol. 32 (1926), p. 159.

¶ Alonzo Church, this Bulletin, vol. 32 (1926), p. 626.

set (I) is irredundant, then set (II) is irredundant and is equivalent to (I), postulate for postulate.* This being so, not only should there be no objection to (II) as a form for irredundant postulates, but (II) should be a *desired normal* form for such postulates. When a set of postulates is in this form, its irredundance can be established by simply exhibiting the necessary independence systems. My postulate sets below are, then, in the normal irredundant form, and they have been so chosen as to be simple and free from the irrelevances usually found in a set blindly made irredundant by Church's mechanical method.

I give three sets of postulates: set A, set B, and set C. The undefined notions in these sets are, respectively,

$$K, \oplus, \odot\ ; \quad K, \mid\ ; \quad K, \oplus.$$

These undefined notions have been used by me before in non-irredundant sets of postulates for the logic of propositions.† For each set I give the necessary independence systems. Since the postulates are in the normal irredundant form, the independence systems for a set of postulates are also irredundance systems for that set and, together with a consistency system, constitute its "complete existential theory."

2. *Set A. Postulates in Terms of Addition and Multiplication.* The set of postulates 1–3 below is the irredundized form of my former set I.‡ The undefined notions for this set are K, $\oplus$, $\odot$.

1. There is a unique K-element 0 such that, for every K-element a,

$$0 \oplus a = a, \quad 0 \odot a = 0.$$

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* Church showed that the postulates (I) *A*, *B*, *C* are equivalent respectively to (III) *A*, *if A then AB*, *if A and AB then ABC*. Set (III) has redundancies which (II) has not.

† Paper I. The sets referred to are numbered I, II, III. Set IV of that paper, consisting of but a single postulate, is of course (trivially) irredundant (and "completely independent").

‡ See Paper I.

2. If the element 0 of Postulate 1 exists, there is a unique K-element $1 \neq 0$ such that, for every K-element a,

$$1 \oplus a = 1, \quad 1 \odot a = a.$$

3. If the K-elements 0, 1 of Postulates 1 and 2 exist, they are the only K-elements.

3. *Irredundance of Set* A. Postulate set A is in the normal irredundant form. Hence the following independence systems $(K, \oplus, \odot)$ prove the irredundance of the set. In these systems, the elements 0, 1 are the Boolean *zero* and the *whole* respectively; e' is the logical *negative* of e; $a+b$ and ab are the logical *sum* and logical *product* respectively.

$\overline{1}$. $K : 0,1$; $\quad a \oplus b = 1$; $\quad a \odot b = ab$.

$\overline{2}$. $K : 0,1$; $\quad a \oplus b = ab$; $\quad a \odot b = 1$.

$\overline{3}$. $K : 0,1,e,e'$; $\quad a \oplus b = a + b$; $\quad a \odot b = ab$.

4. *Set* B. *Postulates in Terms of Rejection.* Postulates $1'$–$3'$ following are the postulates of my former set II* recast in an irredundant form. The notions left undefined are $K, | \cdot$ ($a|b$ may be read: "not-a and not-b").

$1'$. K consists of two elements.

$2'$. If K consists of two elements and a is one of the elements, then $a|a$ is the other element.

$3'$. If K consists of two elements, and a is one of the elements and $a|a$ the other, then $a|(a|a)$ and $(a|a)|a$ are one and the same K-elements.

5. *Irredundance of Set* B. The following independence systems prove the irredundance of set B. The elements and the operations are all Boolean.

$\overline{1'}$. $\quad K : 0$; $\quad a|b = 0$.

$\overline{2'}$. $\quad K : 0,1$; $\quad a|b = ab$.

$\overline{3'}$. $\quad K : 0,1$; $\quad a|b = b'$.

6. *Set* C. *Postulates in Terms of Addition.* Postulates $1''$–$3''$ below are my former set III† made irredundant.

* See Paper I.

† See Paper I.

The undefined notions are K, $\oplus$.

$1''$. There is a unique K-element 0 such that, for every K-element a,

$$0 \oplus a = a.$$

$1''$. If the element 0 of Postulate 1 exists, there is a unique K-element $1 \neq 0$ such that, for every K-element a,

$$1 \oplus a = 1.$$

$3''$. If the K-elements 0, 1 of Postulates 1 and 2 exist, they are the only K-elements.

7. *Irredundance of Set* C. The independence systems proving the irredundance of set C follow. As in the case of sets A and B, the elements and the operations of these systems are all Boolean.

$\overline{1''}$.	$K : 0,1$;	$a \oplus b = 1$.
$\overline{2''}$.	$K : 0,1$;	$a \oplus b = a'b$.
$\overline{3''}$.	$K : 0,1,e,e'$;	$a \oplus b = a + b$.

8. *Consistency and Sufficiency of Sets* A, B, C. The following arithmetic systems* prove that sets A, B, C are consistent:

A. K :0,1 ; $a \oplus b = ab + a + b \bmod 2$; $a \odot b = ab$.

B. K :0,1 ; $a \mid b = ab + a + b + 1 \bmod 2$.

C. K :0,1 ; $a \oplus b = ab + a + b \bmod 2$.

The sufficiency of each of the sets A, B, C for the logic of propositions is obvious from a comparison of A, B, C with my former sets I, II, III,† respectively.

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* See B. A. Bernstein, *Complete sets of representations of two-element algebras*, this Bulletin, vol. 30 (1924), p. 24.

† See Paper I.

THE DETERMINATION OF PLANE NETS CHARACTERIZED BY CERTAIN PROPERTIES OF THEIR LAPLACE TRANSFORMS*

BY C. T. SULLIVAN

In a previous number of this Bulletin,† Professor J. O. Hassler discusses plane nets whose first and minus first Laplace transforms each degenerate into a straight line, and finds their canonical differential equations. The determination of these equations requires the solution of two partial differential equations of the first order in two dependent and two independent variables. Since two of Hassler's conditions are $H=K=0$, it follows from the well known theory of the equation of Laplace that the second of equations (1) below can be integrated by quadratures. It will be shown that the entire system (1) can be integrated by quadratures, and a fundamental set of integrals for the canonical system will be obtained.

In order to avoid unnecessary repetition of explanations and computations already contained in Hassler's paper, or in works to which he refers, we shall confine our attention entirely to the integration of the differential equations of the problem. These equations have the form

$$(1) \qquad \begin{cases} y_{11} = a^{(11)}y_1 + b^{(11)}y_2 + c^{(11)}y, \\ y_{12} = a^{(12)}y_1 + b^{(12)}y_2 + c^{(12)}y, \\ y_{22} = a^{(22)}y_1 + b^{(22)}y_2 + c^{(22)}y, \end{cases}$$

where $y_1=\partial y/\partial u$, $y_2=\partial y/\partial v$, etc., and where $a^{(ij)}$, $b^{(ij)}$, $c^{(ij)}$, $(i, j=1, 2)$, are functions of u and v, defined in the following manner:

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* Presented to the Society, December 27, 1928.

† This Bulletin, vol. 34 (1928), p. 591.

$$
(2)\quad \begin{cases}
a_1^{(22)} + a^{(22)}b^{(11)} = 0, \quad b_2^{(11)} + a^{(22)}b^{(11)} = 0, \\
a^{(11)} = \dfrac{1}{3}(b^{(11)} - 1), \quad c^{(11)} = \dfrac{2}{9}(1 + b^{(11)} + [b^{(11)}]^2) \\
\qquad\qquad - \dfrac{1}{3}(b_1^{(11)} + b_2^{(11)}), \\
a^{(12)} = \dfrac{1}{3}(1 - a^{(22)}), \quad b^{(12)} = -a^{(11)}, \\
c^{(12)} = \dfrac{1}{9}(2a^{(22)}b^{(11)} + a^{(22)} + b^{(11)} - 1), \\
b^{(22)} = -a^{(12)}, \quad c^{(22)} = \dfrac{2}{9}(1 + a^{(22)} + [a^{(22)}]^2) \\
\qquad\qquad - \dfrac{1}{3}(a_1^{(22)} + a_2^{(22)}).
\end{cases}
$$

Thus all the functions $a^{(ij)}$, $b^{(ij)}$, $c^{(ij)}$ are expressible in terms of the two $a^{(22)}$, $b^{(11)}$, and their derivatives, while these are defined by the first two equations. The relations defining $a^{(22)}$, $b^{(11)}$ show that

$$b^{(11)}du + a^{(22)}dv = d\psi(u,v),$$

and

$$a^{(22)} = \psi_2, \qquad b^{(11)} = \psi_1, \qquad a_1^{(22)} = b_2^{(11)} = \psi_{12}.$$

Hence we require the solution of

$$\psi_{12} + \psi_1\psi_2 = 0.$$

On integrating with respect to u and v, in turn, we find

$$e^{\psi}\psi_2 = A(v), \quad \psi = \log(\alpha(u) + \beta(v)),$$

where $\alpha(u)$, $\beta(v)$ are arbitrary functions of their respective arguments. The functions $a^{(22)}$, $b^{(11)}$ are therefore* given by

$$(3)\qquad a^{(22)} = \frac{\beta'}{\alpha + \beta}, \quad b^{(11)} = \frac{\alpha'}{\alpha + \beta},$$

the primes indicating differentiation.

* Hassler acknowledges his indebtedness to G. E. Raynor for these solutions, but does not indicate the method by which they were obtained.

The invariants of the second of equations (1) are

$$-H \equiv a_1^{(12)} - a^{(12)}b^{(12)} - c^{(12)} = -\tfrac{1}{3}(a_1^{(22)} + a^{(22)}b^{(11)}) = 0,$$

$$-K \equiv b_2^{(12)} - a^{(12)}b^{(12)} - c^{(12)} = -\tfrac{1}{3}(b_2^{(11)} + a^{(22)}b^{(11)}) = 0.$$

We now transform equations (1) by means of the substitution

$$y = e^{-V}Y,$$

where

$$V = -\int b^{(12)}du + a^{(12)}dv.$$

The system (1) becomes

$$(4) \qquad \begin{cases} Y_{11} + Y_1 - b^{(11)}(Y_1 + Y_2 + Y) = 0, \\ Y_{12} = 0, \\ Y_{22} + Y_2 - a^{(22)}(Y_1 + Y_2 + Y) = 0. \end{cases}$$

From the second of (4) we find

$$Y = G(u) + H(v).$$

Eliminate $Y_1 + Y_2 + Y$ between the first and third of equations (4), and take cognizance of (3); there results the equation

$$\frac{\alpha'}{G'' + G'} = \frac{\beta'}{H'' + H'}.$$

Since u and v are independent variables, we infer that

$$(5) \qquad G'' + G' = k\alpha, \quad H'' + H' = k\beta',$$

where k is a constant. On integrating these equations by the usual method, we find

$$G(u) = ke^{-u}\int e^{u}\alpha du + le^{-u} + l',$$

$$H(v) = ke^{-v}\int e^{v}\beta dv + me^{-v} + m',$$

where l, m, l', m' are additional constants. Hence the most general expression for Y is

$$Y = k\left[e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv\right] + le^{-u} + me^{-v} + l' + m'.$$

In order that the first and third of equations (4) may be satisfied by this value of Y, the constant $l'+m'$ must vanish. The equations (4) have therefore the following three linearly independent integrals:

$$Y^{(1)} = e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv,$$

$$Y^{(2)} = e^{-u}, \quad Y^{(3)} = e^{-v}.$$

If we substitute for $a^{(12)}$ and $b^{(12)}$ in the expression for V, and effect the resulting quadratures, we find

$$V = -\tfrac{1}{3}(u+v) + \tfrac{1}{3}\log(\alpha+\beta).$$

Consequently we have

$$y = \frac{e^{(u+v)/3}}{(\alpha+\beta)^{1/3}}Y.$$

Finally we have the integrals of the canonical differential equations of the most general plane net possessing the property that its first and minus first Laplace transforms each degenerate to a straight line in the form

$$y^{(1)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}\left[e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv\right],$$

$$y^{(2)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}e^{-u}, \quad y^{(3)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}e^{-v}.$$

Since α and β are arbitrary, it is possible to write these equations in a form entirely free from quadratures.

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ON CERTAIN LOCI OF LINES INCIDENT WITH CURVES AND SURFACES IN FOUR-SPACE

BY B. C. WONG

In this paper is presented a partial list of formulas giving the orders of the ruled hypersurfaces and ruled surfaces whose rulings are incident with given curves and surfaces in 4-space. If the number of incidences is equivalent to six simple conditions, the number of lines having these incidences is finite. The formulas herein presented are analogous to those giving the orders of the surfaces whose rulings are lines satisfying three conditions and the number of lines satisfying four conditions in 3-space.* Some of these formulas are obvious, others require proof. Those that are not herein included are left out for future consideration.

We shall let h be the number of apparent double points, p the deficiency, of a given curve C^m; H the number of apparent double points, P the deficiency, of a 3-space section c^n of a given surface F^n; and t the number of apparent triple points of F^n or the number of lines through a given point meeting F^n three times.†

First consider the hypersurface V^μ whose lines satisfy four conditions. For a line to meet a given curve once is equivalent to two conditions. Hence the ∞^2 lines cutting across two given curves C^{m_1}, C^{m_2} form a hypersurface whose order is obviously

$$\mu = m_1 m_2. \tag{1}$$

* Salmon, *Analytic Geometry of Three Dimensions*, 5th ed., vol. II, §458–475.

† The formula for t is $\mu\nu(\mu-1)(\mu-2)(\nu-1)(\nu-2)/6$ if F^n is the complete intersection of two hypersurfaces of orders μ and ν respectively. If F^n is only a partial intersection, it is necessary to find t by some other means. See B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35 (1929), pp. 339–343.

If the two curves have s points in common, s must be deducted from the above formula. The order, then, is equal to the number of apparent intersections of the two curves.

The order of the bisecant hypersurface of a given C^m is equal to the number of apparent double points of C^m, that is,

$$\mu = h = \frac{(m-1)(m-2)}{2} - p \cdot \tag{2}$$

The multiplicity of C^m on the hypersurface is, as we shall see, $m-2$.

The V^μ whose ∞^2 lines meet a curve C^m and two surfaces F^{n_1}, F^{n_2} each once is of order

$$\mu = 2mn_1n_2. \tag{3}$$

The proof of this formula is analogous to that of the formula for the order of the surface whose rulings meet three given curves in S_3.*

If C^m meets F^{n_1} and F^{n_2} in s_1 and s_2 points respectively and if F^{n_1} and F^{n_2} meet in a curve of order k, then the quantity $s_1n_2+s_2n_1+km$ has to be deducted from the formula.

The bisecants of an F^n that meet a C^m form a locus of order

$$\mu = m[H + n(n-1)/2] = m[(n-1)^2 - P]. \tag{4}$$

The quantity $s(n-1)$ has to be deducted if C^m and F^n have s points in common.

The ∞^2 lines that are incident with four given surfaces of orders n_1, n_2, n_3, n_4 respectively form a V^μ of order

$$\mu = 3n_1n_2n_3n_4\dagger. \tag{5}$$

Each of the four surfaces is of multiplicity equal to the product of the orders of the other three and each of the points common to two of the surfaces is of multiplicity equal to the sum of their orders multiplied by the product of the orders of the other two surfaces.

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* Salmon, loc. cit., §467.

† For the case $n_1=n_2=n_3=n_4=1$, see Bertini, *Projecktive Geometrie Mehrdimensionaler Räume*, 1924, Chap. 8, §§25–36.

The order of the V^μ whose lines meet a given surface F^n twice and two given surfaces F^{n_1}, F^{n_2} each once is

$$(6)\quad \mu = n_1 n_2[2H + n(n-1)/2] = n_1 n_2[(n-1)(3n-4) - 4P]/2.$$

To prove this formula, obtain the order μ' of the hypersurface whose lines are incident with two planes π, π' and with F^n twice. An S_3 through π' meets π in a line l and F^n in a curve c^n. The lines that meet l once and c^n twice form a surface of order $H+n(n-1)/2$. Since l is H-fold on the surface, π is H-fold on $V^{\mu'}$. For the same reason π' is H-fold. Since S_3 meets $V^{\mu'}$ in a composite surface made up of π' counted H-ply and a surface of order $H+n(n-1)/2$, $V^{\mu'}$ is of order

$$\mu' = 2H + n(n-1)/2.$$

Replacing π, π' by F^{n_1}, F^{n_2}, we obtain the formula above.

To obtain the order of the V^μ whose lines are trisecants of an F^n and are incident with an F^{n_1}, we note that, t being the number of apparent triple points on F^n, F^{n_1} is t-fold on the hypersurface. Since $(n-2)(6H+n-n^2)/6$ is the order of the trisecant surface* of an S_3-section of F^n, we have the formula

$$(7)\quad \mu = n_1[t + (n-2)(6H + n - n^2)/6].$$

Now we pass to the scrolls whose rulings are lines satisfying five simple conditions. The order of the scroll of lines meeting a curve C^m once and three surfaces F^{n_1}, F^{n_2}, F^{n_3} each also once is

$$(8)\quad \nu = 3mn_1n_2n_3.$$

To see this, consider the surface $F^{\nu'}$ whose lines meet a given line l and the three surfaces F^{n_1}, F^{n_2}, F^{n_3}. Any S_3 through l meets F^{n_1}, F^{n_2}, F^{n_3} in three curves c^{n_1}, c^{n_2}, c^{n_3} respectively. The order of the surface in S_3 whose lines meet c^{n_1}, c^{n_2}, c^{n_3} is $2n_1n_2n_3$. Since from each point of l, $n_1n_2n_3$ lines can be drawn incident with F^{n_1}, F^{n_2}, F^{n_3}, l is to be regarded as $n_1n_2n_3$-fold

* Salmon, loc. cit., §471.

on $F^{\nu'}$. Hence $F^{\nu'}$ is of order $\nu' = 3n_1n_2n_3$, for it is met by S_3 in the line l counted $n_1n_2n_3$-ply and the $2n_1n_2n_3$ lines that cut across l, c^{n_1}, c^{n_2}, c^{n_3}. If we replace l by C^m, we have formula (8).

The following formulas can be proved in a similar manner:

$$\nu = mn_1[2H + n(n-1)/2] \tag{9}$$

for the order of the scroll whose lines meet C^m once, F^n twice, and F^{n_1} once;

$$\nu = m[t + (n-2)(6H + n - n^2)/6] \tag{10}$$

for the order of the surface whose lines are trisecants of F^n and unisecants of C^m;

$$\nu = n[h + m(m-1)/2] = n[(m-1)^2 - p] \tag{11}$$

for the order of the surface formed by the bisecants of C^m that are incident with F^n; and

$$\nu = 2m_1m_2n \tag{12}$$

for the order of the surface whose rulings meet C^{m_1}, C^{m_2}, F^n each once.

The orders of the surfaces whose generators meet five surfaces of orders n_1, n_2, n_3, n_4, n_5 each once; one surface F^n twice and three surfaces of orders n_1, n_2, n_3 each once; and one surface F^n three times and two surfaces of orders n_1, n_2 each once are, respectively,

$$\nu = 5n_1n_2n_3n_4n_5, \tag{13}$$

$$\nu = n_1n_2n_3(3H + n^2 - n), \tag{14}$$

$$\nu = n_1n_2[t + (n-2)(6H + n - n^2)/3]. \tag{15}$$

Passing to the formulas for the number of lines that satisfy six simple conditions, we first consider the number N of trisecants to a curve C^m. Let C^m be made up of m lines l_i such that each meets its consecutive line, l_1 and l_m being skew. The number of triples of lines that are all skew is equal to the number of lines which meet a proper C^m, assumed rational, three times. This number is $(m-2)(m-3)(m-4)/6$. If C^m is of deficiency p, the quantity $(m-4)p$ is to be de-

ducted. Hence the formula for the number of trisecants to a curve of order m and deficiency p is

$$N = \frac{(m-2)(m-3)(m-4)}{6} - (m-4)p \cdot \tag{16}$$

The number of lines that meet C^m twice and C^{m_1} once is, from (2),

$$N = m_1 h = m_1\left[\frac{(m-1)(m-2)}{2} - p\right]. \tag{17}$$

We now prove that the multiplicity of C^m on its bisecant hypersurface V^h [see formula (2)] is $m-2$. A line l incident with C^m meets V^h in $h-x$ points distinct from C^m, where x is the multiplicity of C^m. There are $h-x$ lines that meet C^m twice and l once. This number plus the number, $(m-2)\cdot(m-3)(m-4)/6-(m-4)p$, of lines that meet C^m three times is equal to the number of lines that meet a curve C^{m+1} of order $m+1$ and deficiency p three times. Hence

$$h - x + \frac{(m-2)(m-3)(m-4)}{6} - (m-4)p$$
$$= \frac{(m-1)(m-2)(m+3)}{6} - (m-3)p,$$

the right-hand member being the result of changing m to $m+1$ in (16). Replacing h by $[(m-1)(m-2)/2]-p$, we have $x=m-2$, which was to be proved.

The following formulas require no explanation. From (1), (3), (4) we obtain, respectively,

$$N = m_1 m_2 m_3 \tag{18}$$

for the number of lines that meet three curves of orders m_1, m_2, m_3, each once;

$$N = 2m_1 m_2 n_1 n_2 \tag{19}$$

for the number of lines incident with two curves of orders m_1, m_2 and two surfaces of orders n_1, n_2;

$$N = mm_1[(n-1)^2 - P] \tag{20}$$

for the number of lines that meet two curves of orders m, m_1 each once and one surface of order n twice. Formulas (5) or (8), (6) or (9), and (7) or (10) yield, respectively,

$$N = 3mn_1n_2n_3n_4 \tag{21}$$

lines meeting a curve of order m once and four surfaces of orders n_1, n_2, n_3, n_4 each once;

$$N = mn_1n_2[2H + n(n-1)/2] \tag{22}$$

lines meeting C^m, F^{n_1}, F^{n_2} each once and F^n twice; and

$$N = mn_1[t + (n-2)(6H + n - n^2)/6] \tag{23}$$

lines that meet F^n three times, F^{n_1} and C^m each once.

We also have the formulas

$$N = nn_1[(m-1)^2 - p] \tag{24}$$

for the number of bisecants of C^m that meet F^n, F^{n_1} each once;

$$N = 5n_1n_2n_3n_4n_5n_6 \tag{25}$$

for the number of lines incident with six surfaces;

$$N = n_1n_2n_3n_4(3H + n^2 - n) \tag{26}$$

for the number of lines meeting F^n twice and four other surfaces each once; and

$$N = n_1n_2n_3[t + (n-2)(6H + n - n^2)/3] \tag{27}$$

for the number of trisecants of F^n that are incident with three other surfaces.

There are other formulas that call for attention. We omit them for the present.

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GROUPS WHICH ADMIT AUTOMORPHISMS IN WHICH EXACTLY THREE-FOURTHS OF THE OPERATORS CORRESPOND TO THEIR INVERSES

BY G. A. MILLER

It is well known that a necessary and sufficient condition that a given group G be abelian is that each of the operators of G correspond to its inverse in one of the possible automorphisms of G, and that in an automorphism of an abelian group all the operators must correspond to their inverses if more than half of them satisfy this condition. Moreover, it is known that in a non-abelian group at most three-fourths of the operators correspond to their inverses in a possible automorphism. The object of the present article is to determine some fundamental properties of the category of groups which separately admit an automorphism in which exactly three-fourths of the operators correspond to their inverses. Such an automorphism will be called a *three-fourths automorphism*.

Suppose that G is a group for which a three-fourths automorphism has been established, and that s_1 and s_2 are any two of its operators which are both non-commutative and correspond to their inverses in this automorphism, while s_α is any operator of G which corresponds to its inverse in the same automorphism. A necessary and sufficient condition that $s_1 s_\alpha$ correspond to its inverse in this automorphism is that s_1 and s_α be commutative. Since $s_1 s_\alpha$ must correspond to its inverse for at least two-thirds of the possible values of s_α it results that s_1 is commutative with exactly one-half of the operators of G, and that these operators constitute an abelian subgroup of G since all of its operators correspond to their inverses in an automorphism. Since an abelian subgroup of index 2 must also correspond

to s_2, and these two subgroups are distinct, it results that G is a non-abelian group which has more than one abelian subgroup of index 2. It therefore contains exactly three such subgroups and has a commutator subgroup of order 2.

In the given three-fourths automorphism of G all of the operators of two of its abelian subgroups of index 2 correspond to their inverses while only half of those of the third such subgroup satisfy the same condition. The remaining operators of this third abelian subgroup obviously correspond in this automorphism to their inverses multiplied by the commutator of order 2. Moreover, the order of G is divisible by 8, and G is the direct product of an abelian group of odd order and a group of order 2^m which admits a three-fourths automorphism. It is obvious that every such direct product admits exactly three three-fourths automorphisms, one for each of its pairs of abelian subgroups of index 2. Hence the following theorem has been established.

THEOREM 1. *If a group admits one three-fourths automorphism it admits exactly three such automorphisms, and it is the direct product of a group of order 2^m which has this property and an abelian group of odd order. Moreover, every such direct product admits a three-fourths automorphism.*

Each of the three three-fourths automorphisms of G is of order 1 or 2, and the three products of these automorphisms in pairs are the three non-identity inner automorphisms of G. As these products are of order 2 the three three-fourths automorphisms of G are commutative and generate a group which is either of order 4 or of order 8. In the former case this group is the group of inner automorphisms of G, and each of the three three-fourths automorphisms of G is an inner automorphism of G, which may be the identity automorphism for one of them, while in the latter case none of the three-fourths automorphisms is an inner automorphism and their continued product is the characteristic outer automorphism of G in which each of the operators of the central of G corresponds to its inverse while each of the remaining operators of G corresponds to its inverse multiplied by the com-

mutator of order 2 contained in G. This continued product is obviously also a characteristic automorphism when the three-fourths automorphisms are inner automorphisms of G.

A necessary and sufficient condition that one of the three three-fourths automorphisms of G be the identity automorphism is that exactly one-fourth of the operators of G have orders which exceed 2. Hence this condition characterizes the groups which are direct products of the octic group and an abelian group of order 2^m and of type $(1, 1, 1, \cdots)$.* The continued product of the three three-fourths automorphisms in this case is the characteristic inner automorphism of order 2 in which all the operators of the subgroup of index 2 generated by the operators of order 4 contained in G correspond to themselves. The other two three-fourths automorphisms in this case are the two non-characteristic inner automorphisms of G. In the other case where the three three-fourths automorphisms of G generate a group of order 4, they must therefore all be inner automorphisms of order 2 and hence G must be the Hamiltonian group of order 2^m. That is, G must be the direct product of the quaternion group and the abelian group of order 2^{m-3} and of type $(1, 1, 1, \cdots)$.

From the preceding paragraph it results that when the three three-fourths automorphisms of G generate a group of order 4 then G is the direct product of an abelian group of order 2^α and of type $(1, 1, 1, \cdots)$, and either the octic group or the quaternion group. When G is the direct product of any abelian group which is not of the given type and one of the two non-abelian groups of order 8 its three three-fourths automorphisms will generate a group of order 8 and of type $(1, 1, 1, \cdots)$, and when the three three-fourths automorphisms of G generate such a group this will also be the case of the direct product of G and any abelian group whatever. While every group which admits three-fourths automorphisms is the direct product of its Sylow subgroups,

* G. A. Miller, *Annals of Mathematics*, vol. 7 (1906), p. 55.

and only one of these Sylow subgroups is non-abelian, this non-abelian subgroup H of order 2^m may itself be the direct product of an abelian group and a non-abelian group. This can obviously not be the case when it has only two independent generators. We shall prove that it is always the case when it has more than three such generators and that it may or may not be the case when it has exactly three such generators.

Let H_0 be the central of H and suppose the operators of H are arranged in co-sets with respect to H_0. We select s_1, s_2 from two distinct co-sets so that their orders are as low as possible, and when their orders are different we assume that the order of s_1 is the larger. If H_0 contains any operator whose order is equal to that of s_1, we select all the operators whose orders are at least equal to this order in any given reduced set of independent generators of H_0. The group generated by these operators has only the identity in common with the group generated by s_1, since s_1 is an operator of lowest order in the co-set to which it belongs. We therefore adjoin s_1 to the operators already selected in the given set of independent generators of H. These operators when s_1 is replaced by s_1^2 appear in a reduced set of independent generators of H_0 if the order of s_1 exceeds 2. If the order of s_1 is 2 the order of s_2 is also 2 and the non-identity commutator of H is an operator of order 2 in the group generated by the selected set of independent generators of H_0. If this operator is not a power of one of the operators of this set they may evidently be re-selected in such a way that it is such a power, and H is the direct product of the group generated by s_1, s_2 and this operator, and the group generated by the rest of the operators in the given set of independent generators of H_0. In what follows we shall therefore assume that the order of s_1 exceeds 2 and hence s_1^2 is an operator in a reduced set of independent generators of H_0.

When the order of s_2 is equal to that of s_1 we may add also s_2^2 to these generators and augment the set thus obtained by other operators in a reduced set of independent generators of H_0. The only case when this cannot be done is when $s_1^2 = s_2^2$

and hence the order of s_1 is 4, but this order may also be 4 when $s_1^2 \neq s_2^2$. In the former case s_1, s_2 generated the quaternion group while they generate a group of order 16 or of order 32 in the latter case. When the order of s_2 is less than the order of s_1 we adjoin to the operators of the given set of reduced independent generators of H_0 others, if any exist, whose orders are equal to or greater than that of s_2. We thus obtain a set of reduced independent generators of H_0 when the order of s_2 is 2. When this order exceeds 2 we add s_2^2 to this set and complete a set of reduced independent generators of H_0 in the usual way. Hence we may state the following theorem.

THEOREM 2. *If H is any non-abelian group of order 2^m which involves more than one abelian subgroup of index 2, then it is possible to select all the operators of a set of independent generators of H in such a way that only one pair of them is composed of non-commutative operators and that the commutator of order 2 is a power of one of these independent generators.*

From this theorem it follows that when the commutator of order 2 in H is a power of s_1 or of s_2, then H is the direct product of the group generated by s_1, s_2, and the group generated by all the remaining operators of the given set of independent generators of H. In all other cases we adjoin to s_1, s_2 the independent generators whose power is the commutator of order 2, and H is then the direct product of the group generated by these three operators and the group generated by the remaining operators in the given set of independent generators of H. This proves the following theorem.

THEOREM 3. *Whenever H has more than three independent generators it is the direct product of an abelian group and a non-abelian group which has at most three independent generators of H for its independent generators.*

When it has three such generators they may be so selected that one of them appears in the central of H.

Whenever a non-abelian group G involves three abelian subgroups of index 2 then either all of these subgroups are

conjugate under the group of isomorphisms of G or two of them are thus conjugate while the third is a characteristic subgroup of G. Hence it results that the group of inner isomorphisms of such a group involves at most one characteristic operator of order 2 under its group of isomorphisms. When the three three-fourths automorphisms of G generate a group of order 8 it therefore results that this group may involve either only one or three operators of order 2 which are characteristic under the group of isomorphisms of G. If it involves only one such operator its other six operators of order 2 are composed of two sets of three conjugate operators under this group of isomorphisms. In the other possible case the four non-characteristic operators of order 2 appear in two sets of two conjugates under the same group.

It was noted above that when the three three-fourths automorphisms of G generate a group of order 4, then the central of G can involve only operators of order 2 besides the identity. If these automorphisms generate a group of order 8 and the central of G is generated by operators of order 2 then the three abelian subgroups of index 2 under G are either all of type $(2, 1, 1, \cdots)$, or two of them are of this type while the third is of type $(1, 1, 1, \cdots)$. In the latter case an operator t which transforms G according to a three-fourths automorphism and has its square in G gives rise to a group which has an abelian subgroup of index 2. This is also obviously the case when t transforms G according to an operator in its group of inner isomorphisms but it cannot happen when the central of G involves operators whose orders exceed 2. This gives rise to the following theorem.

THEOREM 4. *There is one and only one group G of order 2^m, $m>4$, which satisfies the two conditions that its central involves no operator whose order exceeds 2 and that it appears in a group of order 2^{m+1} involving operators which transform exactly three-fourths of the operators of G into their inverses but involving no abelian subgroup of index 2.*

Since the Hamiltonian groups of order 2^m constitute such a well known and important category of groups it may be of interest to note here that it results from the preceding developments that these groups are characterized by the facts that they admit three-fourths automorphisms and that the continued product of these three automorphisms is the identity. This category is also characterized by the fact that it is composed of the groups whose three three-fourths automorphisms are inner automorphisms of order 2. The entire category of Hamiltonian groups may be similarly characterized by the fact that it is composed of the groups whose centrals involve no operators of order 4 and which admit such three three-fourths automorphisms that in their continued product all the operators of the central correspond to their inverses and at least one operator in each central co-set corresponds to itself.

THE UNIVERSITY OF ILLINOIS

THE ROLE OF DIRECTION IN THE VARIATIONS OF A DEFINITE INTEGRAL IN SPACE

BY LOUIS BRAND

1. *Introduction.* In the space problem of the calculus of variations, whether treated in the parametric form as by Bliss and Mason,† or in terms of angle as by Rider‡ or Sakellariou,§ the Euler equations are three in number but connected by a relation. Thus but two of the equations are independent. The object of this paper is to show that two independent Euler equations arise from variations parallel

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* Presented to the Society, April 16, 1927.

† Mason, M. and Bliss, G. A., *The properties of curves in space which minimize a definite integral*, Transactions of this Society, vol. 9 (1908), pp. 440–466.

‡ Rider, P. R., *The space problem of the calculus of variations in terms of angle*, American Journal of Mathematics, vol. 39 (1917), pp. 241–256.

§ Sakellariou, N., *The space problem of the calculus of variations*, Palermo Rendiconti, vol. 44 (1920), pp. 53–68.

to principal normal and binormal, respectively, of the curve giving an extremum. Tangential variations yield nothing but an identity in partial differentiation. The two independent Euler equations may be replaced by a vector equation which involves a certain vector $\boldsymbol{Q}$. This vector plays an important part in the entire theory covering the first and second variations of the integral.

2. *The Euler Equations.* Consider the integral

$$I = \int_0^l F(\boldsymbol{r}, \boldsymbol{t})ds$$

taken along a curve Γ in space joining two fixed points. Here F is a scalar point function of the position vector $\boldsymbol{r}$ and its derivative $d\boldsymbol{r}/ds = \boldsymbol{t}$ (the unit tangent vector) which has continuous partial gradients of the second order with respect to $\boldsymbol{r}$ and $\boldsymbol{t}$. We note in passing that this formulation of the problem applies equally well to variations in space or on a given surface.

Now let the curve Γ, $\boldsymbol{r} = \boldsymbol{r}(s)$, give I an extreme value. We replace Γ by another curve Γ_1:

$$\boldsymbol{r}_1 = \boldsymbol{r} + \epsilon(u\boldsymbol{t} + v\boldsymbol{p} + w\boldsymbol{b}) \tag{1}$$

between the given points, where u, v, w are three independent functions of s of class C' which vanish for $s=0$ and $s=l$, and $\boldsymbol{t}$, $\boldsymbol{p}$, $\boldsymbol{b}$ are unit vectors in the direction of the tangent, principal normal and binormal along Γ, chosen so that $\boldsymbol{tpb}$ form a right-handed set. Thus (1) gives explicit representation to variations in the directions of $\boldsymbol{t}$, $\boldsymbol{p}$ and $\boldsymbol{b}$.

The value of I along Γ_1 is

$$I(\epsilon) = \int_0^l F(\boldsymbol{r}_1, \boldsymbol{t}_1)\phi(s, \epsilon)ds,$$

where

$$\phi(s, \epsilon) = ds_1/ds,$$

and if $\nabla_r F$, $\nabla_t F$ denote partial gradients of F with respect to $\boldsymbol{r}$ and $\boldsymbol{t}$,

$$(2)\quad I'(\epsilon) = \int_0^l \left\{\left[\frac{\partial \boldsymbol{r}_1}{\partial \epsilon}\cdot\nabla_{r_1}F + \frac{\partial \boldsymbol{t}_1}{\partial \epsilon}\cdot\nabla_{t_1}F\right]\phi + F\frac{\partial \phi}{\partial \epsilon}\right\}ds.$$

On differentiating (1) with respect to ϵ and with respect to s and making use of Frenet's formulas:

$$\frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{p}, \qquad \frac{d\boldsymbol{p}}{ds} = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \qquad \frac{d\boldsymbol{b}}{ds} = -\tau\boldsymbol{p},$$

we obtain

$$(3)\quad \frac{\partial \boldsymbol{r}_1}{\partial \epsilon} = u\boldsymbol{t} + v\boldsymbol{p} + w\boldsymbol{b},$$

$$(4)\quad \boldsymbol{t}_1\phi(s,\epsilon) = (1 + \epsilon u' - \epsilon\kappa v)\boldsymbol{t} + \epsilon(\kappa u + v' - \tau w)\boldsymbol{p} + \epsilon(\tau v + w')\boldsymbol{b}$$

where the primes denote differentiation with respect to s. We now differentiate (4) with respect to ϵ:

$$(5)\quad \frac{\partial \boldsymbol{t}_1}{\partial \epsilon}\phi + \boldsymbol{t}_1\frac{\partial \phi}{\partial \epsilon} = (u' - \kappa v)\boldsymbol{t} + (\kappa u + v' - \tau w)\boldsymbol{p} + (\tau v + w')\boldsymbol{b},$$

whence

$$(6)\quad \frac{\partial \phi}{\partial \epsilon} = (u' - \kappa v)\boldsymbol{t}\cdot\boldsymbol{t}_1 + (\kappa u + v' - \tau w)\boldsymbol{p}\cdot\boldsymbol{t}_1 + (\tau v + w')\boldsymbol{b}\cdot\boldsymbol{t}_1.$$

Putting $\epsilon = 0$ in (4), (6) and (5) in turn now gives

$$\phi(s,0) = 1, \qquad \phi_\epsilon(s,0) = u' - \kappa v,$$

$$\left.\frac{\partial \boldsymbol{t}_1}{\partial \epsilon}\right|_{\epsilon=0} = (\kappa u + v' - \tau w)\boldsymbol{p} + (\tau v + w')\boldsymbol{b} = (u\boldsymbol{t})' + (v\boldsymbol{p})' + (w\boldsymbol{b})' + (\kappa v - u')\boldsymbol{t}.$$

Finally we note that for a given $\boldsymbol{r}$, $\boldsymbol{t}$ can only vary over a unit sphere; hence $\boldsymbol{t}\cdot\nabla_t F = 0$. With these results and (3), (2) becomes

$$I'(0) = \int_0^l \{(u\boldsymbol{t} + v\boldsymbol{p} + w\boldsymbol{b})\cdot\nabla_r F + [(u\boldsymbol{t})' + (v\boldsymbol{p})' + (w\boldsymbol{b})']\cdot\nabla_t F + (u' - \kappa v)F\}ds,$$

and by the usual integration by parts

$$I'(0) = \int_0^l \left\{ u\left(\boldsymbol{t}\cdot\nabla_r F - \boldsymbol{t}\cdot\frac{d}{ds}\nabla_t F - \frac{dF}{ds}\right) + v\left(\boldsymbol{p}\cdot\nabla_r F - \boldsymbol{p}\cdot\frac{d}{ds}\nabla_t F - \kappa F\right) + w\left(\boldsymbol{b}\cdot\nabla_r F - \boldsymbol{b}\cdot\frac{d}{ds}\nabla_t F\right)\right\} ds.$$

Putting $v=w=0$, $w=u=0$, $u=v=0$ in turn, the condition $I'(0)=0$ yields the three Euler equations:

$$\boldsymbol{t}\cdot\nabla_r F - \boldsymbol{t}\cdot\frac{d}{ds}\nabla_t F - \frac{dF}{ds} = 0, \tag{7}$$

$$\boldsymbol{p}\cdot\nabla_r F - \boldsymbol{p}\cdot\frac{d}{ds}\nabla_t F - \kappa F = 0, \tag{8}$$

$$\boldsymbol{b}\cdot\nabla_r F - \boldsymbol{b}\cdot\frac{d}{ds}\nabla_t F = 0. \tag{9}$$

3. *The Euler Equation for the Tangential Variation is an Identity.* To show this we differentiate the relation $\boldsymbol{t}\cdot\nabla_t F=0$ with respect to s:

$$\frac{d\boldsymbol{t}}{ds}\cdot\nabla_t F + \boldsymbol{t}\cdot\frac{d}{ds}\nabla_t F = 0.$$

Thus (7) reduces to an identity in partial differentiation,

$$\frac{dF}{ds} = \frac{d\boldsymbol{r}}{ds}\cdot\nabla_r F + \frac{d\boldsymbol{t}}{ds}\cdot\nabla_t F.$$

The two independent Euler equations thus arise from variations parallel to the principal normal and binormal.

4. *The Euler Equation in Vector Form.* Equations (7), (8), (9) are the $\boldsymbol{t}$, $\boldsymbol{p}$ and $\boldsymbol{b}$ components of the vector equation

$$\nabla_r F - \frac{d\boldsymbol{Q}}{ds} = 0, \quad \text{where} \quad \boldsymbol{Q} = F\boldsymbol{t} + \Delta_t F. \tag{10}$$

This equation puts in evidence three cases of immediate integrability, namely when $\nabla_r F$ is zero, parallel to a fixed vector k, or perpendicular to k; in these cases $\boldsymbol{Q}$, $k\times\boldsymbol{Q}$

or $k \cdot \mathcal{Q}$ are respectively constant. Equation (10) is of the second order when $\nabla_t \mathcal{Q} \neq 0$; it degenerates when $\mathcal{Q}$ is a function of r alone, that is, only when F has the form $t \cdot f(r)$.

In a paper presented to the Society,† I have shown that (10) follows directly from the condition $I'(0) = 0$ by using variations of an arbitrary but fixed direction, $r_1 = r + \epsilon v c$, where c is a constant vector. The second variation then takes the form $I''(0) = c \cdot \Phi \cdot c$, where Φ is a tensor of the second order.

To show the part played by the vector $\mathcal{Q}$ in the general theory we merely state the following results. The Weierstrass-Erdmann corner condition is

$$\mathcal{Q}(r, t^+) = \mathcal{Q}(r, t^-).$$

In a *regular* problem the tensor $\nabla_t \mathcal{Q}$, which is planar and symmetric, is not zero; its second scalar invariant (in the sense of Gibbs) must be positive or zero in the case of a minimum. This is the analogue of the Legendre condition.

If $t = t^*$ in a field of extremals and $\mathcal{Q}^* = \mathcal{Q}(r, t^*)$, the Euler equation gives

$$t^* \times \operatorname{rot} \mathcal{Q}^* = 0$$

in this field. The Hilbert invariant integral is

$$I^* = \int \mathcal{Q}^* \cdot t ds.$$

From this we have the fundamental theorem of Weierstrass

$$\nabla I = \int (\mathcal{Q} - \mathcal{Q}^*) \cdot t ds.$$

The integrand $(\mathcal{Q} - \mathcal{Q}^*) \cdot t$ is the E-function. The transversality condition is simply $\mathcal{Q}^* \cdot t = 0$.

In the corresponding problem on a surface of unit normal n, the Euler equation is

† December 1, 1928, *The calculus of variations from a vector standpoint.* See this Bulletin, vol., 35 (1929), p. 153.

$$n \times \left(\nabla_r F - \frac{d\mathcal{Q}}{ds} \right) = 0,$$

when $\mathcal{Q}$ is defined as above. The tensor $\nabla_t \mathcal{Q}$ is now symmetric and linear and its first scalar invariant must be positive or zero in the case of a minimum. In a field of extremals

$$n \cdot \operatorname{rot} \mathcal{Q}^* = 0.$$

The Hilbert integral, the E-function and the transversality condition are the same as before.† The last equation and $\mathcal{Q}^* \cdot n = 0$ show that $\mathcal{Q}^*$ is the surface gradient of a scalar function W. The curves $W = \text{const.}$ are the transversals of the field.

We note, finally, that analogous results hold in the problem of finding an extremum for a surface integral

$$I = \int F(\mathbf{r}, \mathbf{n}) d\sigma$$

bounded by a fixed closed curve. The Euler equation in this case is

$$n \cdot \nabla_r F + \operatorname{div} P = 0 \quad \text{where} \quad P = Fn + \nabla_n F$$

and div denotes a *surface* divergence. That P has the same general form as $\mathcal{Q}$ is apparent. If $n = n^*$ in a field of extremal surfaces and $P^* = P(\mathbf{r}, n^*)$, then

$$\operatorname{div} P^* = 0.$$

The Hilbert integral, the E-function and the transversality condition are respectively

$$\int P^* \cdot n d\sigma, \quad (P - P^*) \cdot n, \quad P^* \cdot n = 0.$$

These results will be developed elsewhere.

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† From this fact we have an immediate proof of a theorem given by Douglas in this Bulletin (vol. 32 (1926), pp. 669–674) to the effect that transversality of an integral relative to a surface Σ is the section by Σ of its space transversality.

EFFECTS OF THE OSCNODE UPON COVARIANT FORMS OF THE RATIONAL PLANE QUARTIC CURVE*

BY J. H. NEELLEY

1. *Introduction.* A recent paper† discusses some effects of the tacnode and ramphoid cusp upon the combinants of two line sections and other associated forms of the rational plane quartic curve. This paper observes those forms for the curve with an oscnode. Well known results have been purposely omitted.

2. *The Ordinary Oscnode.* As previously shown‡ the oscnodal parameters may be taken as $t = \pm a$ if the two tangents of the pencil of lines on the oscnode have contacts at $t = 0$ and $t = \infty$. These tangents and the join of their contacts give the curve the representation

$$(1) \qquad \begin{cases} x_0 = t^4 - a^2t^2, \\ x_1 = t^2 - a^2, \\ x_2 = 4t^3 - 2\sigma t^2 - 4a^2t, \end{cases}$$

where $\sigma = t_1 - a^2t_1^{-1}$, the sum of the parameters of the points of contact of the distinct double line. Two line sections $(\xi x) = 0$, $(\eta x) = 0$ have combinants which are converted into covariant curves of R_2^4 by means of the relations

$$\begin{vmatrix} (a\xi) & (b\xi) \\ (a\eta) & (b\eta) \end{vmatrix} = | \, a \, b \, x \, | \, ,$$

and so forth, where the third-order determinants are formed from equations (1).§ The discriminant of the invariant

* Presented to the Society, September 6, 1928.

† Neelley, *Concerning covariants of the rational plane quartic curve with compound singularities*, this Bulletin, vol. 35, pp. 67–72.

‡ Neelley, *Compound singularities* etc., American Journal of Mathematics, vol. 49, p. 397.

§ Rowe, Transactions of this Society, vol. 12, p. 296.

$\overline{12}^4$ (1, 1) of the pencil $(\xi x)+\lambda(\eta x)=0$ is the quadratic form which becomes the conic g_2,* the envelope of the self-apolar line sections of the curve. The condition that the quartics cut out by (ξx) and (ηx) be derived functions of a quintic is another second-order combinant which becomes the Stahl conic K,† the conic on the vertices of the flex-triangles of the first osculants of R_2^4. The pencil $g_2+\lambda k=0$ is used to study the effects of the singularity with only well known results for this type of the curve.

The fundamental involution is determined by the quartics

$$(2)\quad (\beta t)^4 \equiv 4t^3 + 4a^2t, \quad (\gamma t)^4 \equiv t^4 + 2\sigma t + 6a^2t^2 + a^4.$$

The complete system of covariants of these two quartics has the g_3 of the first quartic and $\overline{12}^4(1, H_2)$ and $\overline{12}^4(2, H_1)$ identically zero. We also note that the form $\overline{12}^2(1, 2)$ is a quartic with the same roots as those of the quartic cut from the curve by the reference line $x_0=0$. We shall observe this form for the special cases of the singularity.

3. *The Oscnodal Cusp.* This type of the oscnode assumes the properties of the ramphoid cusp as well as those resulting from the coincidence of three adjacent nodes. If the nodal parameter is $t=\infty$ and the contact of the distinct (improper) double line on the node is $t=0$, the nodal line, this double line, and a line on $t=0$ can be taken so as to represent the curve by‡

$$(3)\quad \begin{cases} x_0 = e, \\ x_1 = 6ct^2, \\ x_2 = 12at^4 + 12dt, \end{cases}$$

with

$$(4)\quad (\beta t)^4 \equiv 4at, \qquad (\gamma t)^4 \equiv 4at^3 + d.$$

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* Rowe, loc. cit., pp. 298–299; Salmon, *Modern Higher Algebra*, 4th edition, p. 219.

† Stahl, Journal für Mathematik, vol. 101, pp. 300–325.

‡ Winger, American Journal of Mathematics, vol. 36, p. 65.

Consideration of the combinants of (ξx) and (ηx) gives the conics g_2 and K as

$$aex_1^2 + c^2x_0x_2 = 0 \tag{5}$$

and

$$cx_0x_2 = 0 \tag{6}$$

respectively. So K reduces to two sides of the triangle of reference, the oscnodal line being one. Obviously the pencil of conics $g_2+\lambda K$ is a double-contact pencil and the conics of the pencil have contact with R_2^4 at the cusp. The conic on the flexes and the points q_i* is of this pencil and has the equation

$$3\,aex_1^2 - c^2x_0x_2 = 0. \tag{7}$$

As this is on the point (1, 0, 0) and as it has previously been shown that three flexes and one point q coincide at a ramphoid cusp† we have the following theorem.

THEOREM 1. *The oscnodal rational plane quartic curve with a cusp has its points q_i on the distinct double line.*

But more interesting is the fact that the general equation of the Stahl conic N‡ reduces to the square of this distinct double line. Now the conic N may be defined as the locus of a point from which a pencil of lines will cut a pencil of quartics from the curve R_2^4 for which the invariant I_2' vanishes. The line obviously satisfies such a condition as any line $(\lambda x)=0$ and $x_1=0$ give such a pencil of quartics. We have therefore the following result.

THEOREM 2. *The conic N meets the curve in two pairs of coincident points, the points q_i, when the oscnode is a cusp.*

The covariant system of the fundamental quartics (4) has several forms which vanish identically. The quartics

* Rowe, Transactions of this Society, vol. 13, p. 390.

† Neelley, *Compound singularities* etc., American Journal of Mathematic, vol. 49, p. 390.

‡ Stahl, Journal für Mathematik, vol. 101, pp. 300–325.

are self-apolar and each is apolar to the Hessian of the other. This is characteristic of the oscnodal cusp. The ramphoid cusp covariant, the quadratic resulting from operating with the first of the quartics upon the Jacobian of the second and its Hessian,* vanishes here also. Then the quartic $\overline{12}^2(1, 2)$ is the same as that cut out by the line $x_1 = 0$. So again this form is a line section of the curve. The covariant forms of this system are very degenerate and many syzygetic relations exist.

4. *The Knot.* This special form of the oscnode is such that any line on the singular point meets the curve in three coincident points and the oscnodal line meets it in four, all with the same parameter value. The oscnodal line at $t=0$ and the line on the oscnode and the point $t=\infty$ with the tangent at the latter point give R_2^4 as follows:

$$(8) \qquad \begin{cases} x_0 = at^4, \\ x_1 = 4bt^3, \\ x_2 = 6ct^2 + 4dt + e, \end{cases}$$

with

$$(9) \qquad (\beta t)^4 \equiv dt^4 + et^3, \qquad (\gamma t)^4 \equiv ct^3 + dt^2.$$

The conic g_2 has the equation

$$(10) \qquad (4bdx_0 + aex_1)^2 = 0,$$

which is the square of a line on the oscnode. The conic K is the square of the oscnodal line

$$(11) \qquad x_0^2 = 0.$$

So the pencil $g_2 + \lambda K$ is a degenerate one with its vertex at the oscnode. The conic K has all of its intersections with R_2^4 at $t=0$ and since six of those of g_2 and the curve are at the same point the conic on the flexes has at least six intersections with R_2^4 at the oscnode. Now the points q_i are given by $\overline{12}^3(1, 2)$ of the system of the quartics (9) and this form

* Neelley, *Concerning covariants* etc., this Bulletin, vol. 35, p. 72.

is the quadratic $t^2 = 0$. Hence we have the following theorem.

THEOREM 3. *The two points q_1 coincide at the knot on the oscnodal rational plane quartic curve.*

The resultant of the line sections is the very simple equation of the curve given below:

$$(12)\quad a^3ex_1^4 + 16a^2bdx_0x_1^3 + 96ab^2cx_0^2x_1^2 - 256b^4x_0^3x_2 = 0.$$

Again the forms of the system of the fundamental quartics are in many cases dependent. Many relations vanish identically; all of the independent quadratic forms do except $\overline{12}^3(1, 2)$ and $\overline{12}^3(2, H_1)$. Once more we find that $\overline{12}^2(1, 2)$ is apolar to each of the fundamental quartics and is consequently a line section. When formed for types of the curve without an oscnode $\overline{12}^2(1, 2)$ is not a line section, but it has been shown to be for all types with the oscnode, so we suggest the following theorem.

THEOREM 4. *The quartic form $\overline{12}^2(1, 2)$ of the system of the fundamental quartics of the rational plane quartic curve locates four points on a line if the curve has an oscnode, the line being on the oscnode.*

CARNEGIE INSTITUTE OF TECHNOLOGY

TOWNSEND ON REAL VARIABLES

Functions of Real Variables. By E. J. Townsend. New York, Henry Holt, 1928. 405 pp. $5.00.

This book, as indicated in the preface, is a text-book on the subject, and not a treatise. The material used has been divided into seven chapters of approximately equal length, except for the first. They are as follows: Chapter I, *Real Number System*,—a short discussion of real numbers as defined by Cantor and Dedekind; Chapter II, *Theory of Point Sets*,—the usual properties involved in the notions of open and closed sets and measure, chiefly confined to linear sets; Chapter III, *Continuity and Discontinuity of Functions*,—including double and iterated limits, continuity in one or two variables, semi-continuous and pointwise discontinuous functions; Chapter IV, *Derivatives and their Properties*,—derivatives, upper and lower derivatives, law of the mean, partial derivatives, and total differentiability; Chapter V, *Riemann Theory of Integration*,—the usual theory, including improper integrals and integrals involving a parameter; Chapter VI, *Lebesgue and Other Integrals*,—the usual definitions and theory, including primitive functions and a short account of the work of Stieltjes' Hellinger, Denjoy, and Perron; Chapter VII, *Infinite Series*,—uniform and quasi-uniform convergence of series of functions, with applications to differentiation and integration, properties of power series (not including Taylor's series), condensation of singularities, and various methods of handling divergent series. Mention should also be made of the collections of problems at the ends of the chapters and the numerous references to original sources, both of which should be useful to teacher and student.

In criticizing this book it would be natural to compare it with the other two books on the same subject in English, those by Hobson and by Pierpont, but it must be remembered that they are essentially treatises, while the book under consideration is a comparatively short text-book and must be judged accordingly. A discussion of its merits falls naturally under three heads: its aesthetic qualities, the choice of material, and its utility.

With regard to the first the book is excellent. Not only is the author's style good, but he has a manner of presentation that makes the book intensely interesting. The publishers, also, are to be congratulated on their workmanship, as the arrangement and printing are artistic and there is a noticeable freedom from misprints.

The choice of material appeals to the writer in general, but not entirely so. The chief sin of omission is the confining of the point-set theory to linear sets and the restriction of so many of the theorems on functions to functions of one variable. To be sure, most of the generalization is immediate, but in some cases this is not true and the beginner cannot be expected to know when this is the case. It also seems strange to see no discussion of Taylor's series and the conditions for its validity in a chapter devoted to infinite series, especially when nearly eight pages are given to power series. On the other hand it is the writer's opinion that the author has to some extent

yielded to the temptation to make his book encyclopedic in character and would have done better to omit certain topics so as to permit a thorough treatment of the topics mentioned above.

One who expresses an opinion as to the utility of this book will be largely influenced by what he conceives to be the primary aim of a first course in this subject, whether the imparting of a large body of new information or training in rigorous mathematical reasoning. The writer regards this book as superior in regard to the first of these objectives, but weak in regard to the second. The chief trouble is a willingness to allow the student to accept as true without proof facts which appear to be obvious, a failing most difficult to overcome in the average graduate student. For instance, in the definition of interior measure on p. 65 there is no mention of the necessity of proving uniqueness. The casual statements in various places (for example, p. 277) that the work can be readily extended to n dimensions is an encouragement to loose thinking. The paucity of references to previous sections in demonstrations, although improving the style, adds to the difficulty in regard to this matter, since the student is apt to accept the statements made without verifying them. Furthermore, the author himself is not impeccable in some of his statements. We learn on p. 44 that the sum of any number of closed sets is a closed set, which is of course not true unless a finite number is meant. But on pp. 51, 52 we find a set E which is defined as the sum of an enumerable system of perfect sets E_n and read that "since E_n is a perfect set for each value of n, it follows that E is also a perfect set." It is regrettable that a book otherwise so fine should be so marred by this sort of thing.

W. A. Wilson

SECOND EDITION OF NETTO'S KOMBINATORIK

Lehrbuch der Kombinatorik. By Eugen Netto. Second edition, with notes and additional chapters by Viggo Brun and Th. Skolem. Leipzig and Berlin, B. G. Teubner, 1927. viii+341 pp.

This work is volume 7 of the series B. G. Teubners Sammlung von Lehrbüchern aus dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, in which a number of contributors to the Encyclopædie expand their articles in textbook form.

The first edition appeared in 1901 and contained thirteen chapters, which are reprinted without change, pp. 1–258. Th. Skolem contributes notes on these chapters (pp. 309–338) containing simplifications of a number of proofs, as well as extensions of several of the topics treated by Netto. Since no review of the first edition appeared in this Bulletin, it seems appropriate to indicate briefly the contents of the first thirteen chapters. Chapters 1 and 2 give the elementary notions on permutations and combinations, and their connection with the binomial and multinomial theorems. Chapter 3 deals with arrangements of a number of elements with restrictions on the places which the elements may occupy; topics such as the problem of eight queens and Latin squares are treated here. Chapter 4 considers the inversions and sequences in permutations, and Chapter 5

gives an elementary exposition of various partition problems. The analytic treatment of partitions by means of generating functions follows in Chapter 6, and Chapter 7 contains some applications of partitions to problems in analysis. Chapter 8 considers the combinatorial arrangements as sums of symbolic products, and Chapter 9 is concerned with some special arrangements other than permutations and combinations. Chapter 10 deals with triads, and although Netto quotes some early work of E. H. Moore, Skolem's notes on this chapter contain no reference to the researches of Cole, White and their students. In Chapter 11, the Kirkman problem is considered, and Chapter 12 gives applications of combinatorial analysis to probability and questions in formal algebra such as the inversion of a power series. Chapter 13 contains a collection of formulas, mainly identical relations between binomial coefficients. The reviewer has found this collection very useful on a number of occasions. The new subject matter of the second edition begins with Chapter 14 (by Viggo Brun). Here a very general combinatorial concept is introduced, that of the distribution function

$$D(s_0 \cdots s,\ r_0 \cdots r,\ \cdots,\ w_0 \cdots w;\ \sigma_0 \cdots \sigma,\ \rho_0 \cdots \rho,\ \omega_0 \cdots \omega)$$

which is defined as the number of distributions of from s_0 to s black balls, from r_0 to r red balls, $\cdots$, from w_0 to w white balls in a number of urns of which the first is to contain from σ_0 to σ balls, the second from ρ_0 to ρ balls, $\cdots$, and the last, from ω_0 to ω balls. First, a duality theorem is noted, which says that in D the sets of Latin and Greek letters may be interchanged without changing the value of the distribution function. The recurrent formulas and explicit expressions for D are given, and the results are applied to an extension of the work of Möbius on the factorization of integers treated in Chapter 8 (where the Möbius number-theoretic function $\mu(n)$ is introduced). Reference is also made to Brun's own work on the sieve of Eratosthenes. In Chapter 15, Th. Skolem deals with groupments, combinatorial reciprocities, and systems of pairs. A groupment is a distribution in the sense of the preceding chapter with the restriction that no urn shall contain two balls of the same color. The notion of duality is introduced and generalized, and it is shown that the groupments of systems of pairs have important connections with the theory of triads, as well as with the Petersen graphs in analysis situs. Lack of space prevents the reviewer from giving any details of these somewhat complicated things, but it should be stated that both these last chapters form important contributions to combinatorial analysis, and that they may be read without knowledge of the preceding chapters.

T. H. Gronwall

SHORTER NOTICES

The Bakhshâlî Manuscript—A Study in Mediaeval Mathematics. By G. R. Kaye. With photographic facsimiles of the manuscript and transliteration of the text in roman script. Calcutta, Government of India, Central Publication Branch, 1927. Pp. i+156+v. Plates 1+4+42. Price Rs 28 or 43s. 6d.

Bakhshâlî is the name of a village near the city of Peshawar in the north-west corner of India. There in 1881, a farmer unearthed a manuscript of a work on mathematics. The greater portion of it has been destroyed by rough handling and the remains consist of 70 leaves of birch-bark but some of these are mere scraps. That manuscript is now generally known as the Bakhshâlî Manuscript. It is preserved in the Bodleian Library, Oxford. The work of analysing and editing the Manuscript was begun by the distinguished Indologist, Dr. R. Hoernle, but on account of his untimely death, it came to the hands of Mr. G. R. Kaye who has successfully finished it. The editor has added an elaborate and masterly introduction to the text covering 84 pages in print which contains not only an analysis of the contents of the manuscript but also a study of its relation with other Hindu treatises of mathematics as far as possible.

The Bakhshâlî work is a compendium of rules and illustrative examples. It gives also the solutions of the most of the examples together with their verification. It contains material relating to arithmetic, algebra, and geometry (including mensuration). The topics for discussion are found to include fractions, square-root, arithmetical and geometrical progressions, income and expenditure, profit and loss, computation of gold, interest, rule of three, summation of certain complex series, simple equations, simultaneous linear equations, quadratic equations, indeterminate equations of the second degree of a particular type, mensuration and miscellaneous problems. The treatment of all these subjects is commonly included in other Hindu treatises of mathematics. But we miss in it any reference to the treatment of the indeterminate equation of the first degree and the so-called Pellian equation, both of which enter largely into later Hindu works and in the solution of which the Hindus long anticipated the works of Euler and Lagrange.

The most notable mathematical principles in the Bakhshâlî work are the approximate square-root formula,

$$(a^2 + r)^{1/2} = a + \frac{r}{2a} - \frac{(r/(2a)^2)}{2(a+r/(2a))},$$

the calculation of the errors of successive order

$$\epsilon_1 = (r/(2a))^2, \quad \epsilon_2 = \left(\frac{(r/(2a)^2)}{2(a+r/(2a))}\right)^2$$

and a consequent process of reconciliation. The fundamental arithmetical

operations are indicated by putting after, occasionally before, the quantity affected, the abbreviations of the Sanskrit words of respective import. In the case of subtraction, the abbreviation is found to have degenerated into a simple cross-sign (+). The Bakhshâlî work is particularly characterised by the absence of any kind of algebraic symbolism. This has necessitated the preservation of every detail of the working of the solutions of algebraic problems keeping up their generality throughout so that the final statement of the results should clearly present the whole formulas involved. "Indeed the numerical quantities in those problems are treated almost like algebraic symbols," rightly observes Mr. Kaye. Most of those problems have been solved by the method of *false position*. The place of an unknown quantity in a statement of a problem is indicated by a heavy dot (•) and is designated as *śûnyasthāna* or "empty space," meaning thereby that the number to be placed there has not been ascertained. Other matters of interest are: the least common multiple, the solution of the quadratic equation, the exclusive use of the decimal place-value notation with the cipher, the average value and the change ratios of measures.

The time of the Bakhshâlî work is rather uncertain. On paleographical evidence Mr. Kaye holds that it should be referred to a period about the twelfth century. There are clear signs to show that the present manuscript is an imperfect copy of an older manuscript which again is of the character of a perpetual commentary on a still earlier work. The editor has failed to detect it. But it did not escape the keen eyes of Dr. Hoernle. He is of opinion that the present copy was made about the ninth century but the original work must be referred to the early centuries of the Christian era. There are many things in the work to point to the same period. Mr. Kaye has drawn attention to a few instances of foreign influence in the Bakhshâlî work including the one about the approximate square-root stated above which he says he has not found in any other Hindu works. That formula, correct up to the second approximation, was known in the early Jaina canonical works (about 300 B.C. or still earlier) and also reappears in later works including that of Âryabhata (499 A.D.) as has been pointed out by Rodet. Another has been overstated under a similar misapprehension and the rest are minor and of doubtful value. The editor himself does not wish to press upon the latter.

The significance and importance of a publication like the present one are apparent to all lovers of the history of science. And their thanks are certainly due to Mr. Kaye for the great amount of pains that he has taken in explaining and editing the Bakhshâlî manuscript. There are, no doubt, several inaccuracies in his study of the work, some of which are not negligible. It is not possible to point them out in a short review like this. A critical examination of them will be found in a paper by the present reviewer entitled *The Bakhshâlî mathematics* in the Bulletin of the Calcutta Mathematical Society (vol. 21, pp. 1–60). But they do not lessen the labor of the editor. Moreover such inaccuracies are not unusual in case of all pioneer studies of difficult ancient manuscripts.

Bibhutibhusan Datta

Vorlesungen über Algebra. Fourth revised edition of the work with the same title by the late Dr. Gustav Bauer. By L. Bieberbach. Leipzig and Berlin, B. G. Teubner, 1928. x+334 pp.

The well known *Algebra* of Bauer* now appears in a fourth edition remodeled and modernized to such an extent as to form practically a new book. In Section 1, on the fundamental properties of algebraic equations, we may note the thoroughly modern chapter on complex numbers, taken largely from volume 1 of Bieberbach's *Funktionentheorie*, and in Chapter 3, Cauchy's function-theoretic proof of the fundamental theorem of algebra is followed very appropriately by Rouché's theorem, which is used to prove the continuity of the roots as functions of the coefficients. Section 2 deals with determinants, matrices, linear equations, quadratic and bilinear forms, and the proofs are made short and elegant by an extensive use of the vector notation. Section 3 deals with symmetric functions and Tschirnhaus' and Jerrard's transformations in the customary manner. Section 4, which carries the too narrow title "numerical solution of equations," is the most distinctive feature of the book. The first chapter gives various methods for finding upper and lower bounds for the real roots, and then proceeds to Lill's graphical method, which is the most rapid means of obtaining fairly good approximations to the real roots; better approximations may then be obtained by the standard methods which are briefly described. It is pleasing to see less than half a page devoted to Horner's method instead of the inordinate amount of space given to it in American texts. Chapter 2 gives Descartes' rule of signs, and the theorems of Rolle, Budan Fourier, and Sturm, including Hermite's treatment of Sturm's problem by quadratic forms. Chapter 3 contains the method of Cohn (Mathematische Zeitschrift, vol.14) for determining the number of complex roots inside a circle, theorems of Schur and Kakeya, and Schur's discussion of equations with all the roots having negative real parts.

Chapter 4 gives a thorough discussion of Graeffe's method both for real and complex roots. Chapter 5 begins with the classical theorem of Gauss on the relative position in the complex plane of the zeros of $f(x)$ and $f'(x)$, proceeds to theorems of Poulain, Laguerre, Fejér, Grace, Heywood and Walsh, and also gives composition theorems due to Malo and Schur. The reviewer is particularly pleased to see the very readable account in Chapters 3 and 5 of a number of quite modern and interesting things, which are sadly neglected in most other texts.

Section 5, on the algebraic solution of equations, gives a very readable account, mostly along classical lines, of the elements of cyclotomy, substitution groups and the Galois theory; there is a new proof of the irreducibility of the cyclotomic equation due to Späth (Mathematische Zeitschrift, vol. 26). The book closes with an elementary account of continued fractions.

T. H. Gronwall

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* Reviewed in this Bulletin: first edition, vol. 10 (1903–4), p. 257, by L. E. Dickson, and second edition, vol. 17 (1911), p. 428, by A. Dresden.

NOTES

The April, 1929, number of the Transactions of this Society (volume 31, No. 2) contains the following papers: *A note on closest approximation*, by D. Jackson; *Homogeneous polynomials with a multiplication theorem*, by O. C. Hazlett; *Generalized factorial series*, by T. Fort; *On extending a continuous* (1-1) *correspondence (second paper*, H. M. Gehman; *A determination of all normal division algebras in sixteen units*, by A. A. Albert; *Expansions in generalized Appell polynomials, and a class of related linear functional equations*, by I. M. Sheffer; *Systems of infinitely many linear differential equations of infinite order, with constant coefficients*, by I. M. Sheffer; *On general topology and the relation of the properties of the class of all continuous functions to the properties of space*, by E. W. Chittenden; *Properties of functions represented by the Dirichlet series* $\sum (av+b)^{-s}$, *or by linear combinations of such series*, by J. I. Hutchinson; *Concerning zero-dimensional sets in euclidean space*, by R. L. Wilder; *On the linear partial q-difference equation of general type*, by C. R. Adams; *Les fonctions polygènes comme intégrales d'équations différentielles*, by G. Calugaréano.

The April, 1929, number of the American Journal of Mathematics (volume 51, No. 2) contains: *Singular points of vector fields under general boundary conditions*, by M. Morse; *A complete solution of Laplace's equation by an infinite hypervariable*, by P. W. Ketchum; *Functions with assigned initial values*, by W. J. Trjitzinsky; *The abstract identity of modular systems and ideals*, by D. Harkin; *Concerning the cut points of a continuous curve when the arc curve, AB, contains exactly N independent arcs*, by N. E. Rutt; *A direct treatment of systems of linear differential equations whose coefficients have uniform singularities*, by J. A. Nyswander; *On related difference and differential sytems*, by W. M. Whyburn; *On the motion of a rigid body about a fixed point in space of constant curvature*, by J. Pierpont; *On the prime ideals of the general cubic Galois field*, by C. G. Latimer; *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, by O. Zariski.

The opening number of volume 30, series 2, of the Annals of Mathematics (December, 1928) contains: *On cubic fields*, by A. Arwin; *The impossibility of a separation of types of linear odd divisors of binary quadratic forms*, by R. G. Archibald; *A parametric problem of the calculus of variations and its treatment as a problem of Lagrange*, by L. H. McFarlan; *On the fundamental existence theorems for differential systems*, by W. M. Whyburn; *On a class of Taylor's series*, by P. L. Srivastava; *Quartic equations with certain groups*, by R. Garver; *Canonical forms of plane cubic curves under euclidean transformations*, by R. S. Burington and H. K. Holt; *On forms which repeat under multiplication*, by C. G. Latimer; *On the multiple solutions of the Pell equation*, by D. H. Lehmer; *On certain hitherto unsolved cases of the complex multiplication of elliptic functions*, by S. C. Mitra; *On the complex roots of a*

transcendental equation occurring in the electron theory, by J. Pierpont; *Extended results in elimination*, by T. W. Moore; *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, by P. Alexandroff; *Semi-parallel transformations of lines of curvature*, by M. M. Slotnick.

The new Accademia d'Italia consists of the following classes each with fifteen members: Exact and Natural Sciences, Moral Sciences, Arts, Literature. The first thirty members have recently been selected by the government. The representatives of pure and applied mathematics and physics are F. Severi, professor of calculus in the University of Rome, E. Fermi, professor of theoretical physics in the University of Rome, and G. Vallauri, professor of electrical engineering in the Technical School of Turin.

A Congress of Mathematicians of the Slavic Countries will be held at Warsaw, September 23–27, 1929, under the auspices of the head of the Polish government. Professor W. Sierpinski, of the University of Warsaw, is president of the executive committee of the Congress.

Cambridge University has awarded Smith's prizes to H. D. Ursell, of Clare College, and J. M. Whittaker, of Trinity College. Rayleigh prizes have been awarded to J. Hargreaves, of Clare College, J. G. Semple, of St. John's College, and S. Verblunsky, of Magdalene College.

The Italian Society of Sciences has awarded its prize in mathematics for the year 1928 to Professor C. Rosati, of the University of Pisa, for his researches in algebraic geometry.

The Watson medal of the National Academy of Sciences has been awarded to Dr. Willem de Sitter, director of the observatory at Leiden.

The Research Corporation of New York has awarded its research medal to Professor Werner Heisenberg, of the University of Leipzig, for "brilliant scientific achievement."

Professor Erwin Schrödinger has been elected a member of the Prussian Academy of Sciences.

Professor Heinrich Tietze, of the University of Munich, has been elected a member of the Bavarian Academy of Sciences.

Professor Guido Castelnuovo has been elected correspondent of the Paris Académy of Sciences in the section of geometry, as successor to the late Professor Luigi Bianchi.

The following have been elected correspondents of the Academy of Sciences of Turin: Professors L. Tonelli, G. Vitali, A. Signorini, E. Pistolesi, F. Lori, and E. Fermi.

In the presence of the King of Norway, the University of Oslo commemorated, on March 6, 1929, the one hundredth anniversary of the death of

the Norwegian mathematician Abel. On this occasion honorary degrees were conferred on Professors Jacques Hadamard, G. H. Hardy and Paul Painlevé.

Professors Max von Laue, of the University of Berlin, and Arnold Sommerfeld, of the University of Munich, have been elected honorary members of the Leningrad Academy of Sciences.

Among the recently elected fellows of the Royal Society of Edinburgh are the following mathematicians: Dr. F. Bath, lecturer at the University of St. Andrews, and Mr. R. A. Robb, lecturer at the University of Glasgow.

Professor Pieter Zeeman, of the University of Amsterdam, has been elected an honorary member of the Physical Society of London.

The following have been recently elected foreign associates of the National Academy of Sciences: Dr. Willem de Sitter, director of the observatory at Leiden, Professor Arnold Sommerfeld, of the University of Munich, and Professor C. de la Vallée Poussin, of the University of Louvain.

The King of Italy has conferred the decoration of Officer of the Crown of Italy upon Dr. Raymond Pearl, director of the Institute of Biological Research of Johns Hopkins University.

Dr. Paul Dienes, lecturer in mathematics at University College, Swansea, and formerly of the University of Budapest, has been appointed university reader at Birkbeck College.

Dr. D. R. Hartree, of the Cavendish Laboratory, Cambridge, has been appointed to the Beyer chair of applied mathematics at the University of Manchester.

Professor Friedrich Hund, of the University of Rostock, has been appointed professor of mathematical physics at the University of Leipzig.

Dr. Pascual Jordan has been promoted to an associate professorship of theoretical physics at the University of Hamburg.

Associate Professor Theodor Kaluza, of the University of Königsberg, has been appointed professor of mathematics at the University of Kiel.

Dr. Harold R. Phalen, professor of mathematics and instructor in physics at St. Stephen's College, Columbia University, has been appointed dean of the college.

Dr. Tibor Rado has been appointed Lecturer in Mathematics at Harvard for the first term of the year 1929–1930. He will give a course on "Minimal Surfaces and the Problem of Plateau" together with a course on "The Elementary Theory of Differential Equations."

The University of Lwów has conferred an honorary doctorate on Professor W. Sierpínski, of the University of Warsaw.

Professor Leonida Tonelli has been transferred from the professorship

of algebraic analysis at the University of Bologna to the professorship of infinitesimal analysis at the same University.

Professor C. E. Weatherburn, of Canterbury College, Christchurch, New Zealand, has been appointed professor of mathematics at the University of Western Australia.

The following have been appointed docents: Dr. W. Nikliborc and Dr. J. P. Schander, at the University of Lwów; Dr. Zarankiewicz, at the University of Warsaw.

Professor E. R. Hedrick was selected to give the Fifth Annual Faculty Research Lecture at the University of California at Los Angeles. This lecture was given April 26, 1929, on the subject *Logical reasoning in mathematics and in science*.

Professor H. E. Slaught, of the University of Chicago, delivered lectures during March 1929 on *How mankind learned to count* at Haverford College, Lafayette College, Lehigh University, Rutgers University, and Swarthmore College. These lectures were usually sponsored by mathematics clubs. The lectures were given for the purpose of stimulating interest in mathematics and they were well attended.

The original manuscript of Einstein's *Zur einheitlichen Feldtheorie*, recently published in the Sitzungsberichte of the Prussian Academy of Sciences, has been purchased for Wesleyan University by Mr. G. W. Davison and Mr. A. W. Johnston, two of the University's trustees.

A symposium on theoretical physics will be conducted at the University of Michigan, from June 24 to August 16, 1929. The following courses of lectures will be delivered: by E. A. Milne, of the University of Oxford, Problems in astrophysics, and vector and tensor methods in statics and dynamics; by K. F. Herzfeld, of Johns Hopkins University, Statistical mechanics; by Léon Brioullin, of the University of Paris, Quantum statistics; by Edward Condon, of Princeton University, Introduction to quantum mechanics; by P. A. M. Dirac, of the University of Cambridge, Advanced quantum mechanics; by D. M. Dennison, of the University of Michigan, Band spectra.

Assistant Professor H. M. Gehman, of Yale University, has been appointed head of the department of mathematics at the University of Buffalo.

Mr. W. J. Hazard has been promoted to an assistant professorship of mathematics at the University of Colorado.

Assistant Professors Lester S. Hill and Evelyn Walker have been promoted to associate professorships of mathematics at Hunter College.

Miss Marie Litzinger has been promoted to an assistant professorship of mathematics at Mount Holyoke College.

Professor W. V. Lovitt, of Colorado College, has been appointed Dean of Men; he retains his professorship of mathematics.

Assistant Professor A. D. Michal, of the Ohio State University, has been appointed associate professor of mathematics at the California Institute of Technology.

Professor W. E. Milne, who has been on leave as professor of mathematics at Stanford University this year, will return to the University of Oregon next fall. During his absence Dr. H. C. Hicks has served as assistant professor of mathematics at the University of Oregon.

Associate Professor L. L. Smail, of Lehigh University, has been promoted to a professorship of mathematics.

Dr. P. A. Smith has been promoted to an assistant professorship of mathematics at Barnard College, Columbia University.

Dr. W. J. Trjitzinsky has been promoted to an assistant professorship of mathematics at Lehigh University.

Professor James V. Uspensky of the Russian Academy of Sciences has been appointed professorial lecturer at the University of Minnesota for the summer quarter, and part of the spring quarter. During the spring quarter he lectured on *Integration in finite form*. He will lecture on the *Theory of numbers* in the first session of the summer quarter and on the *Theory of probability* in the second session.

Mr. Charles H. Vehse, of Brown University, has been appointed assistant professor of mathematics at the University of West Virginia.

Dr. Louis Weisner has been promoted to an assistant professorship of mathematics at Hunter College.

The following appointments to instructorships are announced:

Barnard College, Columbia University: Dr. Lulu Hofmann, Mr. H. W. Raudenbush.

Brown University, Mr. G. N. Carmichael, Mr. K. G. Fuller of Northwestern University.

Harvard University, Dr. T. F. Cope.

Hunter College, Mr. Aubrey W. Landers, Jr.

The Johns Hopkins University, Dr. Oscar Zariski (Associate).

Swarthmore College, Mr. Aubrey H. Smith.

University of Wyoming, Mr. Richard W. Warner, Mr. C. N. Shuster.

Jorge Duclout, engineer and honorary professor of the theory of elasticity at the University of Buenos Aires, died February 15, 1929, at the age of seventy-five.

Professor Erik Schou, of the Copenhagen Technical School, died November 8, 1928, at the age of fifty-five.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

Bailey (C.). The Greek atomists and Epicurus. Oxford, Clarendon Press, 1929. 9+619 pp.

Beck (H.). Elementargeometrie. Leipzig, Akademische Verlagsgesellschaft, 1929.

Bisacre (F. F. P.). Praktische Infinitesimalrechnung. Berechtigte Deutsche Ausgabe, unter Mitwirkung von E. Trefftz herausgegeben von E. König. Leipzig, Teubner, 1929. 12+364 pp.

Bodewig (E.). See Dickson (L. E.).

Bol (G.). Vlakke Laguerre-meetkunde. (Dissertation, Leiden.) Amsterdam, H. J. Paris, 1928. 130 pp.

Brown (F. G. W.). Progressive trigonometry. Part 1. London, Macmillan, 1928. 10+222 pp.

Carr (H. W.). Leibniz. London, Benn, 1929. 6+222 pp.

Cattaneo (P.). Appunti sui numeri reali. Padova, Zannoni, 1928. 64 pp.

Cisotti (U.). Lezioni di calcolo tensoriale. Milano, Libreria Editrice Politecnica, 1928. 7+94 pp.

Corral (J. I.). Cantidades complejas. (Memorias Cientificas de la Direccion de Montes y Minas, Cuba, tomo 1.) Habana, 1929. 160 pp.

Crantz (P.). Sphärische Trigonometrie. 2te Auflage von H. Hauptmann. Leipzig, Teubner, 1928.

Dickson (L. E.). Höhere Algebra. Autorisierte Deutsche Ausgabe von L. E. Dickson "Modern algebraic theories" herausgegeben von E. Bodewig. Leipzig, Teubner, 1929. 8+242 pp.

Dintzl (E.) und Vaselli (C.). Aufgaben aus der reinen und angewandten Mathematik. Band 2. Wien, Hölder-Pichler-Tempsky, 1928. 107 pp.

Dunne (J. W.). An experiment with time. 2d edition, revised. London, A. and C. Black, 1929. 8+213 pp.

Encyklopädie der Mathematischen Wissenschaften. Band III, Teil 2, Heft 10: Spezielle Algebraische Flächen 3. Ordnung, von W. F. Meyer. Leipzig, Teubner, 1928. 94 pp.

Estéve (R.) et Mitault (H.). Algèbre et trigonométrie. 2 volumes. Paris, Gauthier-Villars, 1928. 552+314 pp.

Fantappiè (L.) See Volterra (V.).

Fettweis (E.). Methodik für den Rechenunterricht. Paderborn, Schöningh, 1929. 272 pp.

Fladt (K.). Elementarmathematik. Band I, Elementargeometrie, Teil 2. Leipzig, Teubner, 1928. 8+181 pp.

Forsyth (A. R.). A treatise on differential equations. 6th edition London, Macmillan, 1929. 18+583 pp.

Fowler (R. H.). The elementary differential geometry of plane curves. 2d edition. Cambridge, University Press, 1929. 9+105 pp.

Fry (T. C.). Elementary differential equations. New York, Van Nostrand, 1929. 10+255 pp.

Fulco (P.). Congruenze algebriche ed esponenziali. Applicazioni. Memoria IIa. Civitavecchia, 1929. 230 pp.

Garabedian (C. A.) and Winston (J.). Plane trigonometry. New York, McGraw-Hill, 1929. 10+306 pp.

Grundel (F.). Die Mathematik an den deutschen höheren Schulen. Teil 2. Leipzig, Teubner, 1929. 6+148 pp.

Haag (J.). Cours complet de mathématiques élémentaires. Tome IV: Trigonométrie. Paris, Gauthier-Villars, 1929.

Haupt (O.). Einführung in die Algebra. 2 Bände. Leipzig, Akademische Verlagsgesellschaft, 1929. 367+15+663+13 pp.

Hauptmann (H.). See Crantz (P.).

Junker (F.) und Witting (A.). Repertorium und Aufgabensammlung zur Differential-Rechnung. Berlin, de Gruyter, 1928. 130 pp.

Kaluza (—.). Vollmotone Funktionen. Halle, Niemeyer, 1928.

Kamke (E.). Mengenlehre. (Sammlung Göschen.) Berlin, de Gruyter, 1928. 160 pp.

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Lietzmann (W.). Überblick über die Geschichte der Elementarmathematik. 2te Auflage. Leipzig, Teubner, 1928.

Meyer (W. F.). See Encyklopädie.

Mitault (H.). See Estève (R.).

Radon (J.). Zum Problem von Lagrange. Leipzig, Teubner, 1928. 27 pp.

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Rose (—.). Die Schulung des Geistes durch den Mathematik- und Rechenunterricht. Leipzig, Teubner, 1928.

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—— Theorie der hoogere-machtscongruenties en der hoogere-machtsresten. Leiden, Gebroeders van der Hoek, 1928. 272 pp.

Study (E.). Denken und Darstellung. 2te Auflage. Braunschweig, Vieweg, 1928. 5+63 pp.

Toft (L.). Definitions and formulae for students (practical mathematics). London, Pitman, 1929. 4+28 pp.

Trefftz (E.). See Bisacre (F. F. P.).

Vaselli (C.). See Dintzl (E.).

Vivanti (G.). Algebra ad uso degli Istituti Tecnici Superiori. Torino, Lattes, 1928. 142 pp.

Volterra (V.). Teoria de los funcionales y de las ecuaciones integrales e integro-diferenciales. Conferencias explicadas en la Facultad de Ciencias de la Universidad, 1925, redactadas por L. Fantappiè. Madrid, Imprenta Clásica Española, 1927. 8+208 pp.

Winston (J.). See Garabedian (C. A.).

Witting (A.). See Junker (F.).

PART II. APPLIED MATHEMATICS

Abbott (W.). Practical geometry and engineering graphics. London, Blackie, 1929. 304 pp.

Angenheister (G.). See Wien (W.).

Armellini (G.). Trattato di astronomia siderale. Volume 1: Parte generale. Bologna, Zanichelli, 1928. 11+393 pp.

Baker (J. F.). See Foster (P. F.).

Balser (L.). Einführung in die Kartenlehre (Kartennetze). Leipzig, Teubner, 1928. 59 pp.

Barbaudy (J.). Les bases physico-chimiques de la distillation. (Mémorial des Sciences Physiques, No. 5.) Paris, Gauthier-Villars, 1928. 66 pp.

Bavink (B.). Die Hauptfragen der heutigen Naturphilosophie. II. Berlin, Salle, 1928. 174 pp.

Becker (A.). See Wien (W.).

Bogaert (E. W. W.) et van den Dungen (F.). Mécanique rationnelle. Tome 1. Bruxelles, Lammertin, et Paris, Hermann, 1928. 332 pp.

Boll (M.). Matière, électricité, radiations. Paris, Delagrave, 1929. 122 pp.

Bottlinger (K. F.). See Wien (W.).

Bouasse (H.). Résistance des fluides. Paris, Delagrave, 1928.

Bouny (F.). Leçons de mécanique rationnelle. Tome 2: Dynamique du point, dynamique des systèmes. Paris, Blanchard, 1929. 8+659 pp.

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lationen (1910–1912). Bearbeitet von A. D. Fokker. Übersetzt von H. Stücklen. Leipzig, Akademische Verlagsgesellschaft, 1929. 9 +180 pp.

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Pacotte (J.). Les méthodes nouvelles en analyse quantique. (Mécanique quantique. Mécanique ondulatoire.) Paris, Blanchard, 1929. 8 +139 pp.

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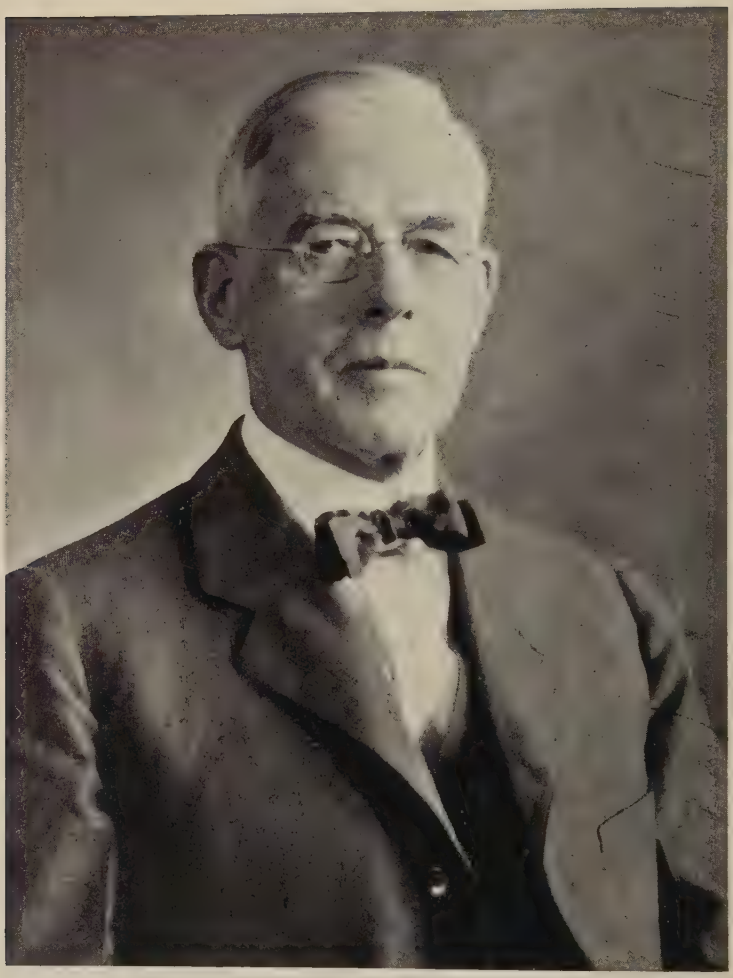

Henry Burchard Fine
1858–1928
President of the American Mathematical Society
1911–1912

THE JUNE MEETING IN BERKELEY

The two hundred seventieth regular meeting of the Society was held at the University of California on Thursday and Friday, June 20–21, 1929, in conjunction with the June 19–22 meetings of the American Association for the Advancement of Science. Sessions were held Thursday morning, Thursday afternoon, and Friday morning. The attendance included the following thirty-three members:

Alderton, J. P. Ballantine, Bateman, Bernstein, Biggerstaff, Cajori, Coleman, Corbin, Daus, Doole, G. C. Edwards, Garver, J. L. Gibson, W. L. Hart, M. W. Haskell, E. R. Hedrick, Hicks, Jerbert, D. N. Lehmer, Lenzen, Leonard, S. H. Levy, McFarlan, F. R. Morris, Pierpont, T. M. Putnam, J. M. Rankin, Pauline Sperry, Morgan Ward, G. T. Whyburn, W. M. Whyburn, A. R. Williams, B. C. Wong.

There was no meeting of the Council or of the Trustees of the Society.

The meeting was presided over at various times by President Hedrick and Professors M. W. Haskell, W. L. Hart James Pierpont, and Harry Bateman.

A luncheon for members and their guests was held at the Women's Faculty Club between the Thursday sessions.

At the Thursday morning session, by invitation of the Program Committee, Professor James Pierpont, of Yale University, delivered an address entitled *Non-euclidean geometry in retrospect.* The address will appear in full in a later issue of this Bulletin.

Titles and abstracts of other papers read at the meeting follow. The papers of the following authors were read by title: Clifford Bell, E. T. Bell, Brown, Cook, Craig, Hollcroft, Jones, Milne, Pierce, Roberts, Robertson, Rutt, Weyl, G. T. Whyburn (second paper), W. M. Whyburn (third paper), Wilder, Woodard, Zippin.

1. Professor J. P. Ballantine: *A vector derivation of Cramer's rule.*

A system of two equations in two unknowns may be written $\alpha x+\beta y=\gamma$, where $\alpha=(a_1, a_2)$ and similarly β, γ are vectors. It is easily seen from a diagram that x is the ratio between the altitudes on the common side β, and hence between the areas of the parallelograms $\gamma\beta$ and $\alpha\beta$. Hence the determinantal formula. The procedure generalizes readily to any number of dimensions. The paper will appear in the American Mathematical Monthly.

2. Professor Harry Bateman: *Some properties of spheroidal harmonics.*

The harmonics of even order associated with an oblate spheroid are expressed as definite integrals involving the corresponding harmonics for a prolate spheroid and some other interesting relations are obtained.

3. Dr. Clifford Bell: *Polygons jointly related to the rational cubics and other curves.*

A study is made of the n-sided polygon having $n-1$ of its vertices on, and all its sides tangent to, a rational cubic, the remaining vertex lying on any one of a finite set of algebraic curves $f_i(x, y, z)=0$ of orders m_i. The number of such n-sided polygons so connected with the cuspidal cubic is $3\sum_{i=1}^{s}m_i$ where s is the total number of curves $f_i(x, y, z)=0$. The number for the nodal cases is given by $3\cdot 2^{n-1}\sum_{i=1}^{s}m_i$. The vertices of the polygons are ordered, and in the cuspidal case it is shown that by a simple magnification of x, y, and z, dependent only on n and the affix of the vertices, any one of the curves of the set $f_i(x, y, z)=0$ can be deformed in such a way as to contain all of the vertices of given affix belonging to polygons associated with this curve and the given cubic. The well known results for triangles in and circumscribed to the rational cubics are obtained by specializing to the case $n=3$ and allowing the curves $f_i(x, y, z)=0$ to become coincident with the cubic.

4. Professor E. T. Bell: *Theorems on total representations as sums of squares or triangular numbers.*

If m, n are coprime integers >0, and $f(x)$ is uniform and finite for integer values of x, $f(x)$ is said to be factorable if $f(mn)=f(m)f(n)$. Denote by $O(n), E(n)$ the total numbers of representations of n as sums of an odd, an even number of squares of integers $\gtrless 0$. Neither of $O(n)$, $E(n)$ is factorable. In a former note (American Mathematical Monthly, vol. 30 (1923), p. 441), it was shown that $O(2n+1)>E(2n+1)$, $E(2n)>O(2n)$ for all $n>0$. Here this property is widely generalized. As a very special case, $(-1)^{n-1}[\sum(-1)^jQ_j(n)/j]$ where $Q_j(n)=$twice the number of representations of n as a sum of j squares with integer roots $\neq 0$, is factorable, where $\sum$ refers to $j=1, 2, \cdots, n$; the sums of the positive, negative terms respectively satisfy the same inequalities as E, O. Further, the number of representations may be replaced by any even function of the actual integers in the respective representations, and the corresponding theorems then stated. The like is carried out for representations as sums of positive

triangular numbers. The results are scarcely obvious by purely arithmetical methods; by the means used in the proofs given here, the theorems may be extended indefinitely.

5. Professor E. T. Bell: *Non-existence theorems on the number of representations of arbitrary odd integers as sums of $4r$ squares.*

This paper appears in full in the present number of this Bulletin.

6. Professor E. T. Bell: *Existence theorems on the number of representations of odd integers as sums of $4t+2$ squares.*

The number $N(n, 2r)$ of representations of n as a sum of $2r$ squares, consists of two parts: first, a polynomial in the real divisors of n; second, a function which, for n general, is not reducible to a polynomial in the real divisors of n. But if n be restricted to be a member of certain arithmetic progressions, the second part may vanish identically. In the preceding paper, the necessary and sufficient conditions that this be so for $N(m, 4r)$, m odd, are determined. A remarkable transcendental equation in r is the decisive condition. This paper closes, negatively, a reasonable conjecture arising from the last section of Eisenstein's classic paper, Crelle's Journal, vol. 34 (1847), p. 35. In order that $N(m, 4r+2)$, $m \equiv 3 \bmod 4$, $r>2$, shall be a polynomial in the real divisors of m, it is necessary (but not sufficient) that

$$105(7^{2r}-1) = (512r^4-1280r^3+1120r^2+440r-162)(3^{2r}-1).$$

A similar condition holds for $m \equiv 1 \bmod 4$. Necessary and sufficient conditions, much more drastic, also are obtained, and thence the "divisor" character of $N(m, 4r+2)$ is settled.

7. Professor E. T. Bell: *On the representation of $4t+2$ as a sum of $4s$ squares.*

See the preceding abstract.

8. Professor E. T. Bell: *On the application of elliptic functions to universal and quasi-universal forms.*

A form f of degree 2 that represents all integers >0, or all integers $\geqq 0$, is called (after Dickson), universal. If f represents all integers of a given type, that type to be finitely definable, f is here called quasi-universal where f is not universal. By the method of paraphrase much information concerning universal or quasi-universal forms can be obtained from extremely simple analysis. By the same method an immediate extension to forms of degree >2 is attainable. The possibilities of this application of elliptic functions being unlimited, only a selection is given, sufficient to show the method.

9. Professor B. A. Bernstein: *On Whitehead and Russell's theory of deduction.*

Whitehead and Russell in their *Principia Mathematica* develop a theory of deduction as basis for the entire structure of the *Principia*. The present author discusses the nature of this theory and its relation to Boolean algebra.

10. Professor B. A. Bernstein: *Application of Boolean algebra to proving consistency and independence of postulates.*

The author shows how the problem of determining the consistency or the independence of a set of postulates can often be reduced to the solvable problem of determining the simultaneity of a set of Boolean relations.

11. Mr. A. B. Brown: *Relations between the critical points of a real analytic function of n independent variables.*

Morse's paper *Relations between the critical points of a real function of n independent variables* (Transactions of this Society, vol. 27, 1925) treats the case of non-degenerate critical points, that is, where the Hessian of the function f is not zero at any critical point. In the present paper the author investigates the case in which this restriction is dropped, but requires f to be analytic and have isolated critical points. Given a critical point at which $f=c$, not a point of minimum or maximum, a small $(n-1)$-sphere with center at the critical point is taken, and a corresponding small constant $e>0$. The kth type number of the critical point, $k=1, 2, \cdots, n-1$, is defined as the $(k-1)$st order connectivity number, diminished by unity, of the complex of points of the locus $f=c-e$ within or on the sphere. Under this count Morse's relations are found to hold.

12. Professor Florian Cajori: *Generalizations in geometry as seen in the history of developable surfaces.*

During the period of primitive intuition in the study of surfaces, only cones and cylinders were known to be developable. There followed three stages of generalization: (1) by the application of infinitesimal analysis to real surfaces, the discovery of other real developable surfaces, as a special type of ruled surfaces; (2) by the freer use, in analysis, of complex numbers, the discovery of developable imaginary surfaces; (3) by transcending the limitations of infinitesimal analysis, the discovery of real developable surfaces which are not ruled.

13. Professor Florian Cajori: *New data on the origin and spread of the dollar mark.*

West Indian manuscripts of 1760 and 1778 link the \$ unmistakably with the Mexican sign p^s for pesos or piastres, the s being brought down upon the p. Nineteenth century manuscripts of Lower California furnish cumulative evidence in support of this origin. The symbol \$ spread from the West Indies and New Orleans to New York, the Mississippi region, and to other places in commercial relations with the West Indies and Gulf Coast.

14. Professor Florian Cajori: *Absurdities due to division by zero. A historical note.*

Bishop Berkeley declared division by zero "shocking to good sense." The first to prove rigorously that division by zero is inadmissible was Martin Ohm. The first openly to derive absurdities from division by zero was Bernard Bolzano. The first to list the derivation of such absurdities as a mathematical recreation was W. W. R. Ball.

15. Professor A. J. Cook: *Pairs of rectilinear congruences with generators in one-to-one correspondence.*

When corresponding generators of a pair of rectilinear congruences do not intersect, a general theory for the projective study of the configuration may be constructed from a system of linear partial homogeneous differental equations of the first order in four dependent and two independent variables. The theory seems to be well adapted for the study of certain correspondences of Fubini, and a complete basis is furnished for their further investigation. The relation of Fubini's work to the earlier work of Darboux is also fully established. In connection with these correspondences a generalization to surfaces is secured for the concept of intersector curve which Lane introduced in his study of pairs of ruled surfaces.

16. Mr. H. V. Craig: *On the simultaneous invariants of two functions.*

This paper contains an investigation of the simultaneous invariants of two functions $F(x,x',x'')$ and $H(x,x')$. The chief result is a theory of parallelism developed from a fundamental tensor whose components are functions of x,x', and x''. The magnitude of vectors and the cosine of the angle between vectors undergoing parallel displacement are constant.

17. Professor P. H. Daus: *Note on the solution of* $x^3+2=y^2$.

The solutions of (1) $x^3+2=y^2$ in integers can be obtained from those of (2) $\alpha^3+3\alpha\beta^2-2\beta^3=1$ and conversely. The fact that (2) has solutions implies that the units in the cubic domain defined by $\theta^3+3\theta-2=0$ have the form $x+\theta y+\theta^2 z$, with $x=X^2+3Y^2$, $y=XY$, $z=Y^2$. The units in the cubic domain are connected by the recursion formulas $s_n=3(x_1-2z_1)s_{n-1}-3(x_{-1}-2z_{-1})s_{n-2}+s_{n-3}$, where (x_1, y_1, z_1) is the fundamental unit and (x_{-1}, y_{-1}, z_{-1}) its reciprocal. Since $s_1=(1, -1, -1)$, these recursion formulas can be used to prove (2) has no solution other than the trivial one $\alpha=1$, $\beta=0$, and (1) has only the solution $x=-1$, $y=\pm 1$. Several proofs of this fact have recently been given by other methods. The method outlined here may be extended to other equations of the type $x^3+a=y^2$.

18. Mr. H. P. Doole: *Certain multiple-parameter expansions.*

This paper deals with the proof of the convergence of the expansion of an arbitrary function in terms of the solutions of certain multiple-parameter

systems of linear differential equations of the first order in two variables. In §2 the solutions of $X_i' + [\lambda a_{i1}(x_i) \pm \mu a_{i2}(x_i)]X_i = 0$, $(i = 1, 2)$, are used with the boundary conditions $X_i(a_i) = \gamma_i X_i(b_i)$, $\gamma_i \neq 0$. In §3 the system $X_i' + (\lambda x_i^{h_i} \pm \mu)X_i = 0$, $X_i(-a) = X_i(b)$, $(i = 1, 2)$, has particular interest in the fact that $x_i^{h_i}$ changes sign in the interval considered. Convergence is shown for the case where h_i is integral, and possible divergence when h_i is such that $x_i^{h_i}$ is complex. In §4 convergence is proved for a certain system which has one auxiliary condition at more than two points of the interval, namely: $X(-1) - 2\alpha X(0) - X(1) = 0$. In §§2–3, the number of parameters might have been extended to n.

19. Professor Raymond Garver: *On the Brioschi normal quintic.*

The most direct reduction of a quintic equation to the Brioschi normal form seems to be that given by Gordan in volume 28 of the Mathematische Annalen, and improved successively by Weber and Dickson. However, the transformation which effects the reduction is valid only when a rather large number of restrictions is imposed on the given quintic. The present paper considers the significance of these restrictions (the matter has apparently not been considered before), and supplies valid transformations for two types of quintics to which the Gordan transformation cannot be applied.

20. Professor W. L. Hart: *Limited trilinear forms in Hilbert space.*

If a point $x(x_1, x_2, \cdots)$ lies in real Hilbert space, then $(\sum_1^\infty x_i^2)^{1/2}$ will be called the modulus of x. The author considers real trilinear forms $A(x, y, z) = \sum_{i,j,k=1}^{\infty} a_{ijk} x_i y_j z_k$ where (x, y, z) are in H-space. By definition, A is limited if, for some $K > 0$, $\left|\sum_{i,j,k=1}^{n} a_{ijk} x_i y_j z_k\right| \leqq K$ for all n and for all (x, y, z) whose moduli are at most 1. It is proved that if, for any specified order of summation, $A(x, y, z)$ converges for all (x, y, z) in H-space, then $A(x, y, z)$ is uniformly bounded for all (x, y, z) whose moduli are at most 1, and hence A is limited. The method used in proving this theorem would apply as well to the proof of the uniform finiteness property for multilinear forms of any multiplicity. By use of familiar methods, it is proved that, if A is limited, then its formal series converges either as a triple series or as an iterated sum, for each possible choice as to the order of summation among the indices (i, j, k). Demonstrations are also given for other easily anticipated properties of limited trilinear forms.

21. Professor M. W. Haskell: *The cylindroid in non-euclidean geometry.*

The cylindroid is the locus of the axes of a pencil of linear complexes. In euclidean geometry this is a surface of the third order and of the third class. In non-euclidean geometry, however, a linear complex will have a pair of axes, which are mutually polar with respect to the absolute, and the locus of the axes of a pencil is a surface of the fourth order and fourth class. We shall use rectangular coordinates, so that the equation of the absolute

is $x^2+y^2+z^2+k^2t^2=0$. If the special complexes of the pencil are real, their axes may be taken symmetrically as the lines

$$y-mx=0, \quad z-ct=0; \quad y+mx=0, \quad z+ct=0.$$

The equation of the cylindroid is then, in point-coordinates,

$$c(m^2+1)xy(z^2/k^2+t^2)=m(1+c^2/k^2)zt(x^2+y^2)$$

and, in plane coordinates,

$$c(m^2+1)\xi\eta(h^2+\theta^2/k^2)=m(1+c^2/k^2)h\theta(\xi^2+\eta^2),$$

which, as $k=\infty$, reduce to the well known equations of the euclidean geometry. The surface is invariant under four infinite sets of collineations and is also autopolar with respect to the four pencils of quadric surfaces

$$x^2+y^2+2\lambda zt=0, \quad xy+\lambda zt=0, \quad x^2+y^2+\lambda(z^2+k^2t^2)=0, \quad xy+\lambda(z^2+k^2t^2)=0.$$

The cylindroid can be shown to be the intersection of two linear complexes and of a quadratic complex.

22. Professor E. R. Hedrick: *Generalizations of Cauchy's integral theorem.*

In this paper, the author indicates generalizations of the well known Cauchy integral theorem for non-analytic functions of a complex variable.

23. Professor T. R. Hollcroft: *Multiple points of algebraic curves.*

In 1750, Cramer determined the maximum number of multiple points that may occur on algebraic curves of orders up to and including eight. In 1839, Plücker derived a general limit to the number of double points of an algebraic curve of given order. For multiple points of orders higher than two, limits have been found only in the case of individual curves. In the present paper, general limits are obtained defining the maximum number of multiple points of given order that may belong to an algebraic curve of given order. Two limits are found. When the order of the multiple point lies within a certain range of values, one of these limits applies; otherwise, the other.

24. Professor Glenn James: *Generalizations of Pascal's and Brianchon's theorems.*

Considering, in the case of Pascal's Theorem, that five vertices of a given hexagon are fixed, this paper studies the locus of the sixth vertex when the opposite sides of the hexagon are not necessarily collinear but various restrictions have been placed upon the triangle which they form. For example, if the determinant of the equations, (in cartesian coordinates), of the intersections of opposite sides of the hexagon is equal to a constant, the vertices of the hexagon describe twelve conics which reduce to the single conic on the vertices of the hexagon when the given constant is zero.

25. Dr. B. W. Jones: *On two regular positive quadratic ternary forms.*

Previous to the appearance of this paper there has been no proof, as far as the author knows, that the form $x^2+8y^2+32z^2$ represents all positive integers of the form $8n+1$ and that $x^2+2y^2+32z^2$ represents all positive integers of the forms $8n+1$ and $8n+3$. The Dirichlet method does not apply in this connection. Here, using some results derived by elliptic functions together with congruence theory, these proofs are given, thus completing the proof of the regularity of these two forms.

26. Professor L. H. McFarlan: *Sets of necessary and sufficient conditions for the most simple problem of the calculus of variations.*

Bolza in his *Vorlesungen über Variationsrechnung* establishes a set of five necessary conditions on an arc joining two fixed points of the plane, which minimizes an integral of the form $\int f(x, y, p)dx$, $p=dy/dx$. He also gives a set of sufficient conditions on such an arc in order that it should have the above property. These sets are not identical. In his treatment of the corresponding problem in the parametric form one set of conditions is given which is shown to be both necessary and sufficient in order that the arc have the minimizing property. In some respects the two problems treated may be shown to differ. For example the minimizing arc in the non-parametric form of the problem must not have a tangent parallel to the y axis. Such a restriction is removed in the parametric treatment of the problem. There exists however a class of problems which may be treated by either method. The existence of a set of necessary and sufficient conditions on the minimizing arc in the parametric problem indicates the existence of one such set in the other form. The problem is studied with the intention of closing the apparent gap existing between the set of necessary and the set of sufficient conditions in the non-parametric form of the problem by transforming the single set of conditions in the parametric problem into the non-parametric form.

27. Professor L. H. McFarlan: *Sufficient conditions in the problem of Lagrange with one variable end point.*

In the problem of Lagrange one considers an integral $\int f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$ and a set of differential equations $\phi_\alpha(x, y_1, \cdots, y_n, y_1', \cdots, y_n')=0$, $(\alpha=1, \cdots, m<n)$. The functions $y_1, \cdots, y_n$ are to satisfy the differential equations $\phi_\alpha=0$ and to render the integral an extreme value. In this paper the values of the functions $y_1(x), \cdots, y_n(x)$ are considered fixed for x_1, while the values for x_2 are restricted by a set of equations $x_2=X(t)$, $y_{12}=Y_1(t), \cdots, y_{n2}=Y_n(t)$. In this paper is given a set of sufficient conditions which when satisfied by the functions $y_1(x), \cdots, y_n(x)$ will cause them to minimize the above integral and satisfy the set of differential equations.

28. Professor L. H. McFarlan: *Problems of the calculus of variations in several dependent variables and their derivatives of various orders.*

The problem of minimizing an integral whose integrand contains x and n dependent variables $y_i (i=1, \cdots, n)$ and their derivatives $y_i', y_i^{(2)}, \cdots, y_i^{(k_i)}$, $(k_i \geqq 0)$, is treated by means of the known theory of the problem of Lagrange in the calculus of variations. Necessary and sufficient conditions on the minimizing arc analogous to those of Legendre, Jacobi, and Weierstrass in the more simple problems are derived. These are readily seen to contain as special cases the results obtained by Forsyth (*Calculus of Variations*, 1927) in which one dependent variable and derivatives of the first and second order enter the integrand. In the most simple problem of the calculus of variations with one dependent variable and its derivative of the first order, one passes from the fixed end-point case to problems in which the end points are required to lie on certain curves in the plane. In this paper the values of the functions y_i and their derivatives up to and including those of order $k_i - 1$ for the values x_1 and x_2 defining the end points of the minimizing arc, are restricted by a system of differential equations of order $k_i - 1$ in the variables y_i. The number of these differential equations is expressible in terms of n and the numbers k_i. For this case one obtains a necessary condition corresponding to the usual transversality condition. For special choice of the differential equations the focal point property is also generalized.

29. Professor W. E. Milne: *On the degree of convergence of the Gram-Charlier series.*

In a previous communication the author has given certain sufficient conditions for an assigned degree of convergence of the series

$$f(x) = \sum_{m=0}^{n} c_m e^{x^2/4}[(d/dx^m)e^{-x^2/2}] + R_n(x).$$

It is now shown that the following conditions are necessary: if $R_n(x) = O(n^{-k-\epsilon})$ for every n, where k is an integer and $\epsilon > 0$, then $f(x)$ must have a continuous derivative of order $2k$; if $R_n(x) = O(n^{-k})$ uniformly in the infinite interval, then $f(x) = O(x^{-2k})$ when x is large. It is also shown that if $f(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition and if $f(x)$ and its derivatives vanish to a suitable order at infinity, then the constants c_m can be chosen so that $R_n(x) = O(n^{-k/2})$.

30. Professor T. A. Pierce: *On an algorithm and its use in approximating roots of algebraic equations.*

If in the generating equations of a continued fraction the operation of addition be replaced by multiplication and that of division into unity be replaced by subtraction from unity we obtain a new algorithm. The expansion by this algorithm of any rational number is periodic. Non-periodic expansions represent irrational numbers. The application of the algorithm in approximating roots of algebraic equations is simple.

31. Mr. J. H. Roberts: *Upper semi-continuous collections of continua some of which are unbounded.*

In an abstract of a paper presented to the Society on December 27, 1928,

I failed to mention a certain hypothesis in the statement of certain results. The corrected statement is as follows: "If G is an upper semi-continuous collection of continua *no one of which separates the plane* S and such that (1) every point of the plane S belongs to some continuum of G, (2) at least two continua of G are unbounded, and (3) the space of elements of G is metric, then the space of elements of G is in continuous one-to-one correspondence with a subset of the plane S. In the present paper the condition that no continuum of G separates S is removed and the space of elements of G is shown to be a subset of a cactoid. A cactoid—as defined by R. L. Moore—is a bounded continuous curve M lying in space of three dimensions and such that (a) every non-degenerate maximal cyclic subset of M is a simple closed surface and (b) no point of M lies in a bounded complementary domain of any subcontinuum of M.

32. Mr. J. H. Roberts: *On a problem of Menger concerning regular curves.*

In 1927 Menger raised the question as to whether or not, for every positive number e, every regular curve M is the sum of a finite number of continua of diameter less than e, the common part of any two of which is at most one point. In the present paper an example is given which shows that the answer to Menger's question is in the negative. It is proved that for the special case where the set of points of M of order greater than 2 is null-dimensional Menger's question can be answered in the affirmative

33. Professors H. P. Robertson and Hermann Weyl: *On a problem in the theory of groups arising in the foundations of infinitesimal geometry.*

The paper appears in full in the present number of this Bulletin.

34. Dr. N. E. Rutt: *On the structure of certain plane continua.*

Let x denote a point and Z a bounded plane continuum which is the sum of a set $[X_\alpha]$ of continua each of which contains the point x, each of which remains connected when x is subtracted from it, and no two of which have any point in common except x. If none of the continua of $[X_\alpha]$ has a bounded complementary domain containing a point of any other continuum of the set $[X_\alpha]$, order is easily assigned among the elements of $[X_\alpha]$, so that with respect to some reference element precedence of elements of $[X_\alpha]$ and clockwise sequences may be defined. The following facts concerning the relations possible among the continua of $[X_\alpha]$ are established. If $[X_i]$ is a clockwise sequence of elements of $[X_\alpha]$, then no element X_a of $[X_\alpha]$ preceding all the elements of $[X_i]$ contains any limit point of $[X_i]$ except x. If $[X_i]$ is a clockwise sequence of elements of $[X_\alpha]$, and if both X_a and X_b are elements of $[X_\alpha]$ preceding all the elements of $[X_i]$, then of the two prime ends of X_a+X_b containing x which are arc-wise accessible from the unbounded complementary domain of X_a+X_b the one which is not a limit of any element of $[X_i]$ is also not a limit of the set of elements $[X_i]$.

35. Dr. Morgan Ward: *The foundations of general arithmetic.*

This paper is the first part of an amplification and extension of two short notes on the same topic (Proceedings of the National Academy, vol. 13 (1927), pp. 748–749; vol. 14 (1928), pp. 907–911). By taking as a basis a set closed under logical multiplication instead of a semi-group closed under ordinary multiplication, a postulational definition of an abstract arithmetic is obtained which insures unique factorization whether the set contains a finite or an infinite number of elements. Furthermore all the classical arithmetics (e.g., ideals of an algebraic field) are instances of this abstract system. A detailed development of the formal properties of the system discloses many analogies with the Boolean algebra of logic. In fact, every finite arithmetic may be interpreted as a set of elements in a Boolean algebra, although the converse theorem is false. Finally the case where the introduction of ideal elements is necessary is studied from the same abstract viewpoint.

36. Professor Hermann Weyl: *On the foundations of general infinitesimal geometry.*

The paper appears in full in the present number of this Bulletin.

37. Professor G. T. Whyburn: *Note on simple closed curves.*

In this paper it is shown that a simple closed curve (in n-space) may be defined as a compact continuum every point of which separates it locally into exactly two components; and in the plane it may be defined as a compact continuum every point of which is accessible from exactly two complementary domains. If these continua are not compact, they are open curves. It is also pointed out that if every point of a continuum M separates it locally into exactly k components, (k finite and >1), then $k=2$, and M is a simple closed curve or an open curve according as it is or is not compact.

38. Professor G. T. Whyburn: *On the set of all cut points of a continuous curve.*

In this paper it is shown that as a consequence principally of the author's results on the structure of a continuous curve relative to its cyclic elements it follows that the set of all cut points of any continuous curve in a metric space is homeomorphic with a certain kind of subset of a plane acyclic continuous curve. It is also shown, conversely, that a subset of this kind of any acyclic continuous curve in a euclidean space of dimension >1 is always identically the set of all cut points of some continuous curve. Thus we obtain a complete characterization of a point set which is topologically equivalent to the set of all cut points of a continuous curve. Incidentally, the results of this paper answer a question raised by C. Zarankiewicz in the Bulletin de l'Académie Polonaise, 1926, p. 362.

39. Professor W. M. Whyburn: *Functional properties of solutions of differential systems.*

In a recent paper (Annals of Mathematics, vol. 30 (1928), pp. 31–39) the author established existence and uniqueness theorems for a general class of differential systems. Using the notation of that paper, the present paper interprets the notion of the solutions of the differential systems being continuous functionals of the functions $f_i(x, y_1, \cdots, y_n)$, $A_{ij}(x, y_1, \cdots, y_n)$, $i, j = 1, \cdots, n$, and proves that under the hypotheses of the above-mentioned existence theorems the solutions of the systems will be continuous functionals of this type. The results here include the well known theorem concerning the continuity of the solutions of linear systems as functionals of the coefficients. The results of the present paper are useful in a study of the effects produced by small perturbations on dynamical systems.

40. Professor W. M. Whyburn: *On the critical sets for functions of n real variables.*

In a former paper, (*Non-isolated critical points of functions*, which appears in the present issue of this Bulletin (701–708), the author defined *critical sets* for functions of n independent real variables in cases where the critical points of these functions were not isolated. *Minimal sets* were characterized and theorems concerning the distribution of these were proved. The present paper continues this classification of critical sets to define several additional classes and extends the former theorems to cover this classification. The definitions and extensions are shown to be consistent with the classification by Morse (Transactions of this Society, vol. 27 (1925), pp. 345–396) when the critical points are isolated.

41. Professor W. M. Whyburn: *Functional inequalities.*

Inequalities of the types $F(x) < \sum_{i=1}^{n} f_i(x) g_i(x)$, $F(x) \leqq \sum_{i=1}^{n} f_i(x) g_i(x)$ where the functions involved are non-negative and continuous on the open or closed, finite or infinite interval X of the real variable x, are studied with a view to determining continuous functions $h_i(x)$ such that $0 \leqq h_i(x) \leqq g_i(x)$, $F(x) = \sum_{i=1}^{n} f_i(x) g_i(x)$ on X, and these functions take on assigned boundary values at points of given subsets of X. Cases where measurability and summability replace continuity in the above statements are treated. Functions of m real independent variables are considered. Applications to proofs of properties of differential equations are given.

42. Professor R. L. Wilder: *Characterizations of continuous curves that are perfectly continuous.*

It was shown by R. L. Moore and the author (see this Bulletin, vol. 29, p. 294) that in order that a continuum M should be a continuous curve, it is necessary and sufficient that every component of an open subset of M should be arcwise connected. The present paper considers those continuous curves which have the property that all of their subcontinua are continuous curves—such curves are called, for the sake of brevity, *perfectly continuous*, or *perfect continuous curves*. It is shown that the above property characterizing continuous curves may be modified so as to characterize perfect continuous curves. If M is any point set, and K is an F_σ which is a subset

of M, then let us call $M-K$ a *quasi-open subset* of M. Then in order that a continuum M in E_n should be a perfect continuous curve, it is necessary and sufficient that every component of a quasi-open subset of M should be arcwise connected. Incidentally, it is shown that every connected subset of a perfect continuous curve in a locally compact metric space is connected im kleinen—a property which also obviously characterizes these curves.

43. Dr. A. R. Williams: *On a type of transformation connected with a pencil of curves of order n.*

The fact that the n^2 base points of a pencil of curves of order n are determined by $n(n+3)/2-1$ of them gives rise to transformations of the plane. It is the purpose of this paper to discuss the locus of the remaining base points when a given number of them are fixed and the others necessary to determine the pencil are taken consecutive on some curve. Use is made of rational surfaces and developables in space of 3 and higher dimensions.

44. Dr. B. C. Wong: *A certain linear ∞^r-system of r-ic hypersurfaces in r-space.*

The hypersurfaces of order r passing through $r+1$ given $(r-2)$-spaces in S_r form a linear ∞^r-system $|V|$. The composite base manifold of $|V|$ is of order $r(r+1)/2$ and is composed of the $r+1$ given, $(r-2)$-spaces and the locus of the ∞^{r-3} lines incident with them. This locus is of order $(r+1)(r-2)/2$. It is shown that any r hypersurfaces of $|V|$ meet in only one distinct point and that, therefore, each hypersurface can be represented rationally on a hyperplane of S_r. It is possible to find r hyperquadric surfaces that will effect the transformation by making any point and the point of intersection of its polar hyperplanes with respect to the r hyperquadric surfaces correspond to each other.

45. Dr. B. C. Wong: *An octavic hypersurface in space of four dimensions.*

This paper deals with the hypersurface V_{r-1} of order 2^{r-1} in r-space represented by the equation $x_0^{1/2}+x_1^{1/2}+\cdots+x_r^{1/2}=0$ and, in particular, with the octavic hypersurface V_3^8 in 4-space. V_{r-1} is the generalization to r-space of Steiner's quartic surface in 3-space. It is touched by each of the $\binom{r-1}{k+1}$ k-spaces $[0<k<r]$ of the coordinate simplex along 2^{r-k-1} coincident $(k-1)$-dimensional varieties of order 2^{k-1}. The representation of V_3^8 and that of its reciprocal hypersurface on a hyperplane are discussed and also that of its general hyperplane section.

46. Professor D. W. Woodard: *On a system of axioms for two-dimensional analysis situs.*

In the author's doctoral dissertation, *On two-dimensional analysis situs with special reference to the Jordan curve theorem*, published in volume 13 of Fundamenta Mathematicae, the following axiom is used: *If P is a point on the boundary of a region R, there exist two simple continuous arcs AP and*

BP such that AP−P and BP−P are subsets of R and S−R′ respectively. Here S represents the set of all points in the space under consideration and R', the set of points contained in R and its derivative. It is proved that the axiom stated above may be replaced by the following weaker axiom: *Every point on the boundary of a region is a limit point of the exterior of the region.* In connection with this axiom an independence example is provided. All of the results of the paper follow from the modified system of axioms. In fact, it is necessary only to make certain additional arguments in the proof of Theorem 7 of the paper.

47. Dr. Leo Zippin: *On acyclic continuous curves.*

The author constructs a plane, bounded, acyclic continuous curve which has a subset homeomorphic with any given acyclic continuous curve. In case the given curve is unbounded it is homeomorphic with the complement in a bounded acyclic continuous curve of a subset of end points. He proves, also, that if B is a closed and totally disconnected subset of a continuous curve C, in order that B be the set of end points of an acyclic continuous subcurve of C it is necessary and sufficient that if M is any bounded continuous curve of C and D a closed and totally disconnected subset of interior points of M, then $M-D$ is connected.

B. A. Bernstein
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THE PROBLEM OF THE CUBIC VARIETY IN S_4*

BY VIRGIL SNYDER

1. *Introduction.* One of the outstanding problems of algebraic geometry that is still unsolved is the classification of Cremona transformations in space of more than two dimensions, and another, closely related to the preceding, is that of possible series of composition.

The theorem of Noether (80), later clarified and interpreted by Segre (90), Castelnuovo (14), Chisini (17), and Alexander (1), has answered the latter problem for the plane; while a large number of papers, represented in particular by the recent work of Hudson (67), Mlodzieioski (73)–(76), and Montesano (79), have made the construction of tables a definite procedure.

The smaller problem of the involutorial transformations has been completely solved by Noether (81), Bertini (6), and others, thus furnishing a weapon of incalculable importance for the study of various applications. Periodic transformations and their groups have been studied from various points of view by Kantor (69)–(70), Wiman (112), and Coble (20)–(21).

The corresponding problem in space is still almost undeveloped. Notwithstanding the excellent report made by Coble (22), the less ambitious one by me (103), the appearance of the extensive treatise by Hudson (67), and the Report of a committee of the National Research Council (106), I wish to speak of one phase of the latter problem. For regular transformations in space of three dimensions, the important theorem of Hudson (66) and a recent memoir of Montesano (78) are distinct steps in advance. When the

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defining elements are chosen independently, without contact conditions, the table of characteristics for a Cremona transformation and for its inverse has been completely constructed.

I wish to confine my attention, however, to the part taken by the cubic variety of four-way space S_4 in the study of involutorial birational transformations of S_3.

The study of the involution defined by the web of quadrics through six points leads at once to the Weddle surface, the locus of the vertex of the quadric cones through the six points, as surface of coincident points. If the quadrics of the web be made projective with the planes of another space S_3', a (1, 2) correspondence is established, and the Kummer surface appears as surface of branch points in S_3'. All the properties of the general transformation and of all particularizations can be made from this point of view. This study of (1, 2) correspondences is an extension of that initiated by De Paolis (27) for S_3, and applied to double planes by Enriques (32) and by Castelnuovo and Enriques (15), thus confirming the result obtained by Noether (81) concerning the number of plane involutions of order two.

But the Kummer surface is the apparent contour of the cubic variety with ten nodes, as has been shown by Segre and further developed by Hudson (68) and by Snyder (101). Moreover, the same surface also appears in a similar rôle in connection with involutions defined by webs of cubic and of certain quartic surfaces. See Sharpe and Snyder (100).

In the same way, the quartic surface with less than sixteen nodes, birationally equivalent to the focal surface of line congruences of order two and class 3, 4, 5, 6 appear as surfaces of branch points of certain involutions, and each is the apparent contour of a cubic variety with one or more actual double points. Thus, to a point on the cubic variety correspond a pair of points in S_3, each point of which uniquely determines the other. These pairs of points determine an involution belonging to V_3.

2. *Rational Involutions.* Given an algebraic curve C in S_n, a necessary and sufficient condition that C is rational is that its points can be put into (1, 1) continuous correspondence with those of a straight line. (Definition.)

If $\rho x_i = f_i(y_1, y_2)$, $(i = 0, 1, \cdots, n)$, define the coordinates (x) of a point on $C(x)$ and y_1, y_2 those of the image point on the line, then f_i is a rational integral polynomial homogeneous in y_1, y_2.

If several sets of values of y_1, y_2 define the same variable point (x), then functions $y_1 = \phi_1(y_1', y_2')$, $y_2 = \phi_2(y_1', y_2')$ can be found such that the relation between (y') and (x) is (1, 1). This is expressed by saying that *every involution on a rational curve is rational.* (Theorem of Lüroth (71)).

Similarly, given an algebraic surface $F(x)$ in S_n, such that $\rho x_i = f_i(y_1, y_2, y_3)$, wherein each f_i is a rational integral form of the same degree in (x), then $F(x)$ is rational. If several sets of values of (y) define the same variable (x), then functions $y_i = \phi_i(y_1', y_2', y_3')$ can be found such that the relation between (y') and (x) is (1, 1). Hence *every involution on a rational surface or in particular on a plane, is rational.* (Theorem of Castelnuovo (11).) The theorem for involutions of order two was already known. See Noether (81) and Bertini (6). More recent contributions to the classification of plane involutions of order higher than two have been made by Howe (65), Hollcroft (64), and Sharpe (99).

Given a three-dimensional manifold $M_3(x)$ and $\rho x_i = f_i(y_1, y_2, y_3, y_4)$ a parametric representation, wherein f_i is a rational integral form, we cannot conclude whether M_3 is rational or not. The condition is necessary but it is not sufficient. If various sets of values (y) define the same variable set (x), it may not be possible to find a set of functions $y_i = \phi_i(y_1', y_2', y_3', y_4')$ such that the relation between (x) on $M_3(x)$ and (y') in S_3' is (1, 1). (Theorem of Enriques (36).)

Illustrations have been given of irrational involutions in S_3 belonging to a three-dimensional variety; others were furnished by Aprile (2) but all are of order larger than two. No illustrations have been found of three-dimensional in-

volutions of order two that are known to be irrational.

In particular, it is not known whether the points of the general cubic variety V_3 in S_4 can be put into (1, 1) correspondence with those of S_3.

It is known that an involution of order two in S_3 belongs to it. The existence of this involution was known to Noether, and later proved by Segre (89) and by Marletta (72). The actual form of the parametric represention is

$$\begin{aligned}
\rho x_1 &= y_1^2(2y_1^3 y_3 y_4 + Ay_3 + By_1),\\
\rho x_2 &= y_1 y_2(\quad\quad " \quad\quad),\\
\rho x_3 &= -\, y_3^2(\quad\quad " \quad\quad),\\
\rho x_4 &= y_1(y_1^3 y_3 y_4^2 + By_1 y_4 - Cy_3),\\
\rho x_5 &= -\, y_1(y_1\, y_4^2 + Ay_1 y_4 + Cy_1),\\
V_3 &= x_1 x_4^2 + x_3 x_5^2 + ax_4 + bx_5 + C = 0,
\end{aligned}$$

$$\begin{aligned}
a &= a_0 x_3^2 + a_1 x_3 + a_2, & A &= a_0 y_3^4 - a_1 y_3^2 y_1 + a_2 y_1^2,\\
b &= b_0 x_3^2 + b_1 x_3 + b_2, & B &= b_0 y_3^4 - b_1 y_3^2 y_1 + b_2 y_1^2,\\
c &= c_1 x_3^2 + c_2 x_3 + c_3, & C &= c_1 y_3^4 - c_2 y_3^2 y_1 + c_3 y_1^2,
\end{aligned}$$

where a_i, b_i, c_i are binary forms of order i in x_1 and x_2.

This form of order 7 is the simplest one consistent with a quadratic locus of invariant points. This result was obtained and the associated involution of order 6 is described by Snyder (104).

3. *Involutions Belonging Multiply to a Line Complex.* Every V_3 contains ∞^2 straight lines. See Enriques (28) and Fano (57)–(58). Let l be a line on V_3, and P any point on V_3.

The plane (P, l) meets V_3 in a residual conic through P which meets l in two points Z_1, Z_2. The line PZ_i is tangent to V_3 at Z_i. Conversely, any line tangent to V_3 at a point Z on l meets V_3 again in a point P.

Between the points P of V_3 and the tangents at points on l there is therefore a (1, 2) correspondence. These tangent lines form a special linear line complex which can be mapped birationally on the points of an S_3. The lines joining a pair of associated points PP' in S_3 always meet the rectilinear image

of l; hence they define a special linear complex. But every line of the complex contains two pairs of associated points. (Snyder (104).)

Other I_2 in S_3 exist which are contained multiply in a rational complex. The simplest one is obtained by constructing a $(1, k)$ correspondence between a pair of lines on a regulus and the quadrics of a pencil. A point P of S_3 determines a quadric passing through it, and hence a pair of directrices (d, d') of the regulus. The transversal of (d, d') through P meets the quadric in P'. The line PP' contains k pairs of associated points, and belongs to a general linear complex. (Snyder (107).)

Consider a line and a pencil of surfaces of order n having the line to multiplicity $n-2$. Make the same $(1, k)$ correspondence between the points of the line and the surfaces of the pencil. The line may be replaced by a conic provided the surface is of order less than 5, or by a space cubic.

A pencil of quadrics and a rational curve of any order may be taken. Similarly, a linear system consisting of a line and of a rational curve of order m meeting it in $m-1$ points may be put in $(1, k)$ correspondence with the quadrics of a pencil. Either the line or the curve may remain fixed. The system may be replaced by a pencil of cubic curves as directrices. All of these forms contain types which belong to the cubic variety as particular cases. Incidentally they all possess properties not shared by other involutions heretofore known.

Another illustration is that of order 7 found by Montesano (77). Given five linear line complexes K_i and a projective form q or g of five elements. An arbitrary plane π of space has five poles O_i as to K_i, and in it is one point O' such that the lines $O'O_i \overline{\wedge} g$. There is one and only one complex K in involution with every K_i. The plane π has a pole O as to K. The relation between O, O' is birational and involutorial.

The line OO' describes a linear complex K and every line of K contains three pairs of conjugate points. There is no surface of invariant points, but a curve of order 8 and genus 5, birationally equivalent to the only fundamental curve of

the system, apart from parasitic lines. It is not known whether this involution belongs to the cubic variety or not. Its properties are different from those of the other types just mentioned.

In particular, if the set of complexes be considered, such that each new one is in involution with five of the given ones (K_i, K), and the corresponding transformation be considered, the rôles of the fundamental curve and of the curve of invariant points will be interchanged. The scheme really furnishes ∞^2 pairs of such complementary involutions, corresponding to the ∞^2 possible values of g.

4. *Infinite Discontinuous Groups.* Various infinite discontinuous groups which belong to V_3 exist. We shall consider six distinct generating operations numbered (a), (b), (c), (d), (e), (f) below. It has not been shown whether others also exist.

(a). *The projection of V_3 upon itself from a vertex A upon it.* The product AB of two such projections is not periodic, nor can it be replaced by another product. Similarly for the product of three or more such projections.

The projection A is Cremonian, and quadratic. An S_3 is transformed into a quadric variety having the polar quadric of A for locus of invariant points. The hyperplanar sections of V_3 are transformed into sextic surfaces all passing through the six lines of V_3 which pass through A. These lines have the property that the image of a point on any one of them is the whole line passing through it.

In the product AB the S_3 sections of V_3 are transformed into surfaces of order 12, having six double basis lines through B, and six conics, all lying on a quartic surface, through another point, the image of A in the operation B. Similarly for products of more projections $A, B, C \cdots$. (Snyder (102).)

(b). *Let l be a fixed line on V_3. A point P of V_3 and l determine a plane which meets V_3 in a residual conic passing through P.*

The line l has a pole L as to this conic. The harmonic homology defined by L and l leaves the conic, and hence also V_3, invariant. The transformation is Cremonian, of order 8. The conjugate of an S_3 contains l to multiplicity 7. The locus of invariant points is a variety of order 5, containing l to multiplicity 4. The image of l itself is a variety of order 7, containing l to multiplicity 6. Every S_3 through l is transformed into itself.

There are ∞^2 lines on V_3, each of which has an associated involution (b). The product of two such involutions, associated with different lines, is not periodic, nor can one such product be replaced by another when the lines are taken arbitrarily.

The product of a transformation of the type (a) and a transformation of the type (b) is in general not periodic. (Snyder (102).)

(c). *Consider a pencil of* S_3 *having* π: $(ax)=0$, $(bx)=0$ *for base, and the line* $l \equiv x_1=x_2=x_3=0$ *on* V_3. Make the points of the line and the hyperplanes of the pencil projective, by associating each S_3 with the point P in which it meets l. Project the cubic surface S_3, V into itself from P. The transformation is Cremonian, of order 4, the fundamental elements are l taken simply, and the basis plane taken twice. The locus of invariant points is a cubic variety containing l, π each simply.

A transformation of this kind exists for every line l on V, and when l has been fixed, the basis plane may be chosen in ∞^6 ways. The product of two or more transformations of this type is not periodic. The product of any transformation of type (c) with one of type (a) or of type (b) or any sequence of (a), (b), (c) is in general not periodic.

The line l may be replaced by a conic, provided the basis plane has one point on it. The variety V_3 contains ∞^4 conics. Moreover, a space cubic with two points on the basis plane or a rational quartic with all three basis points upon it, may be taken. The directrix curve appears as a basis curve with multiplicity equal to its order in every case.

(d). *Assume an arbitrary fixed line m through a point M on V_3, but not lying on V_3.* A point P of V_3 and the line m determine a plane which meets V in a cubic curve through M. The first tangential M' of M, joined to P, meets V_3 in P'. The transformation P, P' is involutorial and Cremonian of order 8. The locus of M' is the surface of intersection of V_3 and the three-dimensional tangent space at M, of order 3. Every plane through m is transformed into itself. The transformation in each such plane is quadratic, a perspective Jonquières type of the third kind. The fundamental line is the tangent to the cubic curve at M', and the locus of invariant points is the polar conic of M' as to the cubic. The images of the hyperplane sections $|x|$ of V_3 are surfaces of order 24, $|8x|$, having M to multiplicity 6, and the six lines of V through M to multiplicity 2. They also contain the locus of the first tangential of M' as the plane describes the linear system through m.

The product of two transformations of type (d) is not periodic. There are ∞^3 points on V_3, and ∞^3 directions m associated with each point. The properties of this transformation should be further studied, both in S_4 and in S_3. Many of them are different from those of involutions heretofore known. The product of a transformation of type (d) with any one of a preceding type or one derived from any combination of the preceding types is in general not periodic.

(e). *Let l, l' be two skew lines on V_3 and π a plane not meeting either.* The plane is the basis of a pencil of S_3. The S_3 through a point P meets (l, l') in (K, K'). The line (K, K') meets V in M. The line PM meets V_3 in P'. The transformation is involutorial, and consists in interchanging P and P'. The locus of M is a rational C_4. The lines l and l' lie in a space Σ_3 which cuts V_3 in a cubic surface F_3 containing l and l'. The pencil (S_3) meets (l, l') in ranges projective with it, hence with each other, hence (K, K') describes a quadric surface H_2 in Σ_3, through (l, l'). The residual intersection of F_3 and H_2 is a rational C_4 which meets (l, l') each in three points.

Either or both lines may be replaced by conics skew to each other, provided π meets each conic in one point, or one may be replaced by a cubic with two points on π, with the other directrix either a line skew to π or a conic with one point on π. When both directrices are cubics, they must have a common point on π. Corresponding forms exist in a three-way space which leave a given cubic surface invariant, each of which presents interesting configurations of fundamental elements.

(f). Another transformation is the following. *Given a line l on V_3 and a plane π of S_4, skew to l.*

The points O of π and the planes through l are projective, and perspective. The plane through l meets V_3 in a residual conic. Consider the harmonic homology having O for center and its pole as to the conic in its associated plane for axis. This procedure defines an involutorial birational transformation which leaves each conic of l in a plane through l invariant, and hence transforms V_3 into itself.

It is Cremonian. The line l and the plane cubic curve section of V_3 by π are the fundamental elements. The plane π may be replaced by any rational surface meeting an arbitrary plane through l in one point apart from those on l.

5. *Continuous Groups.* A curve of genus zero belongs to a three-parameter group of continuous transformations. One of genus 1 has ∞^1 transformations which do not form a group. If the genus exceeds unity, the curve belongs at most to a finite group. Surfaces invariant under ∞^1 linear transformations are rational or reducible to ruled surfaces. If they are invariant under ∞^2 such transformations they are necessarily rational. (Fano (40).) Every algebraic surface, invariant under a transitive continuous group of linear transformations, can be transformed birationally into a plane, or a quadric of S_3, or into a rational cone normal in a certain S_{m+1} in such a manner that the group considered gives rise respectively to a group of plane homographies, or to a group of homographies in S_3, or in S_{m+1}, each leaving the associated form invariant. (Fano (41) and (44).)

Every three-parameter linear group of S_3 leaves a series of forms generated by planes invariant, none containing a simpler one. (Fano (40).)

The three-dimensional varieties of S_4 invariant under ∞^4 continuous linear transformations are classified into five types, which do not include the general cubic variety. (Fano (43).)

Surfaces invariant under a continuous group of Cremona transformations are all reducible to rational surfaces in S_3 invariant under linear transformations. (Fano (45).) Similarly for three-dimensional varieties. (Fano (46).)

Given a homogeneous linear differential equation of order n. If a system of solutions $y_1, \cdots, y_n$ be regarded as projective coordinates of a point of S_{n-1}, the point will describe a rational curve when the coefficients in the given equation are rational functions of x. If the (y) satisfy one or more algebraic relations, the curve will lie on a corresponding number of varieties. The group of rationality of the equation leaves each variety of the system invariant. Finite groups of Cremona transformations in the plane, both according to the order and to the type, in the sense of Kantor and of Wiman have all been determined in this way. (Enriques and Fano (38).) The types of continuous groups in the plane are projective, quadratic and Jonquières, and in S_3 contain these and two other ∞^3 types. The generalized Jonquières groups leave a pencil of planes or a bundle of lines (or both) invariant. (47).

The imprimitive groups of continuous Cremona transformations in S_3 belong as subcases in the types derived by Enriques (29) and Fano (46), (48).

The complete enumeration of continuous Cremona transformations in S_3 can be made by starting from the generating infinitesimal transformation. This has been done and the preceding results confirmed. (49).

A differential equation of the type in question is always solvable when more than one algebraic equation is identically satisfied by the solutions. (50).

The group of rationality may leave a quadratic form in S_{n-1} invariant. (52). For $n=6$, this case can be interpreted in terms of lines of S_3; instead of a curve, the locus is now a ruled surface. For $n=5$, the same interpretation can be made by assuming a linear relation among six solutions. The ruled surface is then contained in a linear line complex. (51). If the discriminant of the quadratic form vanishes (special complex), the equation can be reduced to one of order 4. (54).

Every M_3 invariant under ∞^3 projective transformations is rational. If the group is not integrable, the existence of ∞^1 rational surfaces on it insures its rationality. (53).

These results have been collected and systematically developed by Fano (55). It is shown that the existence of algebraic relations among the fundamental solutions of a linear differential equation is a necessary and sufficient condition for the vanishing of certain invariants belonging to the equation.

On the general quartic three-dimensional variety V_3^4 of S_4, without double points, the system of hyperplane sections is the only linear system of regular surfaces with genera equal to unity and of dimension not less than 2. This variety is not invariant under any birational transformation, with the possible exception of linear ones.

The M_3^6 of S_5, complete intersection of V_4^2 and V_4^3, does not contain such surfaces either. Every birational transformation which leaves it invariant is the product of a finite number of double projections from lines lying on it and of possible linear transformations. (Fano (61).)

6. *Invariants.* The question naturally arises whether algebraic or topological invariants exist, particular values of which can indicate the rationality or irrationality of V_3.

A necessary and sufficient condition that an irreducible algebraic curve in space of any number of dimensions shall be rational is that its genus is zero. (Clebsch (18).)

An algebraic surface has an infinite number of invariants analogous to the genus of a curve, but a necessary and

sufficient condition that a surface shall be rational is that its arithmetic genus p_n and its bigenus P_2 should both be zero. (Castelnuovo (13).) That a surface may have both its geometric genus and its arithmetic genus equal to zero and still not be rational, was shown by Enriques (35).

A system of invariants of three-dimensional varieties was determined by Pannelli (84), analogous to the genera of a surface. (Castelnuovo and Enriques (16).) Some of the fundamental concepts of invariants on hyperdimensional manifolds had been discussed by Noether (82). It was then shown by Severi (97) that all these invariants vanish for the cubic variety of S_4, also for the general quartic V_3^4 of S_4, and for the $M_3^6 \equiv (V_4^2, V_4^3)$ of S_5, hence that the various indices of irregularity are zero. These latter V_3^4, M_3^6 contain only complete intersections. (Severi (92), Fano (59).)

The algebraic surfaces contained in V_3^3 were obtained by Fano (56).

The M_3^6 is the representative of the general cubic line complex of S_3, which has been extensively studied by Voss (111) and by Veneroni (109). In particular, it has ∞^1 linear pencils of lines or quadric reguli, hence M_3^6 contains ∞^1 straight lines. By projecting the manifold from one of them, it can be mapped on S_3 doubly, with a sextic surface of branch points. The two forms, V_3^4 and M_3^6, are birationally distinct. (Fano (59).)

The assumption that either contains a homaloidal system of surfaces leads to a contradiction; hence both of these forms are irrational, that is, they can not be mapped birationally on S_3. (Fano (59).) However, both can be mapped on S_3 by means of an involution that is rational in one sense only.

The actual process was outlined for M_3^6 by Enriques (36), which establishes the important result that irrational involutions in S_3 exist. Other examples of involutions belonging to three-dimensional varieties were given by Marletta (72), and Aprile (2) showed that the Enriques involution is of

order 216, but that the manifold can be mapped on S_3 by means of an involution of order 36. Further conditions for rationality were found by Enriques (37).

These forms are much more general than V_3^3. The general M_3^6 does not contain any planes, since the general cubic line complex of S_3 does not contain a bundle of lines. If now the complex be particularized to contain three distinct bundles of lines (or dually, three plane fields of lines), then the M_3^6 contains three planes belonging to the same system on the quadric variety passing through it. This particularized M_3^6 is birationally identical with the general V_3^3.

Similarly, the V_3^4 can be particularized to the extent of having a double line and still be much more general than the general cubic variety. It may be desirable to study the properties of these particular forms in order to obtain those of the general V_3^3.

The two general forms V_3^4 and M_3^6 are included in the general category M_3^{2p-2} of S_{p+1} (Fano (62)) having surface sections regular with all their genera equal to unity, and curve sections canonical curves of genus p. V_3^4 corresponds to $p=3$ and M_3^6 to $p=4$. The case $p=2$ appears as a double S_3 with a sextic surface of branch points, but this can be obtained by projecting M_3^6 upon S_3 from one of its lines, in a particular form.

The case $p=5$ is represented by the intersection of three general quadratic varieties in S_6. It contains no surfaces other than complete intersections with other forms. This manifold can be mapped on S_3 by an involution of order 4. It contains an infinite number of rational congruences of the first order, of rational curves.

The general $V_3^n : S_1^{n-2}$ of S_4 has been studied from various points of view, but most of the properties found for larger values of n do not exist for $n=3$. (Enriques (33).)

As yet no topological invariants have been found that are characteristic of the general V_3^3, normal in S_4.

Various properties regarding the analytical representation of a variety and of manifolds have been found by Severi

(91), those regarding postulation (93), those concerning complete intersections of non-singular forms (94), (95).

7. *Sections by* $|S_3|$. If all the plane sections of a surface in S_3 are rational, the surface is rational. It is either ruled or is a Steiner quartic surface with three concurrent double lines. (Picard (85), Guccia (63).)

If the sections by $|S_{n-1}|$ of a manifold M_3 in S_n are rational surfaces, M_3 is rational except possibly when M_3 is V_3^3 of S_4, concerning which no conclusion can be drawn, as the method does not apply. (Fano (60).)

Probably the transcendental methods of Bagnera-De Franchis (3)-(5), those of Enriques-Severi (3), and those of Severi (96), (98) as applied to hyperelliptic surfaces, may be extended to three-dimensional varieties. As yet no light has been thrown on V_3^3 from this source. Another possible method of extension is that of Comessatti (23) in his study of real rational surfaces.

8. *Rational* V_3^3. Let there be given a cubic variety $V_3^3(x) = 0$ in S_4, such that each x_i may be expressed in the form $\rho x_i = f_i(y_1, y_2, y_3, y_4)$, each f_i being a rational quaternary form of order N. When these values of x_i are substituted in $V(x) = 0$, it shall be identically satisfied, and that for an arbitrary set of values of the y_i; no other set can be found which will define the same value of (x).

Since $\sum a_i x_i = 0$, $V_3^3(x) = 0$ define a cubic surface, it follows that the $f_i(y)$ must satisfy the following properties.

(a) The system is linear of dimensionality 4.

(b) Any two surfaces of the system intersect in a curve of genus 1. (Plane sections of a cubic surface.)

(c) Any three intersect in three variable points.

From any simple point O_x on $V_3^3(x)$ project the variety on any space $S_3 \equiv \pi$ not passing through O_x. In π, every point is the image of two points of V_3^3; those on the generators of the cone of apparent contour project into the points of the quartic surface of branch points $L_4(x) \equiv u_2^2(x) - u_1(x) \cdot u_3(x) = 0$ in which $u_1 = 0$ is the tangent S_3 to V_3^3 at O_x. The plane

$u_1=0$ in π touches $L_4(x)=0$ along the conic $u_2=0$, which contains the six double points on $L_4(x)=0$, defined by $u_1=u_2=u_3=0$. Between the two three-spaces π and (y) there is a (1, 2) correspondence, in which $L_4(x)=0$ in π is the surface of branch points.

Let the equation of V_3^3 be $u_1x_5^2+2u_2x_5+u_3=0$. Then the projection of V_3^3 upon $x_5=0$ as double S_3 from $O_x\equiv(0, 0, 0, 0, 1)$ has the surface $L_4(x)=0$ of branch points. Let O_y be the image of O_x supposed to be non-singular in $\rho x_i=f_i(y)$. Then (y_0) satisfy the four equations $f_i(y)=0$, $(i=1, 2, 3, 4)$.

In the (1, 2) correspondence between (y) and $x_5=0$, the point O_5 is singular. In the projection upon $x_5=0$ from O_x, the conjugate of O_x is the plane $u_1=0$; it is the intersection with $x_1=0$ of the tangent S_3 to V_3^3 at O_x. The conjugate of O_y is then the f of the ∞^4 system defined by $u_1(f)=0$.

The points of $u_1=0$ in π are of three kinds, those not on L_4 are all conjugate to O_x; those on $u_2=0$, $u_1=0$ but not on $u_3=0$ are all branch points at O_x, as $u_2=0$ defines the quadric cone of inflexional tangents to V_3 at O_x.

Finally, the six double points $u_1=0$, $u_2=0$, $u_3=0$ are images in $x_5=0$ of the six lines of V_3 which pass through O_x.

Any S_3 passing through O_x meets V_3 in a cubic surface which is transformed into itself by interchanging the two points of V_3 on every line through O_x.

Hence in (y), each $f_i(i\neq 5)$ is transformed into itself. The curves $f_k=0$, $f_i=0$, $(i, k\neq 5)$ of genus 1 are the images of the lines of S_3.

The ∞^4 system $|f|$ is not invariant under I_2, and the subsystem is still of grade 3 when the fixed point O_y is adjoined.

A section $\sum a_ix_i=0$ not passing through O_x meets V in a general cubic surface. The three-dimensional cone joining the surface to O_x meets V in a sextic surface, the complete intersection of V and a quadratic variety $|2x|$, obtained from $\sum a_ix_i$ by the quadratic transformation

$$(T)\qquad \begin{cases} \rho x_i' = u_1x_i, & (i=1, \cdots, 4),\\ \rho x_5' = -u_1x_5-u_2. \end{cases}$$

The corresponding transformation in (y) becomes

$$\sigma f_i(y_1', \cdots, y_4') = u_1(f(y))f_i(y),$$
$$\sigma f_5(y_1', \cdots, y_4') = -u_1(f(y))f_5(y) - u_2(f(y)),$$

and for the involution I_2, of associated points,

$$f_i(y') = f_i(y).$$

Thus, the quadratic image of any S_3 section in the transformation T passes through the quadric surface $u_1=0$, $u_2=0$, which is the intersection of the tangent space to V_3 at O_x and the quadric polar of O_x as to V_3.

This is the quadric of inflexional tangents to V_3 at O_x. It meets V_3 in the six lines passing through O_x.

The images of these six lines are six curves through O_y, common to the conjugates of the ∞^4 $|f|$. They are all parasitic curves.

Suppose V_3^3 has a double point not lying in the tangent S_3 at the center of projection. Then in $x_5=0$ the surface of branch points $L_4=0$ has a double point not lying in the singular plane $u_1=0$ associated with the center of projection.

Conversely, if $L_4=0$ has a double point not lying in $u_1=0$, from the vanishing of the first derivatives of L_4 as to x_1, those of V_3^3 can be shown to vanish. Hence we may draw the following conclusion.

A necessary and sufficient condition that the surface of branch points $L_4=0$ has a double point not in the singular plane associated with the center of projection is that V_3^3 has a double point not in the tangent space at the center of projection.

If the values x_i are substituted in L_4, the result is a perfect square $K^2(y)=0$: the surface of contact of V and its three-dimensional tangent cone from O_x.

If V_3^3 has a double point P not at O_x, its image will be an actual double point of $L_4=0$, not on $u_1=0$. In this case, the parametric representation of $V_3^3 : P^2$ can be obtained immediately by projecting V_3^3 from P and cutting the projecting cone by any S_3 not passing through P.

If $u_1 \equiv x_1$, and if u_2 and u_3 contain x_1 at most to the first power, then

$$V_3 \equiv u_1 x_5^2 + 2u_2 x_5 + u_3$$
$$= x_1 x_5^2 + 2(x_1 v_1 + w_2) x_5 + w_3 = 0 \equiv x_1 t_2 + t_3,$$

where v_i, w_i are ternary in x_2, x_3, x_4 of degree i, and

$$t_2 = x_5^2 + 2v_1 x_5 + v_2, \qquad t_3 = 2w_2 x_5 + w_3,$$

represents a cubic variety having a double point at the point (1, 0, 0, 0, 0).

The representation in the space (y) now takes the form

$$\rho x_1 = -t_3(y_2, \cdots, y_5),$$
$$\rho x_i = y_i t_2(y_2, \cdots, y_5), \qquad (i = 2, \cdots, 5).$$

The image of the double point is the quadric surface $t_2 = 0$, and the fundamental element in (y) is the sextic space curve γ_6 of genus 4, $t_2 = 0$, $t_3 = 0$.

The ∞^4 linear system, images of the S_3 sections of V_3, now has the form

$$\sum a_i y_i t_2 - a_1 t_3 = 0;$$

thus it includes all the cubic surfaces passing through γ_6. This system satisfies conditions (a), (b) and (c).

The image in (y) of the vertex $(0, 0, 0, 0, 1) \equiv O_x$ is the point (0, 0, 0, 1), hence the ∞^3 subsystem which defines the (1, 2) correspondence between $x_5 = 0$ and (y) has the form

$$\rho x_1 = -2w_2(y_2, y_3, y_4) y_5 + w_3(y_2, y_3, y_4),$$
$$\rho x_2 = y_2 t_2(y_2, \cdots, y_5),$$
$$\rho x_3 = y_3 t_2(y_2, \cdots, y_5),$$
$$\rho x_4 = y_4 t_2(y_2, \cdots, y_5),$$

$$\sigma y_2 = x_1 x_2,$$
$$\sigma y_3 = x_1 x_3,$$
$$\sigma y_4 = x_1 x_4,$$
$$\sigma y_5 = -(v_1 x_1 + w_2) \pm [(v_1 x_2 + w_2)^2 - x_1(x_1 v_2 + w_3)]^{1/2}.$$

The equations of I_2 in (y) can now be determined at once. The transformation is quartic, monoidal perspective. The surface of invariant points is

$$w_2y_5^2 - w_3y_5 + v_1w_3 + v_2w_2 = 0.$$

Now suppose that V_3^3 is always rational, and that its equation contains a parameter λ, such that when $\lambda = 0$ the corresponding variety acquires a double point at P. Then

$$\rho x_i = \phi_i(y,\lambda) = y_iv_2(y)F(y) + \lambda f_i(y,\lambda), \quad (i = 2, \cdots, 5),$$

$$\rho x_1 = \phi_1(y,\lambda) = -t_3(y)F(y) + \lambda f(y,\lambda).$$

Consider any linear system of at least ∞^4 rational surfaces of order N. By using the common basis elements as part or all of the fundamental elements of the transforming system, suppose the system has been reduced to its simplest form. Then within the system construct a subsystem which satisfies conditions (a), (b), (c). Evidently, by any birational transformation of (y), the properties of $V(x)$ will remain unchanged. Any rational surface may be transformed by means of its adjoints into one having plane sections of genus 0, 1, 2, 3 or of any positive genus, but hyperelliptic. (Enriques (33), (34).) We shall consider only those linear systems which belong to one or another of these five types, I, $\cdots$, V and which satisfy the conditions (a), (b), (c).

Finally, we select an additional basis point to define the (1, 2) correspondence. The five types will be considered in turn, but the three following properties will first be established.

9. *Properties of Basis Elements.* In the ∞^4 system $|f(y)|$, images of the spatial sections $|x|$ of V_3^3, is a subsystem ∞^3 $|\bar{f}(y)|$ having an additional basis point not a basis point of the larger ∞^4 system. This special basis point is the image of an ordinary point O_x on V_3^3. If now V_3^3 be projected upon a space π not passing through O_x from O_x doubly, then between π and (y) exists a (1, 2) correspondence, such that each $\bar{f}(y)$ is transformed into

itself, both images of a point in the image plane in π being in this surface. In the projection of V_3^3 from O_x upon π, the fundamental element is the plane, which is the intersection of the tangent S_3 to V_3^3 at O_x with π. The image of this plane in (y) is a definite $\bar{f}$ of the subsystem.

If $|\bar{f}|$, and therefore $|f|$, has any other simple basis point, there is one surface $\bar{f}$ belonging to it having this point for double point; the point and the nodal surface are conjugate in the associated I_2 of (y). The associated plane in π must therefore touch L at every common point, since its image in (y) is composite; that is, the plane is a singular tangent plane of L and hence contains six double points. In no cases can all these points be in the first singular tangent plane. Therefore we have the following results.

(A) *If* $|f|$ *contains a simple isolated basis point, not that defining* $|\bar{f}|$, *L has double points not in the characteristic singular plane.*

Similarly, if any surface of the system $|\bar{f}|$ is composite, the two components are conjugate to each other in I_2, and the associated plane in π must touch L at every common point; that is, it is a singular plane; hence L contains double points not in the singular plane u_1. (Sharpe and Snyder (100).)

(B) *If* $|\bar{f}|$ *contains a composite surface, L has double points not in the characteristic singular plane.*

Let δ be a curve in (y), which meets the surfaces of $|f|$ only in basis points or in points of basis curves. The surfaces of the system which pass through a point of δ must contain the whole curve. Hence ∞^2 surfaces of $|\bar{f}|$ contain δ. The images of these surfaces are ∞^2 planes having the image of δ in common, hence it must be a point. The curve must lie entirely on K, the surface of coincident points of (y), and the image point is a double point of L. (Segre (89).) If δ does not pass through O_y, this double point on L can not be in the characteristic singular plane of L.

(C) *If* (y) *contains a curve* δ *meeting* $|f|$ *only in basis points,*

and not passing through the basis point O_y of $|\bar{f}|$, L contains a double point not lying in its characteristic singular plane.

10. *Rational Plane Sections.* I. When the plane sections of a surface $f(y)=0$ are rational the system $|f|$ consists of ruled surfaces or of Steiner surfaces, that is, quartics with three concurrent double lines. (Picard (85), Guccia (63).) The simplest case satisfying conditions (a), (b), (c) is that of quadrics through 5 points. (Sharpe and Snyder (100).) The surface $L_4=0$ is now the 16-nodal Kummer surface, hence $V_3(x)=0$ is nodal. In case of surfaces of order greater than 2, the variable curve of intersection is always rational since the double curves of each f_i must belong to the basis curve of the system. (Castelnuovo (12), Enriques (30).)

11. *Plane Sections of Genus* 1. II. The simplest case is $N=3$. Conditions (a), (b), (c) are satisfied; in every case $L_4=0$ has a number of extra nodes. (Sharpe and Snyder (100).) Compare the general properties of linear systems of plane curves of general genus. (Segre (88).)

$N=4$ gives a system of quartic surfaces with a double conic and certain other basis elements. But this can be reduced to the preceding one by a quadratic transformation. For $N=5$ the basis elements are a double C_5 with a triple point. The residual intersection of any two surfaces of the system is an elliptic quintic curve; the system is ∞^6 of grade 5. The conditions (a), (b), (c) will be satisfied only by imposing two additional points; hence L_4 will have another singular plane and V_3^3 has nodes.

For N greater than 5, there is no residual curve. (Del Pezzo (26).) For varieties with curve sections of genus 1, see Scorza (86).

12. *Plane Sections of Genus $p=2$.* III. This will be discussed in connection with general hyperelliptic sections. The surface defined by $|C_6|:8P^2$ is the double quadric cone, which can be mapped on a quartic surface with a double line, which is associated with a nodal variety.

13. *Plane Sections of Genus* $p=3$. IV. The non-hyperelliptic case has been discussed by Castelnuovo (9) and by Scorza (87). Those reducible to quartics all have fundamental lines, hence they lead to nodal varieties. (Noether (83).)

Those of the first species, that is, those containing a net of rational curves of order 4, meeting by twos in a single point, can in general not be reduced to quartics.

The only exception is that of the monoids, represented on the plane by $C_4 : 12P^1$, the 12 basis points all lying on a cubic curve, which is the image of the triple point. Whatever system of basis curves is adjoined to obtain a subsystem satisfying conditions (a), (b), (c), fundamental lines are introduced, hence the associated V_3^3 has one or more double points. The rational quartic curves lie in the planes of the bundle having the triple point for vertex. For a discussion of the involutions which leave invariant a web of monoids, see Snyder (105).

The system $|C_4| \cdot 11P^1$ defines a rational quintic surface with a double cubic curve. All the possible forms of residual basis elements that can satisfy conditions (a), (b), (c) are given in (100); in each L_4, and hence V_3^3, has double points.

14. *Digression. Details of a Quintic Surface.* Case J, Sharpe and Snyder (100), merits further study. The table of characteristics as given is incomplete.

Using the notation there employed, the line h which is the partial image of h' in (x'), with which it is in (1, 1) correspondence, is double on each S_{21} of the web, since h' is double on each S_6' of the system in (x'). Moreover, the curves on H_2, which are the residual images of the points of h', are conics.

The curve β_5 meets H_2 in two points not on γ_3.

The line l joining them is axis of a pencil of planes, each of which cuts from H_2 a parasitic conic.

Two of these have for images in (x') the points of contact of h' and L_4', hence the conics lie on K_{10} and are double on

each S_{21} of the web. Each of the six lines $[R_4, K_{10}] = 6r$ also belongs to the base of $|S_{21}|$; hence we have

$$S_{21}: \quad \gamma_3^8\alpha^4\bar{\alpha}^4\beta_5^5 6r\rho_5 h^2 2c_2^2 ,$$

$$K_{10}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5^2 6r 2c_2 .$$

A plane meets its conjugate S_{21} in a composite curve consisting of (S_1, K_{10}) and of a residual curve

$$\delta_{11}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5^3\rho_5 h^2 2c_2 .$$

Any two S_{21} may also have x_1 curves of order i, parasitic on the web $|S_5|$, and appearing to multiplicity i on $|S_{21}|$. The complete intersection then gives

$$441 = 21 + 192 + 16 + 16 + 125 + 6 + 5 + 4 + 16 + \sum i^3 x_i$$

and the intersection with K_{10} furnishes

$$210 = 10 + 96 + 8 + 8 + 50 + 6 + 8 + \sum i^2 x_i .$$

The only possible solution is $x_1 = 8$, $x_2 = 4$. The 8 lines are divided into two groups of four each, bisecants of β_5 meeting γ_3 and α or $\bar{\alpha}$. Hence we may now write

$$S_{21}: \quad \gamma_3^8\alpha^4\bar{\alpha}^4\beta_5^5 6r\rho_5 h^2 2c_2^2\, 8u_1 4u_2^2 ,$$

$$K_{10}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5\ 6r 2c_2 8u_1 4u_2 ,$$

$$\delta_{11}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5^3\rho_5 h^2 2c_2 4u_2 ,$$

where the interpretation of the symbols in the last equation undergoes an obvious modification.

The genus of δ_{11} is consequently 9, and it intersects K_{10} in 12 variable points. The images of these 12 points are the points of contact of δ' and L'; hence δ', the variable double curve on S_6', image in (x') of S_{21}, is of order 6. The genus of δ' is 2.

The curve δ_{11} has a double point D on h, and the plane of δ_{11} meets the image conic in two points P_1 and P_2. The points D and P_1 form one pair of conjugates, and D and P_2 another; hence δ_6' has a double point where it meets h', the tangents at which are distinct from each other and from h'. The point is a triple point on S_6'. The surface S_6' has a composite double curve consisting of δ_6' and of h', the latter passing through

a double point on the former. The genus of the composite c_7 can be found as follows. From an arbitrary point can be drawn 8 lines meeting δ_6' twice, accounting for 7 apparent double points, and the line to the double point. The plane determined by the point and h' meets δ_6' in 4 other points, hence $h=8+4=12$, $p=3$.

The web $|S_5|$ can be transformed birationally into a web of monoids. Let γ_3 be defined by

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_3}{x_4}.$$

Put $Q_1=x_1x_3-x_2^2$, $Q_2=x_2x_4-x_3^2$, $Q_3=x_1x_4-x_2x_3$. Consider the transformation defined by

$$y_i = Q_ix_4,\ (i=1,2,3),\ y_4 = \sum u_iQ_i,$$

where u_i is linear in x_1, x_2, x_3. The equations define a (3, 3) non-involutorial transformation between (x) and (y) in which the fundamental elements in (x) are γ_3 and a plane cubic $C_3:x_4=0$, $\sum u_iQ_i=0$ meeting γ_3 in three points. The image in (y) of the plane $x_4=0$ is the point $(0, 0, 0, 1)\equiv O$. Every point of γ_3 is transformed into a straight line, the locus of which is a ruled surface R_5 of order 5, trisecants of an elliptic $C_6:O^3$. The image of a point of C_3 is a line through O, the locus of which is the cubic cone $\Gamma_3:C_6$.

The image of an arbitrary line of (x) is a cubic curve through O; a secant of γ_3 has for image a conic through O and a generator of R_5; a bisecant of γ_3 goes into a line through O and two generators of R_5. Thus the congruence of bisecants of γ_3 is transformed into the bundle O. Since every line of the congruence meets S_5 in one point not on γ_3, the image must be a monoid in (y). The complete image of S_5 is of order 15, but R_5^2 is a component; α^2, $\bar{\alpha}^2$ go into double basis lines through O.

The curve β_5 has for image a composite curve of order 15, consisting of a proper curve β_7 of order 7 and of eight lines on R_5. Since β_5 meets $x_4=0$ in 5 points, β_7 has O for five fold point. The residual $\bar{C}_6$ is simple,

$$s_5\colon O^4\alpha\bar{\alpha}\beta_7\bar{C}_6.$$

The variable curve of intersection is a C_{10} having O^6.

Since the variable C_6 meets each α in two points, the image C_{10} meets each image $\bar{\alpha}$ in two points, hence the curve lies on a quartic cone of genus 1, with two fixed double generators, $[\beta_7, \bar{C}_6] = 6$.

15. *Resumption of the Case* $p = 3$. This completes the enumeration of webs of rational surfaces with plane sections of genus 3, belonging to the first species. (Castelnuovo (9).) The normal surfaces are of order $16 - \kappa$, lying in space of $14 - \kappa$ dimensions ($\kappa < 12$). Those that are not normal can be projected into one or another of the types just considered. (Clebsch (19), Cremona (24), Sturm (108).)

The F_6 with a double curve of order 7, genus 3, has an actual triple point on both curve and surface. (Bordiga (7), Veronese (110).)

The system $|C_4| : 10P^1$ defines a sextic surface having a double curve C_7, $p = 3$ with a triple point. The residual curve of order 8 is rational, hence condition (a) is not satisfied. Similarly for surfaces defined by $|C_4|$ having fewer than 10 basis points.

The second species is composed of those surfaces with plane sections of genus 3 which also contain an ∞^2 system of curves of genus 1; this system can be reduced to quartics. (Castelnuovo (9).) If the system of elliptic quartics is composed of plane curves, their planes pass through a point, and the surfaces are quartics having the given point for tacnode.

In every case the surface contains an $\infty^2 I_2$ of points, the lines joining pairs of conjugates being concurrent. From this point the surface can always be projected into a surface of Veronese in S_5, counted twice. The surface is always rational except when the arithmetic genus is -1 and when the curve of invariant points of the g_2^1 from O consists of four concurrent conics. These cases do not concern us. If the curves of genus 1 lying on the surface are space quartics,

the surface can be reduced to a quintic of S_3 having three concurrent double lines, and hence a triple point. The net of quartics is cut from the surface by the net of quadric cones passing simply through the double lines.

The F_8 of S_6, represented by $|C_6|:7P^2$, can be projected into an F_4 of S_3 by choosing the four centers of projection on the same elliptic curve lying on it. The result is a quartic of S_3 with a tacnode and four simple concurrent lines in the tangent plane at the tacnode. (Noether (80), Cremona (24).)

The system of curves $|C_6|:7P^23P^1$ defines an $F_5:3C_1^{\,2}3C_1^{\,1}$, the three double lines being concurrent, and the three simple lines lying in the planes containing a pair of double lines. In the plane representation, let the double points be A, B, C, D, E, F, G and the simple ones P, Q, R.

The plane images of the three double lines are $C_3: A \cdots GPQ$; $C_3:A \cdots GPR$; $C_3:A \cdots GQR$. From the complete intersection of two surfaces of the system (F_5) the three double lines should be subtracted, hence the plane sections are represented by $|C_{12}|:A^4 \cdots G^4PQR$. Thus this characteristic is of grade 29, genus 13, dimension 17. Hence there are nineteen linearly independent surfaces of the system.

In order to secure a subsystem satisfying conditions (a), (b), (c), let (F_5) have an additional basis curve C_m, of order m, genus p, meeting the totality of the double lines in S points; then

$$11m - 2p + 2 - 5s = 26,$$
$$5m - p + 1 - 2s = 14,$$

so that $s=m+2$, $p=3m-17$. (Sharpe and Snyder (100), p. 71.) The residual intersection of genus 1 is met by any third surface in three points, hence it can be reduced to a cubic through six basis points. It has one of the four forms

(α) C_3: $A\,B\,C\,D\,E\,F$,

(β) C_3: $A\,B\,C\,D\,E\,P$,

(γ) C_3: $A\,B\,C\,D\,P\,Q$,

(δ) C_3: $A\,B\,C\,P\,Q\,R$.

In (α), the residual basis curve C_m is

$$C_9: \qquad A^3 \cdots F^3G^4PQR.$$

The corresponding basis curve on the surfaces $|f(y)|$ is C_7, $p=4$, meeting each double line in 3 points. Through O can be drawn two other bisecants of C_7; these lines are fundamental, having five fixed points of intersection with each $f(y)$ of the system. Hence $L_4(x)$ has additional double points, and consequently V has a double point.

(β). The basis curve is now of order 6, genus 1, meeting two of the double lines each in 3 points and the third in 2 points. Hence as before, two additional bisecants can be drawn from O; these lines are fundamental, giving rise to double points on $L_4(x)$, and hence on V.

(γ). The basis curve is now composite, consisting of two conics which do not intersect, each meeting all the double lines once, and of a line skew to both conics, meeting one double line. The additional secant of both conics from O accounts for double points on V.

(δ). This case is impossible, as the residual basis curve is of order 4 and genus -5.

There is one more possible case of systems $|f(y)|$ where plane sections of genus 3 exists, namely, that having a plane representation of the form $|C_7|$; $7P^22P_1$. (Caporali (8).)

The double basis curve of the system $|f(y)|$ consists of a line C_1 and of a sextic γ_6, $p=2$, having two double points on the line. One surface of the system $|f(y)|$ must be composite, hence L_4 and also V_3 have double points.

The cases of systems of species 3 and of species 4, Castelnuovo (9), reduce to systems of quartics with point singularities more complicated than in that of type 2, but without basis lines. (Noether (83).)

The first surface is represented by $|C_6|:7P^24P^1$, all 11 basis points lying on a cubic curve γ. Let the singular point be (0, 0, 0, 1), and the tangent plane be $x_1=0$. Two surfaces of the system meet in a curve C_{16} of order 16, with 8 branches through O, all lying in $x_1=0$. The image of a plane section

through O consists of γ and of another cubic with the symbol C_3: $ABCDEFG$. The plane representation of C_{16} is $C_{12}:7P^4$. When this is composite, one component being of genus 1 and variable, it can be reduced to the $C_3:A\cdots F$, and the fixed basis curve has for image $C_9:A^3\cdots F^3G^4$, $p=4$. In space the variable curve is a C_6 with three branches through O; the basis curve is then C_{10} with five branches through O. Since two trisecants of C_{10} can be drawn through O, these lines are fundamental, giving rise to double points on V_3^3.

It was assumed that the variable cubic in the plane representation had for basis points A, B, C, D, E, F; if instead one or more be chosen from the simple basis points of the system, practically the same argument may be repeated, with the same conclusion.

Another case is that of F_4 represented by $|C_7|:P^39P^2$, all the basis points lying on a cubic curve, and another is the F_4 represented by $|C_8|:8P^3P^2P^1$, the ten basis points lying on a cubic. Both of these can be considered as the preceding case, with the same result.

This completes the case $p=3$, non-hyperelliptic. When the net of curves on each surface of the system is composite, the plane sections of genus 3 are hyperelliptic. In this case the F_{16} is represented by $|C_6|:P^4P^2$ or by $|C_{8-k}|:P^{6-k}(3-k)P^2$, $(k=0, 1, 2, 3)$, the double points being adjacent to the P^{6-k}. They will be considered in connection with the general hyperelliptic case. (De Franchis (25).)

16. *General Hyperelliptic Sections.* V. From Castelnuovo (10) and Enriques (31) we know that every surface with hyperelliptic plane sections or by sections S_{n-1}, if it lies in S_n, is rational. If the genus of the sections exceeds 1, the surface contains a rational ∞^1 system of conics, such that through any point of the surface passes just one conic of the system. If the section is of order n and genus π, the surface cannot belong to a space larger than $S_{n-\pi+1}$; if it belongs to a space of lower dimensionality, it is the projection of a normal surface of $S_{n-\pi+1}$.

The planes of the ∞^1 conics on the surface form a variety of order not larger than $n-\pi-1$.

The surface can be mapped on a plane such that the plane sections (or those of S_{r-1} if in S_r) are represented by curves of order k having a common point of multiplicity $k-2$, and possibly other basis points. Hence if the genus is π, the maximum order is $4\pi+4$. Every such surface of lower order can be obtained from a surface of order $4\pi+4$ of $S_{3\pi+5}$ from $4\pi+4-n$ points upon it, and possibly from other points not on the surface.

In order to obtain a subsystem satisfying conditions (a), (b), (c), it is necessary to impose simple basis points in every case. When double points exist among the additional basis points, the curves can be reduced to those of lower order. Several cases of $|C_k|:P^{k-2}$ have been considered in connection with other problems. Let x be the number of simple basis points, N the order of the surface, m the order of its double curve. Then for (k, x, N, m) (4, 8, 4, 1) see (100), p. 61; (4, 7, 5, 3) see (100), p. 72; (4, 6, 6, 8) see (8), p. 202; (5, 11, 5, 3) see (100), p. 71; (5, 10, 6, 7) see (8), p. 207.

The system $|F_N|:C_1^{N-2}\ \gamma$ can be treated as in (100) p. 62, and the monoidal cases of any order necessarily have fundamental lines.

If the assumption made in § 8 is justifiable it follows that the general cubic variety of S_4 is irrational.

17. *Reduction to Monoidal Types.* From the preceding representation, follows at once the proof of the following theorem.

All (1, 2) *quaternary correspondences of genus* 1, *in which the quartic surface of branch points has a singular plane and at least one double point not in this plane, can be expressed in terms of a* (1, 1) *correspondence between the points of an* S_3' *and the pairs of points of a perspective monoidal involution of order two in another space* S_3.

Let there be a second involution I_2 in (x), having the same surface of branch points L_4' in (x'). Since the points of V_3

and of (x) are in (1, 1) correspondence, we have in any case $y_i = f_i(x_1, x_2, x_3, x_4)$, $i = 1, 2, 3, 4$. For example, by including the vertex P on V_3 from which the variety is projected on the double S_3, we may write

$$y_i = F_2(x)x_i; \quad y_5 = F_3(x).$$

Let the pairs of points of I_2 also be represented on V_3. Since the two surfaces of coincident points K_1 of I_1 and K_2 of I_2 are each in (1, 1) correspondence with L_4', they can be superposed one on the other.

Then I_2 is transformed into a second monoidal involution, having a second double point Q_x on K_2 for vertex. The image on V_3 of Q_x is a second double point, hence L_4' has a second double point. The line joining the double points on V_3 lies entirely upon it; every plane through it meets V_3 in a residual conic, image of a line of the bundle Q_x in (x). This line contains an infinite number of pairs of conjugate points with double points on K_x. Hence the conic on V_3 contains an infinite number of pairs of points forming an involution in which the double points are the residual points of intersection of the plane of the conic with the curve of contact K_V of the tangent cone from P. Thus, by I_2^V every point of K_V remains invariant; by projection upon $x_5' = 0$, the associated point on L_4' is a branch point as before.

If in (x) there exist several involutions having the same quartic surface of branch points with a singular plane, by the above process they can all be reduced to perspective monoidal involutions by the same transformation.

When L_4' has only one double point not in the singular plane, the web of invariant cubic surfaces in (x) has a simple point and a general C_6, $p = 4$, for basis elements. It is described as case D of cubics in Sharpe and Snyder (100), p. 58. It does not appear again in a transformed form in the enumeration there given.

If L_4' has two double points not in the singular plane, $K(x)$ has a second double point, the quadric F_2 containing C_6 and

the cubics of the web touch each other, C_6 has an actual double point and is now of genus 3.

Similarly, L_4' may have 3 or 4, . . . or 10 double points not in its singular plane. In the latter case it is the 16-nodal Kummer surface, and K may be reduced to the symmetroid.

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ON THE LOCATION OF ROOTS OF POLYNOMIALS AND ENTIRE FUNCTIONS*

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On account of its importance, the problem of the location of the roots of a polynomial occupies the central position in the theory of equations. For the most part the attention of mathematicians has been confined to the real roots of real polynomials—i.e., polynomials with real coefficients. So general has been the assumption of the reality of the polynomial that it has often, unfortunately, not been stated in theorems in which it is vital; for example, when it is said that imaginary roots enter in conjugate pairs. Correspondingly, the tools for the investigations were ordinary algebra and the calculus. When, however, one's thought is extended to the location of imaginary roots and to unreal polynomials, the border line is crossed, as the treatment involves almost inevitably considerations and methods belonging to the theory of analytic functions. It is in the overlapping region belonging alike to the theory of equations and that of analytic functions that we will stray today.

After the notable advances of Budan, Fourier, Cauchy, Laguerre and others, investigation of the root-location of polynomials and entire functions was largely neglected for other work. With the new century has come a new activity. It is my pleasure to bring before you today some of the important advances made and some theorems discovered since 1900. This will, I hope, be the more profitable because the new results lie scattered and untied through the periodical literature and have not yet become a part of current texts.† Some of them, though little known today, are destined to become classic.

* An address presented at the invitation of the program committee at a Symposium held at the meeting of the Society in Chicago, March 29, 1929.

† Except Bieberbach's noteworthy *Vorlesungen über Algebra*, 1928.

Time limitations force me to omit certain aspects and developments which I had purposed originally to include. Reluctantly I omit inclusion of the line of work* inaugurated by Landau and Montel. The underlying thought in this work is to give $p+1$ coefficients in an equation

$$(1) \quad P_n(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0, \quad (a_0 \neq 0),$$

leaving the other coefficients arbitrary, and to seek the limitations thereby imposed upon the roots. Thus Montel supposes the first $p+1$ consecutive coefficients $a_0, \cdots, a_p$ to be given in this equation or, more generally, in one of any degree with $n+1$ non-zero terms, and shows that then the p roots of smallest absolute value are conditioned to lie in a circle whose radius depends only on the values of these given coefficients and on n. I omit also consideration of special kinds of polynomials, computational methods for calculating the roots, polynomials with a parameter, chains of polynomials arising through continued fractions or other algorithms, extensions of Descartes' Rule of Signs or of the Hermite-Poulain Theorem, etc. I have not sought to give today an encyclopaedic report but rather so to select the material as to secure some unity of theme and thought.

At the end of the report is appended a bibliography which includes also some omitted topics allied to those considered here. Literature references to the bibliography are indicated in the report by bracketed numbers.

Part I

One succession of modern theorems was inaugurated by the Gauss-Lucas Theorem on the location of the roots of the derivative of any polynomial, real or imaginary. This theorem, though so fundamental, is strangely enough but little known popularly.

Gauss-Lucas Theorem. *If in the complex plane of $z=x+iy$ we mark off the convex polygon whose vertices are*

* See the bibliography.

roots of a given polynomial and which contains either within or upon its boundary all other roots of the polynomial, then none of the roots of the derived polynomial will lie exterior to this polygon, and a root will fall upon its boundary only when coincident with a multiple root of the given polynomial on the boundary.

This theorem was definitely stated and proved by Lucas [63] in 1879 and perhaps can be best known by his name, but it follows as an immediate corollary from an observation of Gauss [62] appended as a *handschriftliche Bemerkung* after a memoir in his *Collected Works*. Gauss says that the roots of the derivative of any given polynomial are the positions of equilibrium in a plane field of force created by placing unit-masses at the roots of the given polynomial which repel (or just as well attract) with a force inversely proportional to the distance. A mass r is, of course, to be placed at each r-fold root. For proof of this Equilibrium Theorem of Gauss* we have merely to observe that the roots of the derivative are given by the equation

$$(2) \qquad \frac{P_n'(z)}{P_n(z)} \equiv \sum \frac{1}{z - \alpha_j} \equiv \sum \frac{1}{r_j e^{i\theta_j}} = 0, \qquad (j = 1, \cdots, n),$$

where the α_j are the roots of the given polynomial and the r_j, θ_j denote the lengths and arguments of the vectors drawn from these roots to a point z. Then, if each θ_j is changed into $-\theta_j$ to convert the terms into their conjugates, each term in (2) becomes a vector representing a force due to a unit mass situated at α_j which repels inversely as the distance, and the equation gives the positions of equilibrium, where there is no force. Now it is obvious that if we produce any side of the convex polygon indefinitely and z is a point within the half plane not containing the polygon, the components of force at z perpendicular to the side will all be directed away from the polygon, thereby reinforcing each other so that equilibrium at z is impossible. Like considera-

* Rediscovered by Lucas, Comptes Rendus, vol. 89 (1879), pp. 224–226.

tions apply when z lies on a side and is not coincident with a root of the given polynomial. The theorem of Lucas thus follows as a corollary.

Beside this theorem of Gauss-Lucas we may now place a beautiful result published without proof by Jensen [72] in 1913, which gives much more definite information regarding the derived polynomial when the given polynomial is *real.*

JENSEN'S THEOREM. *If circles are described having as diameters the segments connecting pairs of conjugate roots of the given polynomial, then no imaginary root of the derivative of the polynomial can lie exterior to all these circles.*

The first published proof [73] of this theorem we owe to Walsh (1920–1921); it is likewise based on the above Equilibrium Theorem of Gauss. Walsh sets a repulsive unit-mass at each root of the given polynomial and shows that at any point not on the real axis and exterior to the circle having as diameter the segment joining a pair of conjugate imaginary roots α_j, $\bar{\alpha}_j$, the net force due to the masses at these two points will have a non-vanishing component perpendicular to the real axis and directed away from this axis. The like is obviously true of the repulsive force due to the masses upon the real axis. Hence at any point exterior to all the Jensen circles and not on the real axis these components reinforce one another and equilibrium is impossible. Consequently no imaginary root of the derived polynomial can lie exterior to all the Jensen circles. Walsh apparently did not notice that the introduction of repulsive masses was an unnecessary trapping and that without essential modification the proof could be thrown into the following elementary form [74], quite possibly that of Jensen himself. Let $\alpha=\beta+i\gamma$, $\bar{\alpha}=\beta-i\gamma$ be a pair of conjugate imaginary roots of $P_n(z)$. The corresponding component in (2) is

$$\frac{1}{(x+iy)-(\beta+i\gamma)}+\frac{1}{(x+iy)-(\beta-i\gamma)}$$
$$=\frac{2(x-\beta)[(x-\beta)^2+y^2+\gamma^2]-2iy[(x-\beta)^2+y^2-\gamma^2]}{[(x-\beta)^2+(y-\gamma)^2]\cdot[(x-\beta)^2+(y+\gamma)^2]}.$$

The coefficient of i is opposite in sign to y if $(x-\beta)^2+y^2-\gamma^2>0$; that is, when z lies outside the circle $(x-\beta)^2+y^2=\gamma^2$. For a real root β the component is

$$\frac{1}{x+iy-\beta}=\frac{(x-\beta)-iy}{(x-\beta)^2+y^2},$$

which also has a sign opposite to that of y for the coefficient of i. Hence at any point exterior to all the Jensen circles and not on the real axis the coefficient of i in P_n'/P_n does not vanish. This proves the theorem.

Walsh supplements Jensen's Theorem by showing that in any segment of the real axis which is free from roots of the real polynomial $P_n(z)$ and which does not penetrate into any of its Jensen circles there can lie at most only one root of its derivative $P_n'(z)$. Further, if any Jensen circle has no point in common with any other Jensen circle in part exterior to it and contains exactly k roots of $P_n(z)$, then it will contain at least $k-1$ roots of its derivative and at most, $k+1$ roots. I may add that I find that if we amalgamate each series of overlapping circles into a single connected region so as to form Jensen regions exterior to one another, the conclusion still holds with respect to these regions.

These results can be readily established analytically without following the Gaussian suggestion of introducing repulsive masses at the roots of the given polynomial, but that method is undeniably illuminating and suggestive, and has the advantage of being independent of the selection of the coordinate axes. Bôcher [68] applied the method to the Jacobian

$$\begin{vmatrix} \dfrac{\partial\phi}{\partial z_1} & \dfrac{\partial\phi}{\partial z_2} \\ \dfrac{\partial\psi}{\partial z_1} & \dfrac{\partial\psi}{\partial z_2} \end{vmatrix}$$

of two binary forms $\phi_{p_1}(z_1, z_2)$, $\psi_{p_2}(z_1, z_2)$ of degrees p_1 and p_2 respectively. It is easy then to see that the roots $z=z_1/z_2$

of the Jacobian, other than multiple roots of either form or common roots of both forms, are the equilibrium positions in a field of force due to unit masses placed at the roots of ϕ and of ψ which respectively attract and repel with a force inversely proportional to the distance. Continuing the application, Walsh has made an extended study of the location of the roots of the Jacobian. In a number of memoirs he has also applied the Gaussian idea to derive some nice theorems relating to the roots of the derivative of a polynomial. For example, he shows [71] that if m_1 roots of a polynomial lie in a circle of radius r_1 and center α_1 and all the remaining m_2 roots lie in a circle of radius r_2 and center α_2, all the roots of the derivative lie in these two circles and a third circle whose center is $(m_1\alpha_2+m_2\alpha_1)/(m_1+m_2)$ and whose radius is $(m_1r_2+m_2r_1)/(m_1+m_2)$. If, furthermore, these circles are all external to one another, then the three circles contain respectively m_1-1, m_2-1, and one root respectively.

I may add that the Gaussian method and various resulting conclusions—for example, Lucas and Jensen's theorems—apply not only to the derivative of the given polynomial $P_n(z)$ but also to any polynomial $\psi_{n-1}(z)$ of degree $n-1$*, for which we have a decomposition into partial fractions of the form

$$\frac{\psi_{n-1}(z)}{P_n(z)} = \sum_{j=1}^{n} \frac{A_j}{z-\alpha_j},$$

in which the A_j are all positive.

The next theorem which I wish to cite relative to the roots of the derivative of a polynomial is due to Grace and arose from an attempt to obtain a generalization of Rolle's Theorem valid for two unreal roots of a polynomial. According to Rolle's Theorem, if a *real* polynomial vanishes at two different points of the real axis (or has the same value at these points), there will lie at least one root of its derivative between them. Thus two real roots of a real polynomial *fixate*

* For extension of the Jensen Theorem, the polynomial $\psi_{n-1}(z)$ is, of course, to be taken real as well as $P_n(z)$.

one root of the derivative. In what direction should one look for a corresponding result for imaginary roots and unreal polynomials? Doubtless this question occurred to others before Grace, and we may conjecture that it was on account of its difficulty that no answer was obtained till the present century. In November 1901, in a notable five-page article [75], Grace obtained the following important and strange theorem.

THEOREM OF GRACE. *If a and b are any two roots of any polynomial, real or unreal, of degree n, then within or on a circle having the center $(a+b)/2$ and a radius one-half of $|a-b|$ ctn (π/n) there will be at least one root of the derived polynomial.*

This can scarcely be called a generalization of Rolle's Theorem inasmuch as it does not reduce to Rolle's Theorem for the special case of a real polynomial with two real roots. Further, in anticipation of the second part of my report today, I may add that it is not a theorem that admits of extension to entire functions, since as n increases indefinitely, the radius of the circle also increases indefinitely. No information could therefore be afforded regarding the derivative of the given entire function except that it had a root somewhere in the complex plane. But this conclusion would be false since the entire function e^z-1 has a periodic system of zeros, while its derivative is devoid of zeros.

Grace's proof of his theorem was essentially invariantive in character. If in an algebraic equation

$$(1) \qquad P_n(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0,$$

we put $z = x_1/x_2$, the left-hand member after dropping the denominator is converted into an algebraic form in x_1 and x_2. If we take a second equation

$$(3) \qquad b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n = 0,$$

the two equations may be said to be *apolar* when the corresponding bilinear forms are apolar; that is, if their coefficients are connected by the equation

$$(4)\quad a_0b_n - \frac{1}{C_1^n}a_1b_{n-1} + \frac{1}{C_2^n}a_2b_{n-2} - \cdots + (-1)^n a_nb_0 = 0.$$

Starting from Lucas' Theorem, Grace first establishes between two apolar equations a most important and far reaching relation which I shall term Grace's Apolarity Theorem.

GRACE'S APOLARITY THEOREM. *If two algebraic equations are apolar, then any circle in the z-plane which contains all the roots of either equation will contain at least one root of the other.*

The term circle is here to be understood in the sense of the theory of functions and may be indifferently the closed interior or the closed exterior of a circle, or a half plane inclusive of its boundary.

It will be noticed that if any two of the equations (1), (3), (4) are given, we can immediately write down the third. Hence the theorem says that if any algebraic equation (1) is given whose coefficients are connected by a homogeneous linear relation (4), we can immediately write down a third equation (3) such that every circle which contains all the roots of (1) will contain at least one root of (3), and reciprocally.

Grace applies this thorem to the derivative of (1). If we substitute in (1) any two of its roots α and β and subtract, we obtain a homogeneous linear relation

$$a_1(\alpha - \beta) + a_2(\alpha^2 - \beta^2) + \cdots + a_n(\alpha^n - \beta^n) = 0$$

between the coefficients of the derived equation. The equation (3) which corresponds to this linear relation is

$$(\alpha^n - \beta^n) - n(\alpha^{n-1} - \beta^{n-1})z + C_2^n(\alpha^{n-2} - \beta^{n-2})z^2 + \cdots + (-1)^{n-1}C_1^n(\alpha - \beta)z^{n-1} = 0,$$

which may be written

$$(5)\qquad (\alpha - z)^n - (\beta - z)^n = 0.$$

Hence at least one root of the derivative $P_n'(z)$ must lie in

any circle containing all the roots of (5). By extracting the nth root in (5) we get for its roots

$$z = \frac{\beta\omega_j - \alpha}{\omega_j - 1}, \qquad (j = 1,2, \cdots, n-1)$$

where $\omega_j \equiv \cos(2j\pi/n) + i \sin(2j\pi/n)$ denotes an nth root of unity. This can be easily reduced to the form

$$z = \frac{\alpha+\beta}{2} + i\frac{\alpha-\beta}{2}\operatorname{ctn}\left(\frac{j\pi}{n}\right),$$

which shows that the n roots of (5) all lie on a straight line perpendicular to the segment joining α and β at its middle point. The most remote of these roots is at a distance one-half of $|\alpha-\beta|$ ctn (π/n) from this middle point. Consequently a circle described around this mid-point as center and with a radius equal to this distance contains all the roots of the equation (5) apolar to the derivative $P_n'(z)$ and hence by Grace's apolarity theorem encloses at least one root of $P_n'(z)$. Thus two roots α and β of a given polynomial fixate one root of the derivative in the manner stated.

In 1907 this theorem was re-derived, apparently independently, by Heawood [76]. He noted the fact that for $n>3$ the region in which lies at least one roct of the derivative need not be extended equally in every direction from the center $(\alpha+\beta)/2$ to the perimeter of the circle. When $n=4$, the minimum region is that covered by two circles of radius one-third of $|\alpha-\beta|\operatorname{ctn}(\pi/n)$ and having their centers on the perpendicular to the segment joining α and β at its middle point and at a distance one-sixth of $|\alpha-\beta|\operatorname{ctn}(\pi/n)$. For greater values of n the form of the curve bounding the region is very complicated and scarcely specifiable. Heawood established Grace's Theorem by showing that the root of the derivative nearest to the mid-point $(\alpha+\beta)/2$ attains its maximum distance only when all the roots of the derivative are equal; that is, only when $P_n'(z) \equiv c(z-\gamma)^{n-1}$, so that we may write $P_n(z) \equiv c(z-\gamma)^n/n + c_1$. Substituting the roots α

and β in turn, we find $(\alpha-\gamma)^n \equiv \beta-\gamma)^n$, and γ is consequently a root of equation (5). The theorem then follows as before.

The most interesting successor of Grace's memoir is one published in 1922 by Szegö [77] with the modest title: *Bemerkungen zu einem Satz von J. H. Grace ueber die Wurzeln algebraischer Gleichungen.* He derives Grace's Theorem on apolar polynomials on the basis of what he calls a *Faltungssatz*, which I shall call his *Coincidence-Theorem.* This runs as follows.

Szegö's Coincidence Theorem. *If S_1 denotes the sum of the products of $z_1, z_2, \cdots, z_n$ taken j at a time and these are any n quantities satisfying the equation*

$$a_0 + a_1 S_1 + \cdots + a_n S_n = 0, \tag{6}$$

then any circle which contains all the roots of the equation

$$a_0 + C_1^n a_1 z + C_2^n a_2 z^2 + \cdots + a_n z^n = 0 \tag{7}$$

will contain at least one of any of the n quantities satisfying (6).

It will be noticed that equation (7) results from (6) when the z_j are all coincident. Szegö's proof, not altogether easy, is by means of mathematical induction. An obvious consequence of his theorem is that if any set of the z_j satisfying (6) all lie *in* (that is, on or within) a circle, not all the roots of (7) lie without. We have thus Walsh's theorem.

Theorem of Walsh. *If any circle contains a set of values of $z_1, \cdots, z_n$ satisfying* (6), *it must contain at least one root of* (7).

This result, in a somewhat different formulation, was published almost simultaneously as Theorem I and the "main result" of one of Walsh's principal papers [78] and was established by geometrical considerations involving continuity. It is not, however, to be regarded as a subordinate corollary of Szegö's Theorem, inasmuch as the latter is obviously also an immediate corollary of the former. The relation of the two theorems was not pointed out by Walsh.

The usefulness of the Szegö-Walsh Theorem is revealed

by very different sets of theorems which the two authors rapidly deduce from it. I have time to mention only one or two of each. As a typical theorem of Walsh, I select the following [78, p. 173]: Let

$$(8) \qquad (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)-A=0$$

be an equation with a parameter A; then if $\alpha_1, \cdots, \alpha_n$ are all situated in a circle of radius r and center α, the n roots of (8) will all be contained in n circles whose centers $\alpha+A^{1/n}$ form a regular polygon with center at α and which have the common radius r. If none of these circles overlap, there will be just one root in each circle. The second of his theorems which I wish to cite is a special case of his "Main result" and is set off as his Theorem II.* This says that "if the points $\alpha_1, \alpha_2, \cdots, \alpha_k$ lie in a circular region C and if z is any point exterior to C, the root of the equation in α

$$(9) \qquad \frac{1}{z-\alpha_1}+\frac{1}{z-\alpha_2}+\cdots+\frac{1}{z-\alpha_k}=\frac{k}{z-\alpha}$$

lies in C." Rephrased in the language of Gaussian attractions, it asserts that when unit masses are placed at k points $\alpha_1, \cdots, \alpha_k$ lying in a circle and the entire mass k is then concentrated at a single point α so as to produce the same resultant force at a given exterior point z, the point α must lie somewhere in the circle.

I have cited this second theorem of Walsh because, in considering a very different matter, I have been led to the same equation but with z as the unknown in place of α. If a polynomial equation $P_n(z)=0$ and its first derived equation are both transformed by the same entire linear substitution $z=az'+b$, they pass over into a new equation and its first derived equation. This is not the case when a fractional linear substitution

$$(10) \qquad z=\frac{az'+b}{cz'+d}, \qquad (c\neq 0),$$

* Curtiss [78a] points out that the theorem is a "result which Laguerre expressed in homogeneous coordinates."

is applied. Nevertheless, various theorems regarding the location of the roots of a polynomial and its derivative may continue to hold in the transformed region with respect to the transformed polynomial and its derivative. Consider, for instance, a theorem less specific than the above Gauss-Lucas Theorem, according to which a circle or half-plane which contains all the roots of $P_n(z)$ also contains all the roots of its derivative. Apply now the transformation (10). In place of $P_n(z)$ we get the new polynomial

$$Q_n(z') \equiv (cz' + d)^n P_n\left(\frac{az' + b}{cz' + d}\right). \tag{11}$$

Suppose that the transformation converts the interior of the circle or half-plane into the interior of a new circle C'. Then the new circle which necessarily contains all the roots of $Q_n(z)$ will contain also all the roots of $Q_n'(z)$. The like conclusion fails, however, when the transformed region is the exterior of C'. Similarly, if Jensen's Theorem were not invalidated by a transformation converting the half-plane into the interior of C', we would have the following conclusion: When the roots of a polynomial $Q_n(z)$ which do not lie on the perimeter of a given circle C' are pairs of roots of equal multiplicity symmetrically located* with respect to C', all the roots of its derivative not on the circle must lie in the circles whose diameters are the segments joining the pairs of symmetrically situated roots. It is easy to show the falsity of such a conclusion by an example to the contrary. Jensen's Theorem, however, is an immediate result of two applications of the following true statement: The roots of a real polynomial which do not lie in a half-plane bounded by a real axis lie in the Jensen circles. The corresponding statement after the above transformation is that when a polynomial $Q_n(z)$ has its roots located in the manner just explained, the roots of the derivative so far as they do not lie on or within

* Two symmetrically situated points are inverse points with respect to the circle.

C' must lie in the circles whose diameters are the segments joining pairs of symmetrically situated roots of $Q_n(z)$, and the validity of this statement has been established by Walsh through the method of repulsive particles.

These examples suffice to show the need of some criterion for determining whether a theorem regarding the mutual location of the roots of a polynomial and its derivative will hold also for the transformed polynomial and its derivative after a transformation (10). To answer this question, suppose $P_n(z)$ replaced by (11) in consequence of the transformation (10). The new derived equation is

$$Q_n'(z') \equiv nc(cz' + d)^{n-1}P_n\left(\frac{az' + b}{cz' + d}\right) + (ad - bc)\cdot(cz' + d)^{n-2}P_n'\left(\frac{az' + b}{cz' + d}\right) = 0,$$

which may be successively written

$$\frac{nc(cz' + d)}{ad - bc} + \frac{P_n'(z)}{P_n(z)} = 0,$$

and

$$(12) \qquad -\frac{n}{z - \dfrac{a}{c}} + \sum \frac{1}{z - \alpha_1} = 0, \qquad (i = 1, \cdots, n),$$

where the α_i are the roots of the original polynomial $P_n(z)$. It thus appears that the new derived equation $Q_n'(z') = 0$ results by application of the linear transformation (10) to the polynomial equation (12). Now (12) is Walsh's equation (9) regarded as an equation of $(n-1)$th degree in z with given α_i and α. It may be written in the form

$$(12a) \qquad nP_n(z) - (z - \alpha)P_n'(z) = 0,$$

where α denotes the point $z = a/c$ which by the transformation passes over into $z' = \infty$. For given α, there corresponds to each pair of polynomial equations

$$P_n(z) = 0, \quad Q_{n-1}(z) \equiv nP_n(z) - (z-\alpha)P'_n(z) = 0,$$

a class of substitutions (10) with $a/c=\alpha$ which replaces them by a polynomial equation $Q_n(z')=0$ and its first derived equation. Conversely, for the latter pair, there is a class of (inverse) substitutions which replaces them by pairs $P_n(z)=0$, $Q_{n-1}(z)=0$ with given α.

The question whether a theorem regarding the root location of $P_n(z)$ and its derivative is valid after the transformation for $Q_n(z)$ and its derivative therefore reduces to the determination whether the original theorem is valid for $P_n(z)$ and $Q_{n-1}(z)$ before the transformation as well as for $P_n(z)$ and $P_n'(z)$. It will be observed that α may have three qualitatively different positions with respect to the interior of a circle or half-plane. It may lie without, within, or on its boundary. If this interior is to be transformed by (10) into the interior of a new circle C', the point α must be taken without the circle C or the half-plane, since it passes over into $z'=\infty$.

So far as I know the above relationship has not been previously pointed out. I find, however, that Gonggrÿp [99] has considered the polynomial $nP_n(z)-zP_n'(z)$ which arises by application of $z=1/z'$ to $P_n'(z)$. He calls it the "*reduite*" of $P_n(z)$ and gives several theorems relating to its roots.

As an example of the general interrelation just pointed out, let us take Walsh's "Theorem 2" previously cited, which says that when z is exterior to a circle containing all the roots of $P_n(z)$, the point $\alpha=a/c$ in (9) must lie in the circle. The word circle has here the general sense of the closed interior or exterior. An equivalent statement is that when the roots of $P_n(z)$ are all in the circle and a/c lies without, the z-roots of (12) must all lie in the circle. By the transformation (10) this gives the restricted Lucas Theorem that when a circle contains all the roots of a polynomial $Q_n(z')$, it contains also the roots of its derivative. Conversely, Walsh's Theorem follows from this by the inverse linear transformation.

I have not time to follow the matter further but return now to the Coincidence Theorem of Szegö. It has been already

noted that he uses it to obtain Grace's Apolarity Theorem. His easy deduction is reversible, so that the two theorems, so very different in appearance, are equivalent. Szegö then goes on to obtain in an interesting manner a wealth of theorems. The only one of these which I shall cite is a theorem of Egerváry:

THEOREM OF EGERVÁRY. *Let*

$$A(x) \equiv a_0 + C_1^n a_1 z + C_2^n a_2 z^2 + \cdots + a_n z^n = 0, \tag{13}$$

$$B(x) \equiv b_0 + C_1^n b_1 z + C_2^n b_2 z^2 + \cdots + b_n z^n = 0 \tag{14}$$

be two equations whose roots all lie in (within) the unit circle. Then the roots of the composition-equation

$$C(x) = a_0 b_0 + C_1^n a_1 b_1 z + \cdots + a_n b_n z^n \tag{15}$$

all lie in (within) the same circle.

For proof [77], denote by $\beta_1, \beta_2, \cdots, \beta_n$ the roots of (14), and by ξ any root of (15), so that

$$a_0 b_0 + C_1^n a_1 b_1 \xi + C_2^n a_2 b_2 \xi^2 + \cdots + a_n b_n \xi^n = 0. \tag{16}$$

Then if we substitute $-\xi/z$ for z in (14), the resulting equation has the roots $-\xi/\beta_1, \cdots, -\xi/\beta_n$. The condition that this equation and (13) shall be apolar is precisely the equation (16). Then by Grace's apolarity theorem some one of these roots, say $-\xi/\beta_i$, must lie in the unit circle when all roots of (13) so do. This is impossible if ξ lies without (respectively without or on) the unit circle, since $|\beta_i| \leqq 1$ (respectively $|\beta_i| < 1$) by hypothesis. The theorem therefore follows.

If we restrict the hypothesis further and suppose that all roots of (13) and of (14) lie on the perimeter of the unit circle, then in applying the theorem just proved, we can take at pleasure the circle to be the interior or the exterior of the unit circle. It follows that no root of (15) can lie without, respectively within; consequently the roots of the composition equation will lie on the perimeter of the unit circle when those of (13) and (14) do.

Very similar in character are theorems given by Malo* (1895) and Schur (1914). Malo's theorem [88] says that if the roots of both of the equations

$$a_0 + a_1 z + \cdots + a_m z^m = 0, \tag{17}$$

$$b_0 + b_1 z + \cdots + b_n z^n = 0 \tag{18}$$

are all real and those of the former are all positive or all negative, then the roots of the composition equation

$$a_0 b_0 + a_1 b_1 z + \cdots + a_k b_k z^k = 0, \tag{18a}$$

in which k denotes the smaller of the two degrees m and n, are also real; furthermore, they are all different if $a_0 b_0 \neq 0$ and $n \leqq m$. Schur [90] draws the same conclusion for another composition equation:

$$a_0 b_0 + 1! a_1 b_1 z + 2! a_2 b_2 z^2 + \cdots + k! a_k b_k z^k = 0.$$

If now we look upon (17) as any arbitrarily given equation with real roots of like sign, we may regard $b_0, b_1, \cdots, b_n$ as a set of multipliers which by composition with (17) will always produce an equation (18a) with only real roots. Pólya and Schur [92] call any sequence of real multipliers with this property a factor sequence of the *second kind*. A sequence of real factors

$$b_0, b_1, b_2, \cdots \tag{19}$$

is said to be of the *first kind* when operating on all equations (17) with real roots ($n = 1, 2, \cdots$) they produce only composition equations (18a) with real roots. Laguerre in his memoirs gave various special sequences with such properties; for instance,

$$1, q, q^4, q^9, q^{16}, \cdots, \qquad |q| \leqq 1,$$

is a sequence of the first kind. Pólya and Schur make a most interesting study of the conditions which must be imposed upon the infinite set (19) of real numbers to furnish a se-

* Malo's theorem is extended by Jensen [72] to series.

quence of the first or second kind. If, for example, (19) is a sequence of the first kind, all non-zero elements must form a single uninterrupted succession, finite or infinite in length, the terms of which are all of same sign or of alternating sign. The two authors obtain both an algebraic and a transcendental necessary and sufficient criterion that it shall be a factor sequence of the first or second kind respectively. According to the algebraic criterion, (19) is a factor sequence of the first kind if and only if all roots of the equations

$$b_0 + C_1^n b_1 z + C_2^n b_2 z^2 + \cdots + C_n^n b_n z^n = 0, \quad (n = 1,2,3, \cdots),$$

are real and of like sign, while it is one of the second kind when they are merely real. By the transcendental criterion it is necessary and sufficient that the real series

$$b_0 + \frac{b_1}{1!}z + \frac{b_2}{2!}z^2 + \frac{b_3}{3!}z^3 + \cdots$$

should converge for all values of z and represent an entire function which has only roots of the same sign and is of genre 0 except for a possible factor $e^{\gamma z}$, in which γ has the sign opposite to that of the roots. For a factor sequence of the second kind this series must be of genre 0 or 1 except possibly for a factor $e^{-\gamma^2 z^2}$ (γ real) and have only real roots.

Somewhat earlier Petrovitch [91] considered factor sequences (19) having the property that when operating upon real polynomials of even degree possessing only imaginary roots they produce always polynomials without real roots. A class of sequences specified by him with this property has, when somewhat restricted, the form

$$b_n = \int_c^d f(t)t^n dt, \qquad (n = 1,2, \cdots),$$

where c and d are arbitrary fixed real limits independent of n and $f(t)$ is an arbitrary real function of invariable sign between c and d. For if for $m = 2n$ we denote the polynomial (17) by $P_{2m}(z)$, the composition equation will be

$$0 = a_0b_0 + a_1b_1z + \cdots + a_nb_nz^n \equiv \int_c^l f(t)P_{2n}(tz)dt,$$

which can not be satisfied for any real value of z inasmuch as both factors of the integrand are of invariable sign. A special set of factors is also obtained by putting $b_n = n!$.

Part II

An infinite series $a_0+a_1z+a_2z^2+\cdots$ convergent over the entire finite plane of z is, in a certain sense, a generalization of a polynomial of the nth degree $a_0+a_1z+\cdots+a_nz^n$. We call it an *entire function*. The futility of efforts to carry over theorems regarding polynomials to entire functions without restriction upon the latter is, I suppose, well known to you. A familiar example is afforded by the Fundamental Theorem of Algebra. Though a polynomial of nth degree has n roots, an entire function need not have any zeros at all. The familiar exponential function $e^z = 1+z+z^2/2!+\cdots$ is an example of such an entire function. Again, take the familiar theorem that if a polynomial has only real roots, its derivative has only real roots. By contrast, $(z+1)e^{z^2}$ is a real entire function whose only zero $z=-1$ is real, but its derivative $(2z^2+2z+1)e^{z^2}$ has two imaginary zeros [132]. The question therefore arises: What sort of limitations must be imposed upon entire functions to enable us to extend to them our theorems on the location of roots of polynomials?

Investigation in this direction was inaugurated by Laguerre in the 80's or earlier. Since then little advance has been made until recently when some very interesting extensions have been obtained, notably by Pólya. Before reporting on these I must recall to you certain results and ideas familiar to the analyst. First of all, I must mention the theorem of Hurwitz that when a sequence of polynomials $P_n(z)$, $(n=1, 2, \cdots)$, or analytic functions converges uniformly over a closed region, the roots of the limiting analytic function within the region are the limits of the zeros of the polynomials in the sequence. In the vicinity of a p-fold

root of the limit lie exactly p roots of $P_n(z)$ for sufficiently large values of n. An interesting application of this indispensable theorem is to the exponential series: in any given finite region of the plane there can lie no roots of the sum $S_n(z)$ of the first n terms for sufficiently large values of n.

Next, I must mention Laguerre's concept of the *genre* of an entire function. According to Weierstrass's Factor-Theorem every entire function can be factored into the uniformly convergent form:

$$F(z) = e^{G(z)} z^r \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{H_n(z)}, \tag{20}$$

where

$$H_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n},$$

and where $G(z)$ is an entire function, polynomial or constant, r is a non-negative integer, and m_n is an appropriate positive integer which may increase indefinitely with n. When the degree m_n is limited and $G(z)$ is a polynomial, the function is said to be of finite genre, the genre p being then the highest (necessary) degree in the exponents of e within or without the product sign. The entire function (20) differs in structure from a polynomial by the presence of exponential factors. Hence, as Borel has remarked, one may expect that the properties of entire functions will depart more and more from those of polynomials as we increase the degree of the exponential factor.

For $p = 0$ we have

$$F(z) = Cz^r \prod \left(1 - \frac{z}{a_n}\right),$$

so that

$$\frac{d \log F(z)}{dz} \equiv \frac{F'(z)}{F(z)} = \frac{r}{z} + \sum \frac{1}{z - a_n}, \tag{21}$$

where Π and Σ are uniformly convergent over any closed finite region of the plane not containing any roots a_n of the entire function. Correspondingly, we have for $p = 1$,

$$F(z) = C_1 e^{cz} z^r \prod \left(1 - \frac{z}{a_n}\right) e^{z/a_n},$$

$$\frac{F'(z)}{F(z)} = c + \frac{r}{z} + \sum \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right); \tag{22}$$

and for $p = 2$,

$$F(z) = c e^{c' z^2 + c'' z} z^r \prod \left(1 - \frac{z}{a_n}\right) e^{z/a_n + (1/2)(z/a_n)^2},$$

$$\frac{F'(z)}{F(z)} = c'' + 2c'z + \frac{r}{z} + \sum \left(\frac{1}{z - a_n} + \frac{1}{a_n} + \frac{z}{a_n^2}\right). \tag{23}$$

When $F(z)$ is a function of finite genre and its Weierstrassian product (20) is devoid of the plaguy exponential factor before the product sign,—that is, when $G(z)$ is a constant—we call the function *primitive* or *canonical*.

We are ready now for the extension of polynomial theorems to entire functions. Consider first the simple theorem that when the roots of a polynomial are all real, the like is true of the derivative. The simple example already given shows that this may fail for an entire function of as low genre as 2. In the case of the real polynomial with only real roots we locate all the $n-1$ roots of the derivative by Rolle's Theorem which puts one between each two successive roots of the given polynomial, but the argument fails in the case of a real entire function because the derivative of an entire function may have an infinite number. In the case of real entire functions of genre 0 or 1 the result is still valid. For first, it can be established that there can be no imaginary roots of $F'(z)$, either by applying the equilibrium theorem of Gauss to the right-hand members of (21) and (22) or, what amounts to the same thing, by observing that the sign of the imaginary component of each term is opposite to y in $z = x + iy$, since c and the a_n are real, so that $F'(z)$ will not vanish unless $y = 0$. Next, if we differentiate the right-hand members of (21) and (22) their terms will all be negative for a real value of z. Consequently $F'(x)/F(x)$ is a monotone

decreasing function of x, and can vanish but once between two successive roots of the denominator $F(x)$. Thus the roots of the derivative of a given real entire function of genre 0 or 1 are all real and alternate with those of the function itself, as in the case of a given real polynomial.

The argument just indicated fails in the case of entire functions of genre 2 because of the additional terms introduced in (23). When, however, c' is negative and Π is a canonical product of genre 1 or 0 so that the term $(z/a_n)^2$ is lacking in Σ, the argument carries through. We are led thus to pick out the same special class of functions of genre 2 which we ran across at the close of Part I, namely, real entire functions which are of genre 0 or 1, except for an exponential factor $e^{c'z^2}$ in which c' is negative.

I digress a moment at this point to mention a more general result that was just obtained. A real entire function of genre p with q imaginary roots possesses a derivative of the same genre which in addition to the roots accounted for by the theorem of Rolle can have at most $p+q$ imaginary and other real roots. A rigorous proof of this theorem of Laguerre is given by Borel [132].

I pass next to the consideration of another noteworthy case in which the derivative of the real entire function $F(z)$ possesses only real zeros which alternate with those of $F(z)$ on either side of the origin. It is that in which $F(z)$ is a canonical function, limited by a restriction placed upon it which at first sight appears very curious, viz., *that it shall have no zero at the origin.* The result is discussed in Vivanti's 1907 report on entire functions but without attributing it to any particular person. I find it established in a paper [133] of Cesaro (1884) not mentioned in Vivanti's bibliography. Cesaro's exceedingly simple proof is as follows: For every zero of the derivative of such a canonical $F(z)$ of genre p we have

$$0 = \frac{F'(z)}{F(z)} = \sum_i \left(\frac{1}{z - \alpha_i} + \frac{1}{\alpha_i} + \frac{z}{\alpha_i^2} + \cdots + \frac{z^{p-1}}{\alpha_i^p} \right),$$

which may be written

$$0 = z^p \sum_i \frac{1}{\alpha_i^p (z - \alpha_i)}. \tag{24}$$

Now when all the roots α_i of $F(z)$ are real and the genre p is an even integer, the coefficient of i in every term of Σ in (24) has a sign opposite to that of y. Therefore the equation can not be satisfied for an imaginary value of z. On the other hand, if p is an odd integer, we may put the equation into the form

$$0 = z^p \sum_i \frac{\alpha_i z}{\alpha_i^{p+1}(z - \alpha_i)}, \tag{25}$$

and then the same conclusion follows, inasmuch as for real α the imaginary part of the coefficient of $\alpha z/(z-\alpha)$ is

$$-\frac{i\alpha^2 y}{(x - \alpha)^2 + y^2},$$

which again is opposite in sign to y. In consequence, $F'(z)$ has no imaginary roots. Furthermore, the derivative of each term under the summation sign in (24) for even p and in (25) for odd p is negative for real $z \neq \alpha_i$. In each case Σ is therefore a monotone non-increasing function and consequently between any two roots α_i of $F(z)$ on the same side of the origin there lies one and only one of its derivative.

It must not be thought that the conclusions obtained concerning the reality of roots of the derivative will carry over to higher derivatives, for the suppositions underlying the proof regarding the given entire function ordinarily do not hold regarding the derivative. If, for instance, the given function is canonical, the derived function is not necessarily so. Again, a function of genre p has a derivative of genre p at most, and an example has been given of a function of genre p with a derivative of genre $p-1$. The question therefore arises: What are then the necessary and sufficient conditions to be imposed upon a given entire function possessing only real roots in order that its derivatives of all

orders shall likewise have only real roots? This very difficult question has recently been tackled by Pólya.

The nature of the answer seems to have been suggested by the following theorem.

THEOREM OF LAGUERRE.* *If a sequence of polynomials whose roots are all real converges over every finite region of the z-plane,† the limit of the sequence is an entire function of genre* 0 *or* 1, (*by the above theorem of Hurwitz, possessing only real zeros*), *or the same multiplied by a factor* e^{cz^2} *in which c is real. Furthermore, if the roots of the polynomials are all of like sign, the limit, except for a possible factor* e^{az} *with real a, will be a function of genre* 0 (*possessing only roots of the same sign*).

This theorem and others to be presently mentioned seem to me noteworthy in that they establish the genre of the function not on the basis of the asymptotic character of the coefficients or roots but by use of functional properties.

The derivatives of the polynomials in the theorem are of the same character as the polynomials themselves, having likewise only real roots (respectively real roots of the same sign), and also by a well known theorem on analytic functions converge uniformly over every finite region of the z-plane. The conclusions of the theorem can therefore be applied to succeeding derivatives of the entire function. Thus on the given hypothesis all derivatives have only real roots (respectively real roots of the same sign).

This application of Laguerre's Theorem stops short, however, of answering the question proposed. For, on the one hand, we are not sure that all real entire functions with real zeros are obtainable as limits of sequences of polynomials with real roots. On the other hand, there exist *unreal* entire functions possessing only real zeros. To get such a function we have merely to take in (20) an unreal $G(z)$ along with a canonical product Π having only real zeros. Further-

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* *Oeuvres*, vol. 1, 1882, pp. 174–177.

† Apparently uniform convergence is intended.

more, the possibility that all the succeeding derivatives should have only real roots, if any, can be actually realized. The function $c'(e^{ibz}-c)$, in which we take b real and $|c|=1$, is an unreal entire function possessing an infinite periodic system of real zeros and no others, while its successive derivatives have no zeros at all. It is easy to see that it is of genre 1. Pólya cites $c'e^{icz}$ (c real) as a second instance, this being an unreal function which with its successive derivatives is devoid of imaginary zeros. He [120a] conjectures that with just these two exceptions all entire functions which together with their derivatives have only real zeros must be of the form

$$cz^r e^{-\gamma^2 z^2+dz}\prod(1-d_n z)e^{d_n z}, \tag{26}$$

in which all constants except c are real and Σd_n^2 is convergent. Note that we have here exactly the first kind of functions mentioned in the above Theorem of Laguerre with the added restriction that the coefficient of z^2 in the exterior exponential factor is negative or zero. This conjecture Pólya was able to prove on the supposition that the entire function is of finite genre and has only a finite number of zeros. It can then be written in the form $e^{Q(z)}P(z)$ where P and Q are polynomials, so that we have then a purely polynomial problem. He showed further that when all the roots of the entire function (except $z=0$) are of positive sign, we must take $\gamma=0$ and a non-positive d; accordingly the function except for such a possible exterior factor e^{dz} is of genre zero. In 1923 Alander [124] extended this conclusion to any entire function of finite genre without the limitation to a finite number of zeros. This he did by showing that the function of finite genre, unless it has one of the two exceptional forms above noted, can have only a finite number of zeros, so that Pólya's conclusion is applicable.

The sufficiency of the form (26) for the reality of the roots of all the successive derivatives Pólya established neatly by setting up a sequence of polynomials

$$\phi_n(z) = cz^r\left(1 - \frac{\gamma^2 z^2}{n}\right)^n$$

$$\cdot\left(1 + \frac{d + d_1 + \cdots + d_n}{m_n}\right)^{m_n} \prod_{\nu=1}^{n}(1 - d_\nu z), \ (n = 1, 2, \cdots),$$

which with proper choice of the positive integers m_n converge uniformly to the function (26) in every finite region of the z-plane. The character of the limiting function then follows by Laguerre's Theorem given above.

It is interesting to note another closely related result of Pólya [136]. When the entire function has the form $e^{Q(z)}P(z)$ in which $Q(z)$ and $P(z)$ are polynomials, the number of imaginary roots of its nth derivative increases indefinitely with n if $Q(z)$ is of higher degree than 2, and likewise if it is of degree 2 unless the coefficient of z^2 in $Q(z)$ is negative.

The interesting theorem of Laguerre above quoted was based upon the hypothesis of a sequence of polynomials with real roots which converged uniformly in every finite region of the plane. Pólya [120b] has notably lightened the hypothesis by showing that it is sufficient to demand that they converge uniformly in some finite region containing the origin. One thinks here of Stieltjes's Theorem that if a sequence of analytic functions converges uniformly in any given region and is bounded in absolute value over any larger region embracing the given region, then it also converges uniformly over this larger region. Pólya does not use this theorem but is able to base the extension of the region of uniform convergence directly upon the reality of the roots provided the limit is not identically zero. In a subsequent paper [130] he lightens even this latter requirement and asks only that the roots of the polynomials shall all lie within some sector having its vertex at the origin.

When the polynomials under consideration are segments $S_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ of a power series, these results of Pólya follow immediately from a remarkable theorem of Jentzsch [141]. If, namely, the series has a finite radius of convergence $R > 0$, every point of the circle of convergence

is a limit point of the set of roots of the equations $S_n(z)=0$, $(n=1, 2, \cdots)$. Suppose now that there is a sector with vertex at the origin free from the roots of $S_n(z)=0$. Then if the polynomial segments $S_n(z)$ converge in a region including the origin, it will follow that R is infinite, since otherwise the theorem is violated. Later Szegö [142] has given an extension of the theorem in which the sequence is no longer limited to a polynomial sequence obtained by segmenting a power series.

Jentzsch [143] has obtained a theorem which derives the existence and genre of an entire function from suppositions regarding the roots of its segments without making the slightest demand regarding the convergence of the sequence $S_n(z)$, $(n=1, 2, \cdots)$. Denote by $\beta_{1,n}, \beta_{2,n}, \cdots, \beta_{n,n}$ the n roots of $S_n(z)$. Then if from and after some fixed value of n these roots satisfy the inequality

$$\sum_{i=1}^{n} \frac{1}{|\beta_{in}|^{\rho}} < M, \qquad (n = r, r+1, \cdots), \tag{27}$$

where ρ is some positive integer and M a fixed positive number, the series will converge over the entire plane and its limit will be an entire function of genre equal to or less than ρ. As a particular application of this result consider any series whose segments $S_n(z)$ have only real roots. By the theory of symmetric functions we have

$$\sum_{i=1}^{n} \frac{1}{|\beta_{in}|^{2}} = \frac{a_1^2 - 2a_0a_2}{a_0^2}.$$

Here $\rho=2$, and M is any number greater than the right-hand member. It follows, therefore, if the polynomial sections $S_n(z)$ of a formally given series have only real roots, the series converges to a limit, and the limit is an entire function of genre 2 at most. Pólya [120b] (1913) cuts down the genre to 1 at most, and to 0 when the roots are all of the same sign. Jentzsch states also, without giving proof, in his 1914 Comptes Rendus note [144], that his conclusion will hold

also for any sequence of polynomials $a_{n0}+a_{n1}z+\cdots+a_{nn}z^n$, $(n=1, 2, \cdots)$, whose roots satisfy the inequality (27), provided that it converges uniformly in some region of the plane. In the same year Lindwart and Pólya [128] considered such a sequence under the same supposition but with the added limitation (presumably unnecessary) that the region of uniform convergence shall include the origin, and proved not only that it has a limit which is an entire function of genre equal to or less than ρ but that also the limit is actually a function of genre $\rho-1$ at highest except for a possible factor e^{cz^ρ}.

I turn lastly to a topic as yet but too little worked, the determination of the conditions which must be imposed upon the coefficients of an entire function to secure a specified distribution of its roots. In the case of a real polynomial of the nth degree, it has been long known that if its roots are *distinct*, the number of pairs of imaginary roots is equal to the number of changes of sign in the sequence

$$(28)\qquad s_0, \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}, \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}, \cdots, \begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \cdot & \cdot & \cdots & \cdot \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{vmatrix},$$

where $s_0=n$, and s_i denotes the sum of the ith power of the roots. The roots will be all real if and only if all of these determinants are positive. If the members of (28) are all positive when we advance every subscript by unity, we have, as Grommer [121] has shown, the necessary and sufficient condition that the roots should all be distinct and positive, provided that there is no root at the origin.

With this proviso, it is obviously permissible to replace the roots by their reciprocals, or, in other words, to substitute s_{-i} for s_i. This is necessary before we can think of generalizing the condition so as to get a corresponding result for entire functions. But the indefinitely continued sequence (28) is obviously unusable even for an entire function of

genre 0 with an infinite number of roots, for though the sum of the ith powers of the reciprocals of the roots will converge for all values of i, the element $s_0 = n$ increases indefinitely with the degree of the polynomial. Fortunately, in place of (28) we can use the sequence

$$(29)\qquad s_{-2},\ \begin{vmatrix} s_{-2} & s_{-3} \\ s_{-3} & s_{-4} \end{vmatrix},\ \cdots,\ \begin{vmatrix} s_{-2} & s_{-3} & \cdots & s_{-n-1} \\ \cdot & \cdot & \cdots & \cdot \\ s_{-n-1} & s_{-n} & \cdots & s_{-2n} \end{vmatrix}$$

to test the reality of the roots of a polynomial of nth degree. The roots are all real when and only when the terms of the sequence (29) are either all positive or are positive up to a certain member inclusive, after which they are all zero [121, p. 114]. The elements s_{-i} therein contained have a significance for entire functions of genre 0 or 1, the sum of the ith powers of the reciprocals of the roots being convergent for $i \geqq 2$. According to Grommer, Hurwitz considered the question whether the roots must all be real when the terms of (29) continued indefinitely are all positive for such a function. But the elements s_{-i} in (29) may be assigned a significance for any entire function. If, in fact, we take an entire function $F(z) = a_0 + a_1 z + \cdots$ without a zero at the origin and expand the negative of its logarithmic derivative into a series

$$(30)\qquad -\frac{F'(z)}{F(z)} = s_{-1} + s_{-2} z + s_{-3} z^2 + \cdots,$$

the successive coefficients are connected by linear recurrence formulas in which both the s_{-i} and the a_i appear linearly. When the function is of genre 0 the s_{-i} are actually equal to the sum of the ith powers of the reciprocals of the roots by virtue of the relation

$$-\frac{F'(z)}{F(z)} \equiv -\sum_{i=1}^{\infty} \frac{1}{z-\alpha_i} = \sum_{i=1}^{\infty} \left(\frac{1}{\alpha_i} + \frac{z}{\alpha_i^2} + \cdots \right).$$

Whether or not this be the case, let the determinant sequence (29) continued indefinitely be formed with elements taken from the series (30). With this generalization of the s_{-i}

Grommer took up the problem at Toeplitz's suggestion and proved that when the indefinitely continued sequence (29) is merely restricted by the requirement that all its terms shall be positive, the entire function will be of genre 0 or 1 except possibly for a factor $e^{-\gamma^2 z^2}$ (γ real). His proof is complicated, involving the use of quadratic forms, continued fractions and Stieltjes integrals. Later, in 1920, Kritikos [122] under the instigation of Pólya re-derived the result by a simpler method. Grommer derives also a more general theorem, according to which the necessary and sufficient condition that a real entire function $F(z)$, for which $F(0) \neq 0$, shall have an infinite number of roots all real and simultaneously shall be an entire function of genre $2m-1$ or less, except for a factor $e^{-\gamma^2 z^{2m}}$ (γ real), is that all the determinants

$$\begin{vmatrix} s_{-2m} & s_{-2m-1} & \cdots & s_{-2m-k+1} \\ \cdot & \cdot & \cdots & \\ s_{-2m-k+1} & s_{-2m-k} & \cdots & s_{-2m-2k+2} \end{vmatrix}, \quad (k = 1, 2, \cdots),$$

shall be positive.

It may not be without interest to say a word or two regarding the opposite possibility that all the roots shall be imaginary. Long ago I showed [2] very easily that if

$$(31) \quad a_0, \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}, \cdots, \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \cdot & & \cdots & \cdot \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix},$$

$$(a_i = 0 \text{ for } i > n),$$

are all positive, all the roots of the polynomial

$$P_n(z) = a_0 + a_1 z + \cdots + a_n z_n$$

are imaginary if n is even, or all but one if n is odd. The condition is sufficient but not necessary. It follows that if for an entire function $a_0 + a_1 z + a_2 z^2 + \cdots$ all the terms of the sequence (31), indefinitely continued, are positive, it is an imaginary function with only imaginary zeros. A diver gent power series may also be consistent with the requirement

of the positive character of (31), as will be seen by increasing a_0, a_2, a_4, sufficiently rapidly.

This I trust is sufficient to indicate the kind of theorems which are being obtained by extension of the theory of equations into the province of the theory of analytic functions. The mine has been yet imperfectly worked and much ore remains to be extracted.

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I. Roots Real, or Real and of Same Sign.

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119. J. I. Hutchinson, *On a remarkable class of entire functions*, Transactions of this Society, vol. 25 (1923), pp. 324–332.
(This article connects with Nos. 117 and 118.)

120. G. Pólya, (a) *Bemerkung zur Theorie der ganzen Funktionen*, Jahresbericht der Vereinigung, vol. 24 (1915), pp. 392–400.

(b) *Ueber Annäherung durch Polynome mit lauter reellen Wurzeln*, Rendiconti di Palermo, vol. 36 (1913), pp. 279–295.

(c) *Algebraische Untersuchungen über ganze Funktionen vom Geschlechte Null und Eins*, Journal für Mathematik, vol. 145 (1915), pp. 224–249.
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121. J. Grommer, *Ganze transzendente Funktionen mit lauter reellen Nullstellen*, Journal für Mathematik, vol. 144 (1914), pp. 114–165.
(With this Nos. 122 and 123 connect.)

122. N. Kritikos, *Ueber ganze transzendente Funktionen mit reellen Nullstellen*, Mathematische Annalen, vol. 81 (1920), pp. 97–118.
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124. M. Alander, *Sur les fonctions entières qui ont tous les zéros sur une droite*, Comptes Rendus, vol. 176 (1923), pp. 158–161.

125. A. Wahlund, *Zum Beweise einer Beziehung zwischen den Koeffizienten der ganzen Funktionen von der Höhe Null, die lauter negative oder lauter positive Nullstellen besitzen*, Arkiv för Matematik, Astronomi, och Fysik, vol. 18 (1924), No. 22, pp. 1–5.

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128. Lindwart and Pólya, *Ueber einen Zusammenhang zwischen der Konvergenz von Polynomfolgen und der Verteilung ihrer Wurzeln*, Rendiconti di Palermo, vol. 37 (1914), pp. 297–304.

129. Y. Okada, *A theorem on power series*, Tôhoku Mathematical Journal, vol. 15 (1919), pp. 278–279.
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See also Nos. 91, 123.

K. Roots of the Derivatives.

132. E. Borel, *Leçons sur les Fonctions Entières* (1900), see Chapter 2, *Les Idées de Laguerre*, pp. 24–47.

133. Cesaro, *Sur les fonctions holomorphes de genre quelquonque*, Comptes Rendus, vol. 99 (1884), pp. 26–27.

134. L. Desaint, *Sur quelques points de la théorie des fonctions*, Annales de l'École Normale Supérieure, (3), vol. 14 (1897), see pp. 336–344.

135. L. Leau, Chapter 6, *Racines des dérivées de certaines fonctions réelles de genre fini, Complément à un théorème da Laguerre*, Annales de l'École Normale Supérieure, vol. 23 (1906), pp. 102–120.

136. G. Pólya, *Ueber die Nullstellen sukzessiver Derivierten*, Mathematische Zeitschrift, vol. 12 (1922), pp. 36–60.

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137. M. B. Porter, *On a theorem of Lucas*, Proceedings of the National Academy of Sciences, vol. 2 (1916), pp. 247–248, and pp. 335–336.

138. M. Alander, (a) *Sur le déplacement des zéros des fonctions entières par leur dérivation*, Thèse, Upsala, 1914.

(b) *Sur les zéros extraordinaires des dérivées des fonctions entières réelles*, Arkiv för Matematik, Astronomi och Fysik, vol. 11 (1916)–17), No. 15, pp. 1–18.

(c) *Sur les zéros des dérivées des fonctions rationnelles et d'autres fonctions méromorphes*, Arkiv för Matematik, Astronomi och Fysik, vol. 14 (1919–20), No. 23, pp. 1–30.

(d) *Sur les zéros complexes des dérivées des fonctions entières réelles*, Arkiv för Matematik, Astronomi och Fysik, vol. 16 (1922), No. 10, pp. 1–19.

(e) *Sur les fonctions entières non-réelles*, Arkiv för Matematik, Astronomi och Fysik, vol. 18 (1924), No. 12, pp. 1–9.

139. M. Biernacki, *Sur le déplacement des zéros des fonctions entières par leur dérivation*, Comptes Rendus, vol. 175 (1922), pp. 18–20.

L. Miscellaneous.

140. F. Lukács, *Eine Eigenschaft des Konvergenzkreises der Potenzreihen*, Archiv der Mathematik und Physik, (3), vol. 23 (1915), pp. 34–35.

141. R. Jentzsch, (a) *Untersuchungen zur Theorie der Folgen analytischer Funktionen*, Dissertation, Berlin, 1914; and Acta Mathematica, vol. 41 (1918), pp. 219–251.

(b) *Fortgesetzte Untersuchungen über die Abschnitte von Potenzreihen*, Acta Mathematica, vol. 41 (1918), pp. 253–270.

(Jentzsch proves that every point on the circle of convergence of a Taylor's power series is a limit point of roots of its segments. In No. 140 Lukacs showed that there was at least one such limit point on the circle. An extension of Jentzsch's result is given by Szegö in the following article.)

142. G. Szegö, (a) *Ueber die Nullstellen von Polynomen, die in einem Kreise gleichmassig konvergieren*, Sitzungsberichte der Berliner Matematischen Gesellschaft, vol. 21 (1922), pp. 59–64.

(b) *Ueber die Nullstellen der Polynome einer Folge, die in einem einfach zusammenhängenden Gebiete gleichmässig konvergiert*, Göttinger Nachrichten, 1922, pp. 137–143.

143. R. Jentzsch, *Sur l'extension d'un théorème de Laguerre*, Comptes Rendus, vol. 158 (1914), pp. 780–782.

144. G. Pólya, *Ueber das Graeffesche Verfahren*, Zeitschrift für Mathematik und Physik, vol. 63 (1915), pp. 275–290.

Comptes Rendus, vol. 156 (1913), pp. 1145–1147.

See also Nos. 11, 72

The University of Wisconsin

AN ANNOUNCEMENT REGARDING FACTOR STENCILS

BY D. N. LEHMER

An edition of fifty sets of Factor Stencils has been completed and will shortly be distributed by the Carnegie Institution of Washington under whose auspices the work of construction has been done. The device is intended to facilitate the finding of factors of numbers of an order as high as two billion and a half.

The theory of the stencils is based on the well known fact that if R is known to be a quadratic residue of a number N then the factors of N are to be found in certain linear forms. Thus if -1 is known to be a residue of N, the factors of N are all of the form $4n+1$; and if 2 is a residue of N, the factors of N are of the form $8n \pm 1$; and so on. These linear forms may be combined and a set of linear forms deduced from them to limit the number of trials necessary to determine the character of the number N.

As the number R increases, however, the number of forms to be considered also increases so that the combination of the forms corresponding to two numbers R such as 113 and 199 would involve some 11,088 resulting forms. The straightforward combination of the linear forms becomes impossible except for very small values of R. Nevertheless large values of R will exclude the same proportion of trials as small values, and the stencil device is intended to make it possible to use values of R as high as ± 238. As each residue discovered serves to reject approximately half the number of trials, it is seen that for a number of the order of 2,000,000,000 where some 5,000 trial divisors must be examined, one residue reduces the number to about 2,500 trials; two to about 1,250; three to 625; four to 312; five to 78; six to 39; seven to 20; eight to 10; nine to 5; and ten to 2. The finding, therefore of

ten quadratic residues of N ought to be sufficient to find the factors of a number of the order of two billion. For smaller numbers, of course, a smaller list will serve. The only difficulty lies in the combination of the linear forms belonging to the residues. This difficulty is completely solved by the factor stencils. One does not seek to find the resultant forms of all the forms belonging to the residues, but only the primes that may lie in them. This is accomplished by a plan which is believed to be entirely new, and the work of examining a number is practically complete as soon as the list of residues is found. The construction of the Factor Stencils is as follows.

The first page of the list of primes published by the Carnegie Institution of Washington in 1914 contains 5,000 primes beginning with 1 and ending with 48,593. Any composite number not greater than 2,361,279,649, which is the square of 48,593, will have at least one of its factors in this list of primes. The list is arranged in fifty columns with one hundred rows in each column. A sheet of paper for a stencil is then ruled to show 5,000 cells, 50 columns and 100 rows. For a given R, holes are punched in those cells corresponding to primes which have R for a quadratic residue. Thus for $R=-1$ holes appear in the cells corresponding to the position in the list of primes of the numbers 1, 5, 13, 17, 29, 37, $\cdots$, and for $R=2$ the holes appear in the cells corresponding to the numbers 1, 7, 17, 23, 31, 41, 47, 71, 73, $\cdots$; the first set all having -1 for a residue and the second all having 2 for a residue. If these two stencils are superposed it is clear that those holes which appear through both indicate primes which are in both sets and have both -1 and 2 for residues. The first holes that thus appear are those corresponding to the primes 17 and 31. If now the stencil for a third residue is laid on the other two approximately half the remaining holes are covered and the holes that shine through the three stencils indicate primes that have all three numbers for quadratic residues. The problem of finding the factors of a number within the range of the first 2,000,000,000 numbers is reduced to the finding of some eight or ten quadratic residues.

In the book which accompanies the stencils various methods of finding quadratic residues are discussed and the method by means of the expansion of the square root of the number in a continued fraction is found to be by far the most effective. Various examples are given illustrating the power of the stencils, and a reproduction of the first page of the list of primes accompanies the work.

The plan of the stencils was first conceived in November 1924 and has been carried on since then under grants from the Carnegie Institution of Washington. Plans for the distribution of the sets are not yet completed, but every effort will be made to place them where they will be of most use.

THE UNIVERSITY OF CALIFORNIA

ON A PROBLEM IN THE THEORY OF GROUPS ARISING IN THE FOUNDATIONS OF INFINITESIMAL GEOMETRY*

BY H. P. ROBERTSON AND H. WEYL

In another paper in this issue,† the fundamental problem of infinitesimal geometry is formulated as the problem of uniquely associating with an arbitrary coordinate system on the manifold M a normal coordinate system on the tangent plane T_P by means of the fundamental coefficients of displacement on M.

The importance of the other aspect of this problem raised by O. Veblen and H. P. Robertson, yet remains: to associate a transformation of the given group $\mathfrak{G}$ with an arbitrary tranformation of the coordinates x in such a way that it gives rise to a representation by $\mathfrak{G}$, that is, that to composition of arbitrary transformations of x corresponds composition of the associated transformations of $\mathfrak{G}$. From

* Presented to the Society, June 21, 1929.

† This issue, pp. 716–725.

this standpoint the problem is a pure group-theoretic one which is not at all concerned with a "connection" on M. We consider in the following the projective group as an example.

More precisely, the problem is the following. Take the point P under consideration as the origin of coordinates, that is, consider the relative coordinates $x_k - x_0{}^i$, where $P = P(x_0)$. An arbitrary transformation of these relative coordinates x^i has the form

$$(1) \qquad s: \qquad \bar{x}^i = a_k^i x^k + \tfrac{1}{2} a_{kl}^i x^k x^l + \cdots .$$

The associated n-dimensional projective transformation σ, which shall also leave the origin fixed, is expressed in homogeneous coordinates η in accordance with the normalization (4) of the paper mentioned above. We take as coefficients a_k^i of σ the coefficients a_k^i of (1); the coefficients a_k are then to be determined. The condition imposed is that the association $s \rightarrow \sigma$ shall yield a representation. We may add the additional condition that σ depend only on the coefficients a_k^i and a_{kl}^i of s of first and second orders.

We are then naturally led to the problem of representing the group $\mathfrak{G}$ of all transformations by linear transformations, whereby the accessory condition that the transformation corresponding to s shall depend on the terms of the first two or first three orders etc. may be imposed. This problem is to be attacked as follows. We first restrict ourselves to the subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$ consisting of linear transformations

$$\bar{x}^i = a_k^i x^k .$$

We know the representations of $\mathfrak{G}_0$ and we shall assume that the given representation $s \rightarrow \sigma$ of $\mathfrak{G}$ by N-dimensional matrices σ decomposes into known irreducible representations when considered as a representation of $\mathfrak{G}_0$ (that is certainly the case when we restrict ourselves to the unimodular transformations of $\mathfrak{G}_0$; we take it to be the case for the full linear group for the sake of simplicity). Let these irreducible constituents be of orders o, o', $\cdots$, respectively; i.e. to the infinitesimal dilatation $dx^i = x^i$ correspond the dilatations

$d\xi^\iota = o\xi^\iota$, $d\xi^\iota = o'\xi^\iota$, $\cdots$, respectively in these representations. (It follows from the general theory that in an irreducible representation a dilatation must correspond to a dilatation in the original group.)

Furthermore, we employ Lie's infinitesimal method. To an infinitesimal transformation s

$$dx^i = a_k^i x^k + \frac{1}{2!} a_{kl}^i x^k x^l \tag{2}$$

corresponds an infinitesimal transformation

$$d\xi^\iota = \alpha_\kappa^\iota \xi^\kappa \tag{3}$$

of the given representation. But the α_κ^ι are now *linear* homogeneous functions of the a_k^i, a_{kl}^i, $\cdots$. The condition that σ be a representation requires in addition that to the commutator of two transformations s corresponds the commutator of the corresponding transformations ξ. We consider in particular the two transformations

$$s: \quad dx^i = \frac{1}{2!} a_{kl}{}^i x^k x^l + \cdots \quad \text{and} \quad C: \quad d'x^i = c_k^i x^k. \tag{4}$$

If the matrices of the corresponding σ (3) be A and Γ, the infinitesimal σ with matrix $A\Gamma - \Gamma A$ shall correspond to the commutator of (4). We proceed in three steps, by first taking as C the dilatation

$$d'x^i = x^i, \quad \text{that is,} \quad c_k^i = \delta_k{}^i,$$

then the more general linear "principal transformation" $d'x^i = \lambda_i x^i$ (not summed!) with arbitrary λ_i, and finally the unrestricted C.

The commutator of (2) and the dilatation is obtained by multiplying the term of nth degree in (2) by $(n-1)$. The set of variables ξ^ι is divided into subsets the members of which are transformed among themselves under Γ, and in particular are only multiplied by o, o' etc. by the transformation $\Gamma = \Gamma_d$ which corresponds to the dilatation. The rectangle $[A]$ of the matrix A, in which the set o intersects the

set o', is multiplied by $o'-o$ on going over to $A\Gamma_d-\Gamma_d A$ from A. The α_{κ^ι} in this rectangle must consequently be multiplied by $o'-o$ when the a_k^i, a_{kl}^i, etc. are replaced by $0\cdot a_k^i$, $1\cdot a_{kl}^i$ etc. Consequently if $o'-o$ is not a whole number, 0, 1, 2, $\cdots$, [A] must vanish identically, and if $o'-o=n-1$, [A] can depend only on the terms of order n in (2). In particular the square [A] of A in which the set o intersects itself can only depend on the a_k^i.

We can choose the coordinates in representation space in such a way that to the principal transformations $dx^i=\lambda_i x^i$ correspond principal transformations $\xi^\iota \to \Lambda_\iota \xi^\iota$; the "weights" Λ_ι are linear forms in the λ_i. In the commutator of (2) with this principal transformation a_{kl}^i is replaced by

$$a_{kl}^i \ldots (\lambda_k + \lambda_l + \cdots - \lambda_i)$$

while the corresponding process for the σ changes α_{κ^ι} into $\alpha_{\kappa^\iota}(\Lambda_\kappa - \Lambda_\iota)$. Consequently the α_{κ^ι} can depend only on those $a_{kl}^i \cdots$ for which $\lambda_k+\lambda_l+\cdots-\lambda_i$ is identical with $\Lambda_\kappa-\Lambda_\iota$.

Equivalent representations, which differ only in the choice of the coordinate system in which they are expressed, shall naturally be considered as the same. But as soon as the irreducible representations into which the given representation of $\mathfrak{G}_0$ is decomposed is fixed the coordinate system is fixed except for multiplication of the variables of each subset by a non-vanishing arbitrary constant.

Let us return to the case considered above, in which $N=n+1$ and the $(n+1)$-dimensional representation of $\mathfrak{G}_0$ is decomposed into the n-dimensional $s \to s$ and the one-dimensional which associates the identity $\xi^0=\xi^0$ with every s. Here $o=1$, $o'=0$; we must consequently have in (3)

$$\alpha_k{}^i = a_k{}^i, \qquad \alpha_0{}^0 = 1, \qquad \alpha_0{}^i = 0,$$

and α_k^0 depends only on the terms a_{rs}^i of second order. Furthermore, α_k^0 must be linear form in those a_{rs}^i only for which $\lambda_r+\lambda_s-\lambda_i=\lambda_k$. We must therefore have

$$\alpha_k^0 = \sum_i H_{ik} a_{ik}^i$$

where the H_{ik} are constants independent of s. In order to determine the H_{ik} we now consider the commutator of

$$dx^i = \tfrac{1}{2}a^i_{kl}x^k x^l \quad \text{and} \quad d'x^i = c_k{}^i x^k$$

and obtain

$$\sum_{i,r} H_{ik}(a^i_{rk}c_i{}^r + a^i_{ir}c_k{}^r - a^r_{ik}c_r{}^i) = \sum_{i,r} c^r_k H_{ir} a^i_{ir}$$

or

$$\sum_{i,r} a^i_{rk}(H_{ik} - H_{rk})c_i{}^r = \sum_{i,r} b^r_k(H_{ir} - H_{ik})a^i_{ir}.$$

Equating the coefficients of $c_i{}^k$ on both sides for $i \neq k$ we see that $H_{ik} = H_{kk}$; H_{ik} does not depend on i and can be written H_k. The equation then reduces to

$$\sum (H_r - H_k)c^r_k = 0$$

which tells us that H_k is independent of k, $H_k = H$. Hence

$$\alpha^0_k = H \sum_i a^i_{ik}.$$

When $H \neq 0$ we can multiply the variable ξ^0 by an appropriate constant in such a way that in this new coordinate system

$$H = -\frac{1}{n+1}.$$

We thus find that there are two distinct possibilities: (1) $\alpha^0_k = 0$ or the affine tangent plane, (2) the one obtained by Robertson and, from another standpoint, in the paper by H. Weyl on pages 716–725 of this issue, that is, the semi-osculating projective plane.

PRINCETON UNIVERSITY

NOTE ON LINEAR TRANSFORMATIONS OF n-ICS IN m VARIABLES*

BY A. D. CAMPBELL

Let us consider the n-ic in m variables

$$F(x_1, x_2, \cdots, x_m) = 0. \tag{1}$$

If we subject (1) to the linear transformation

$$\begin{aligned} \rho x_1 &= a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1m}x_m', \\ \rho x_2 &= a_{21}x_1' + a_{22}x_2' + \cdots + a_{2m}x_m', \cdots, \\ \rho x_m &= a_{m1}x_1' + a_{m2}x_2' + a_{m3}x_3' + \cdots + a_{mm}x_m', \end{aligned} \tag{2}$$

we obtain

$$\begin{aligned} F(&a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1m}x_m', a_{21}x_1' + a_{22}x_2' \\ &+ a_{23}x_3' + \cdots + a_{2m}x_m', \cdots, a_{m1}x_1' + a_{m2}x_2' \\ &+ a_{m3}x_3' + \cdots + a_{mm}x_m') = 0. \end{aligned} \tag{3}$$

Note that in the expansion of (3) the coefficient of the term in $x_i'^n$, $(i=1, 2, 3, \cdots, m)$, is $F(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$. A necessary and sufficient condition for this coefficient to vanish is that the point $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ shall lie on the geometric locus of (1). To obtain the coefficient of such a term as $x_i'^r x_j'^{n-r}$ in the expansion of (3) we can put

$$\begin{aligned} &x_i' x_j' \neq 0, \; x_1' = x_2' = x_3' = \cdots = x_{i-1}' \\ &\quad = x_{i+1}' = x_{i+2}' = \cdots = x_{j-1}' = x_{j+1}' = \cdots = x_m' = 0, \end{aligned}$$

then use Taylor's Expansion on

$$\begin{aligned} F(&a_{1i}x_i' + a_{1j}x_j', \; a_{2i}x_i' + a_{2j}x_j', a_{3i}x_i' \\ &+ a_{3j}x_j', \cdots, a_{mi}x_i' + a_{mj}x_j') \equiv F(X_1 + X_1', \\ &X_2 + X_2', X_3 + X_3', \cdots, X_m + X_m'), \end{aligned} \tag{4}$$

* Presented to the Society, August 29, 1929.

where $X_1 = a_{1i}x_i'$, $X_1' = a_{1j}x_j'$, $X_2 = a_{2i}x_i'$, $X_2' = a_{2j}x_j'$, etc. We find the coefficient of $x_i'^{r}x_j'^{n-r}$ in the group of terms*

$$\frac{1}{(n-r)!}\left(\frac{\partial F}{\partial X_1}X_1' + \frac{\partial F}{\partial X_2}X_2' + \frac{\partial F}{\partial X_3}X_3' + \cdots + \frac{\partial F}{\partial X_m}X_m'\right)^{(n-r)}$$

$$\equiv \frac{(x_i'^{r}x_j'^{n-r})}{(n-r)!}\left(\frac{\partial F}{\partial a_{1i}}a_{1j} + \frac{\partial F}{\partial a_{2i}}a_{2j} + \frac{\partial F}{\partial a_{3i}}a_{3j} + \cdots + \frac{\partial F}{\partial a_{mi}}a_{mj}\right)^{(n-r)}, \tag{5}$$

where $\partial F/\partial a_{1i}$ means $\partial F/\partial X_1$ with X_1 replaced by a_{1i}, X_2 by a_{2i}, X_3 by a_{3i}, $\cdots$, X_m by a_{mi}, and similarly for $\partial F/\partial a_{2i}$, $\partial F/\partial a_{3i}$, $\cdots$, $\partial F/\partial a_{mi}$. Hence we may conclude that a necessary and sufficient condition for the vanishing of the coefficient of $x_i'^{r}x_j'^{n-r}$ in the expansion of (3) is that the point $P_j(a_{1j}, a_{2j}, a_{3j}, \cdots, a_{mj})$ shall lie on the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ with respect to the locus of (1).

We obtain the coefficient of the term in $x_i'^{r}x_j'^{s}x_k'^{t}\cdots x_l'^{u}x_p'^{v}$ (where $r+s+t+\cdots+u+v=n$) in the expansion of (3) by the following device. We can write (3) as

$$F(Y_1 + Y_1', Y_2 + Y_2', Y_3 + Y_3', \cdots, Y_m + Y_m') = 0, \tag{6}$$

where

$$Y_1 = a_{1i}x_i', Y_1' = a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1i-1}x_{i-1}' + a_{1i+1}x_{i+1}' + a_{1i+2}x_{i+2}' + \cdots + a_{1m}x_m',$$

$$Y_2 = a_{2i}x_i', Y_2' = a_{21}x_1' + a_{22}x_2' + \cdots + a_{2i-1}x_{i-1}' + a_{2i+1}x_{i+1}' + \cdots + a_{2m}x_m',$$

etc. We take the collection of terms

$$\frac{1}{(n-r)!}\left(\frac{\partial F}{\partial Y_1}Y_1' + \frac{\partial F}{\partial Y_2}Y_2' + \frac{\partial F}{\partial Y_3}Y_3' + \cdots + \frac{\partial F}{\partial Y_m}Y_m'\right)^{(n-r)} \tag{7}$$

* See Goursat-Hedrick *Mathematical Analysis*, vol. 1, pp. 107–108. For the Galois fields, see A. D. Campbell, *The polar curves of plane algebraic curves in the Galois fields*, this Bulletin, vol. 34 (1928), pp. 361–363. The methods of this paper may be readily generalized to the polars of an n-ic in m variables.

in the expansion of (6). All the terms with $x_i'^r$ as a factor must come from (7). We can write (7) in the form

$$\frac{x_i'^r}{(n-r)!}\left(\frac{\partial F}{\partial a_{1i}}Y_1' + \frac{\partial F}{\partial a_{2i}}Y_2' + \frac{\partial F}{\partial a_{3i}}Y_3' + \cdots + \frac{\partial F}{\partial a_{mi}}Y_m'\right)^{(n-r)}. \tag{8}$$

If we equate (8) to zero we obtain the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$ with respect to the locus of (1). We can also write (7) in the form

$$\frac{x_i'^r}{(n-r)!}F'(Y_1', Y_2', Y_3', \cdots, Y_m'), \tag{9}$$

where F' is a function of the $(n-r)$th degree. We put

$$Y_1' = Z_1 + Z_1', Y_2' = Z_2 + Z_2', \cdots, Y_m' = Z_m + Z_m', \tag{10}$$

where

$$Z_1 = a_{1j}x_j', Z_1' = a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1i-1}x_{i-1}' + a_{1i+1}x_{i+1}' + a_{1i+2}x_{i+2}' + \cdots + a_{1j-1}x_{j-1}' + a_{1j+1}x_{j+1}' + a_{1j+2}x_{j+2}' + \cdots + a_{1m}x_m', Z_2 = a_{2j}x_j',$$

etc. Expanding (9), we find that all the terms in the expansion of (3) that have the factors $x_i'^r$ and $x_j'^s$ must be in the collection of terms

$$\frac{x_i'^r}{(n-r)!(n-r-s)!}\left(\frac{\partial F'}{\partial Z_1}Z_1' + \frac{\partial F'}{\partial Z_2}Z_2' + \frac{\partial F'}{\partial Z_3}Z_3' + \cdots + \frac{\partial F'}{\partial Z_m}Z_m'\right)^{(n-r-s)} \equiv \frac{(x_i'^r x_j'^s)}{(n-r)!(n-r-s)!}\left(\frac{\partial F'}{\partial a_{1j}}Z_1' + \frac{\partial F'}{\partial a_{2i}}Z_2' + \frac{\partial F'}{\partial a_{3j}}Z_3' + \cdots + \frac{\partial F'}{\partial a_{mj}}Z_m'\right)^{(n-r-s)}. \tag{11}$$

If we equate (11) to zero we shall have the $(n-r-s)$th polar of the point $P_j(a_{1j}, a_{2j}, a_{3j}, \cdots, a_{mj})$ with respect to the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$ with respect to the locus of (1).

Next we take

$$Z_1' = W_1 + W_1', \; Z_2' = W_2 + W_2',$$
$$Z_3' = W_3 + W_3', \cdots, Z_m' = W_m + W'_m$$

in (11), where

$$W_1 = a_{1k}x_k', \; W_1' = a_{11}x_1' + a_{12}x_2' + \cdots + a_{1i-1}x_{i-1}'$$
$$+ a_{1i+1}x_{i+1}' + \cdots + a_{1j-1}x_{j-1}' + a_{1j+1}x_{j+1}' + \cdots$$
$$+ a_{1k-1}x_{k-1}' + a_{1k+1}x_{k+1}' + \cdots + a_{1m}x_m', \; W_2 = a_{2k}x_k',$$

etc. We repeat the above processes until we finally reach the collection of terms having all the factors $x_i'^{r}$, $x_j'^{s}$, $x_k'^{t}$, $\cdots$, $x_l'^{u}$, and $x_p'^{v}$. Therefore, we see that for the coefficient of the term $x_i'^{r}x_j'^{s}x_k'^{t}\cdots x_l'^{u}x_p'^{v}$ in the expansion of (3) to vanish we must have the point $P_p(a_{1p}, a_{2p}, a_{3p}, \cdots, a_{mp})$ on the $(n-r-s-t-\cdots-u-v)$th polar of $P_l(a_{1l}, a_{2l}, \cdots, a_{ml})$ with respect to the $\cdots$ th polar of $\cdots$, $\cdots$, $\cdots$, with respect to the $(n-r-s-t)$th polar of $P_k(a_{1k}, a_{2k}, \cdots, a_{mk})$ with respect to the $(n-r-s)$th polar of $P_j(a_{1j}, a_{2j}, \cdots, a_{mj})$ with respect to the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ with respect to the locus of (1).

It is noteworthy that this discussion applies to the ordinary complex or real domains and also to the Galois fields.

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NON-EXISTENCE THEOREMS ON THE NUMBER OF REPRESENTATIONS OF ARBITRARY ODD INTEGERS AS SUMS OF $4r$ SQUARES*

BY E. T. BELL

1. *Introduction.* The theorems stated in §3 and proved in §4 will be more significant if we first outline some known results and devise a definition which they suggest. In §5 an interesting problem is proposed, to which the method of this paper is at least partly applicable.

2. *Simplicity of the Number of Representations of an Integer as a Sum of $4r$ Squares.* Let n, r be given integers. The number $N(n, r)$ of one-rowed matrices $(x_1, x_2, \cdots, x_r)$ of integers $x_j \gtreqless 0$ $(j=1, 2, \cdots, r)$ such that $x_1^2+x_2^2+\cdots+x_r^2=n$ is called, as customary, the number of representations of n as a sum of r squares; $N(0, r)=1$. Henceforth let n be an arbitrary integer >0, and m an odd integer >0. Denote by $\zeta_j(n)$, $j\geqq 0$, the sum of the jth powers of all the divisors of n, $\zeta_j(0)=1$ by convention; and by $\xi_j(n)$ the sum of the jth powers of all the divisors $\equiv 1 \bmod 4$ of n minus the like sum for the divisors $\equiv 3 \bmod 4$. Write $(-1\,|\,m) \equiv (-1)^{(m-1)/2}$. Then, either from the analysis of Bulyguin† or otherwise, it is known that the general structure of $N(m, 2r)$ is as follows:

$$N(m,4r) = a\zeta_{2r-1}(m) + F_r(m),$$

$$N(m,4r-2) = [b + c(-1\,|\,m)]\xi_{2r-2}(m) + G_r(m),$$

where a, b, c are numerical constants (independent of m) different from zero; $F_j(m)=G_j(m)=0$ $(j=1, 2)$, and $F_r(m)$, $G_r(m)$, when $r>2$, are sums of homogeneous polynomials in the integers $y_1, y_2, \cdots, y_{2t} \gtreqless 0$ such that

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* Presented to the Society, June 20, 1929.

† Bulletin de l'Académie de St. Petersburg, 1914, pp. 389–404.

$$y_1^2+y_2^2+\cdots+y_{2t}^2=m,$$

where $t<2r$ for $F_r(m)$ and $t<2r-1$ for $G_r(m)$. For example,

$$F_6(m) = p\sum_8(y_1^8-28y_1^6y_2^2+35y_1^4y_2^4)+q\sum_{16}(y_1^4-3y_1^2y_2^2),$$

where $\sum_8, \sum_{16}$ refer respectively to all sets of solutions of

$$y_1^2+y_2^2+\cdots+y_8^2=m, \qquad y_1^2+y_2^2+\cdots+y_{16}^2=m,$$

respectively, and p, q are numerical constants different from zero. Precisely similar theorems hold for $N(2^\alpha m, 2r)$, $\alpha>0$; the constants a, b are then to be replaced by $a+A2^{(2r-1)\alpha}$, $b+B2^{2(r-1)\alpha}$ respectively, where A, B are numerical constants different from zero; $F_r(2^\alpha m)$, $G_r(2^\alpha m)$ replace $F_r(m)$, $G_r(m)$ and are defined in the same way as the latter with $2^\alpha m$ in place of m throughout.

We shall call $a\zeta_{2r-1}(m)$, $[b+c(-1|m)]\,\xi_{2r-2}(m)$ the *simple parts* of $N(m, 4r)$, $N(m, 4r-2)$ respectively, and similarly for $N(2^\alpha m, 2r)$. The remaining parts will be called *compound.*

DEFINITION. *If $\alpha>0$, $\beta\geqq 0$ are constant integers, and if the compound part of $N(\alpha t+\beta, 2r)$ either vanishes or is identically zero for all integers $t\geqq 0$, $N(\alpha t+\beta, 2r)$ is said to be simple.*

For example, $N(t, 2r)$ is simple when and only when $r=1, 2, 3, 4$. In these cases the compound part vanishes identically; that is, it is absent. On the other hand, $N(2t, 12)$, $N(4t+3, 10)$ are simple because the compound parts of $N(n, 12)$, $N(n, 10)$, neither of which is identically zero, vanish when $n=2t, n=4t+3$ respectively. These six examples exhaust the known instances of simple $N(\alpha t+\beta, 2r)$.

We shall discuss the simplicity of $N(4t+\beta, 2r)$. Evidently the following enumeration of cases is exhaustive; it is made to fit the subsequent analysis:

$$N(4t+\beta,\ 4r-j), \qquad (\beta=0,1,2,3;\ \ j=0,2).$$

In other papers, cited presently, all cases except $N(4t+1, 4r)$, $N(4t+3, 4r)$ have been disposed of. We may therefore restrict the discussion to these.

3. *Theorems on* $N(m, 4r)$. We shall prove the following.

THEOREM 1. *If* $N(m, 4r)$, $m \equiv 1 \bmod 4$, *is simple,* $N(m, 4r) = 8r\zeta_{2r-1}(m)$.

THEOREM 2. *If* $N(m, 4r)$, $m \equiv 3 \bmod 4$, *is simple,*

$$N(m,4) = \frac{32r(2r-1)(4r-1)}{3(3^{2r-1}+1)}\zeta_{2r-1}(m).$$

THEOREM 3. *If* $m \equiv 1 \bmod 4$, *and* $N(m, 4r+4)$, $r>1$, *is simple, it is necessary (but not sufficient) that*

$$3\cdot 5^{2r+2} = 512r^4 + 768r^3 + 352r^2 + 168r + 225.$$

THEOREM 4. *If* $m \equiv 3 \bmod 4$, *and* $N(m, 4r+4)$, $r>2$, *is simple, it is necessary (but not sufficient) that*

$$105(7^{2r+1}+1) = (512r^4 - 256r^3 - 32r^2 + 856r + 210)(3^{2r+1}+1).$$

THEOREM 5. $N(m, 4r+4)$ *is simple for no* $r>1$.

THEOREM 6. $N(m, 4r)$ *is simple when and only when* $r = 1, 2$.

To state the next theorem, let k be the modulus of the elliptic function sn u. Then, as is well known or easily seen by Maclaurin's theorem, the coefficient of $(-1)^r u^{2r+1}/(2r+1)!$ $(r \geqq 0)$ in the expansion of sn u is

$$S_{2r+1}(k^2) \equiv \sum_{t=0}^{r} s_t(r)\, k^{2t},$$

where the $s_t(r)$ $(t=0, 1, \cdots, r)$ are integers >0. The expansion being unique, the integers $s_t(r)$ are uniquely defined. Let $\binom{p}{j}$ denote the coefficient of x^j in the expansion of $(1+x)^p$ if $p>0$ and $0 \leqq j \leqq p$, and zero otherwise. Then

$$\sigma_\mu(r) \equiv \sum_{t=0}^{\mu} 2^{2t} \binom{2r+1-2t}{\mu - t} s_t(r)$$

is an integer >0 whenever $r \geqq 0$, $\mu \geqq 0$ are integers.

THEOREM 7. *If* $m \equiv 1 \bmod 4$, *a necessary and sufficient condition that* $N(m, 4r+4)$, $r \geqq 0$, *shall be simple, is that a constant* λ *(independent of* m*) shall exist such that*

$$\sigma_{2\mu}(r) = \lambda\binom{4r+4}{4\mu+1}, \qquad (\mu = 0,1,\cdots,r);$$

if such a λ *exists,* $\lambda N(m, 4r) = 2\zeta_{2r+1}(m)$.

THEOREM 8. *If* $m \equiv 3 \bmod 4$, *a necessary and sufficient condition that* $N(m, 4r+4)$, $r \geqq 0$, *shall be simple, is that a constant* λ (*independent of* m) *shall exist such that*

$$\sigma_{2\mu+1}(r) = \lambda\binom{4r+4}{4\mu+3}, \qquad (\mu = 0,1,\cdots,r);$$

if such a λ *exists,* $N(m, 4r+4) = 2\zeta_{2r+1}(m)$.

4. *Proofs.* It is readily seen that Theorems 7, 8 imply all the rest. We shall require the following explicit values of $s_t(r)$ $(t = 0, 1, 2, 3)$ for all integers $r \geqq 0$, due to Hermite:*

$$s_0(r) = 1\,;\quad 2^4 s_1(r) = 3^{2r+1} - 8r - 3\,;$$

$$2^8 s_2(r) = 5^{2r+1} - (8r-4)3^{2r+1} + 32r^2 - 32r - 17\,;$$

$$2^{12} s_3(r) = 7^{2r+1} - (8r-12)5^{2r+1} + (32r^2 - 88r + 30)3^{2r+1} - \tfrac{1}{3}(256r^3 - 1056r^2 + 752r + 471).$$

Assume for a moment that Theorems 7, 8 are true. Then the value of λ is found by taking $\mu = 0$. Theorems 1, 2 result. Take $\mu = 0, 1$ in Theorems 7, 8; Theorems 3, 4 then follow by a simple but rather tedious calculation in an obvious manner. Theorem 5 follows from Theorems 3, 4 on remarking that for the appropriate (small) value of $r > 2$ the left-hand members of the equalities stated in Theorems 3, 4 become and remain greater than the respective right-hand members; the few values of r not thus rejected are thrown out by Theorems 7, 8. (The details of a similar calculation are given in the paper cited presently.) Theorem 6 is then immediate. It remains then only to prove Theorems 7, 8.

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* Oeuvres, vol. 3, p. 237; stated without indication of proof. As certain others of Hermite's values contain misprints, the above were calculated by the method of Gruder, Wiener Sitzungsberichte, vol. 126, IIa (1917), and found correct.

It was proved in a former paper* that

$$4\sum_m q^{m/2}\zeta_{2r+1}(m) = \sum_{t=0}^{r} s_t(r)\theta_2^{4t+2}\theta_3^{4r+2-4t}, \quad |q| < 1,$$

for all integers $r \geqq 0$, where $\sum_m$ refers to all odd integers $m > 0$, and θ_2, θ_3 are the usual elliptic theta constants. It was also proved that for each value of the integer $r \geqq 0$, the set of integer coefficients $s_t(r)$ $(t = 0, 1, \cdots, r)$ can not be replaced by any other set of $r+1$ integers >0. Apply to the above the identities

$$\theta_2^2(q^{1/2}) = 2\theta_2\theta_3, \qquad \theta_3^2(q^{1/2}) = \theta_2^2 + \theta_3^2$$

from the transformation of the second order. A short reduction gives

$$2\sum_m q^{m/4}\zeta_{2r+1}(m) = \sum_{\mu=0}^{2r+1} \sigma_\mu(r)\theta_2^{2\mu+1}\theta_3^{4r+3-2\mu}.$$

From the uniqueness of the set of coefficients $s_t(r)(t = 0, 1, \cdots, r)$ and the manner in which the last identity has been obtained, it follows by a simple contradiction that the $\sigma_\mu(r)$ $(\mu = 0, 1, \cdots, 2r+1)$ can not be replaced by any other set of integers >0. Replace q by q^4, and denote by $N(n, r, h)$ the number of representations of n as a sum of r squares, precisely h of which are odd. Since a sum of r squares, precisely h of which are odd, is $\equiv h \bmod 4$, we compare coefficients of like powers of q and separate cases modulo 4. Thus

$$m \equiv 1 \bmod 4\colon\ 2\zeta_{2r+1}(m) = \sum_{\mu=0}^{r} \sigma_{2\mu}(r)\frac{N(m, 4r+4, 4\mu+1)}{\binom{4r+4}{4\mu+3}};$$

$$m \equiv 3 \bmod 4\colon\ 2\zeta_{2r+1}(m) = \sum_{\mu=0}^{r} \sigma_{2\mu+1}(r)\frac{N(m, 4r+4, 4\mu+3)}{\binom{4r+4}{4\mu+3}}.$$

* To appear shortly in the Journal of the London Mathematical Society. The paper dealing with the remaining cases of $N(4t+\beta, 2r)$ has not yet been published.

By what precedes, the $\sigma_j(r)$ can not be replaced by any other sets of integers >0. But, obviously,

$$m \equiv 1 \bmod 4\colon\ N(m, 4r+4) = \sum_{\mu=0}^{r} N(m, 4r+4, 4\mu+1)\,;$$

$$m \equiv 3 \bmod 4\colon\ N(m, 4r+4) = \sum_{\mu=0}^{r} N(m, 4r+4, 4\mu+3).$$

Hence Theorems 7, 8 follow.

5. *Statement of a General Problem.* The theorem that $N(t, 2r)$ is simple when $r=1, 2, 3, 4$ is due to Eisenstein.* That $N(t, 2r)$ is simple only for those values of r does not seem to follow from his statements; the simplicity when and only when $r=1, 2, 3, 4$ can be proved readily by the methods of this paper. It follows from the theorems of the present paper and the others cited in §4 that $N(t, 2r)$, $N(4t+\beta, 2r)$ are simple only for the values of β, r stated in §1. Further, it is clear that the complete theory of the simplicity of $N(\alpha t+\beta, 2r)$ is known for $\alpha=1, 2, 4$. This suggests the following problem. *Find all values of r (if any) for which $N(\alpha t+\beta, 2r)$ is simple, where α is an arbitrary integer >0 and different from* 1, 2, 4. The obvious generalization in which the ζ, ξ functions are replaced by other given functions of divisors might also be considered.

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* Journal für Mathematik, vol. 35 (1847), p. 135.

NON-ISOLATED CRITICAL POINTS OF FUNCTIONS*

BY W. M. WHYBURN†

The isolated critical points of real functions of n independent real variables have been treated in an elegant manner by Marston Morse.‡ This treatment obtains definite relations between the numbers of critical points of $n+1$ types that appear in a bounded portion of the space of the independent variables. Morse requires his functions to have continuous third partial derivatives in the neighborhoods of the critical points and imposes conditions that are sufficient to insure the existence of at most a finite number of such points in the domain under consideration. In the present note§ we consider functions that have continuous first partial derivatives and may have an infinite number of critical points; or even continua of such points, in the given domain. As a special case of our results we obtain the *minimax principle* of Birkhoff in the modified form given by Bieberbach.‖

Let S denote the space of the n real variables $x_1, x_2, \cdots, x_n$, and let $(x) = (x_1, x_2, \cdots, x_n)$ denote a point in this space. Let R be a bounded,¶ open, and connected point set in S and let C, the boundary of R, be connected. Let the real function $f(x_1, \cdots, x_n)$ be single-valued and continuous

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† National Research Fellow in Mathematics.

‡ Transactions of this Society, vol. 27 (1925), pp. 345–396.

§ The author is indebted to Marston Morse for helpful suggestions concerning the material of this paper.

‖ *Differentialgleichungen*, Berlin, Springer, 1927, p. 140.

¶ The terms of classical point set theory are used in their usual sense. The distance between two points is given by a generalization of the ordinary distance formula in the plane. If K denotes a point set, then K' is used to denote K together with all of its limit points.

throughout R' and furthermore let $f_i(x_1, \cdots, x_n) = \partial f/\partial x_i$, $(i = 1, \cdots, n)$, be continuous in R'. On C, let $f(x) = K$, a constant, and at all points of $R = R' - C$ let $f(x)$ be less than K.

A point of R at which $f_1(x) = f_2(x) = \cdots = f_n(x) = 0$ is called a critical point of $f(x)$ and the value of $f(x)$ at such a point is called a critical value of $f(x)$. Let H denote the set of all critical points of $f(x)$ in R.

THEOREM 1. *For each point p of H there exists a connected subset $M(H, p)$ of H that contains p and has $f(x) = f(p)$ at each of its points and finally, such that $M(H, p)$ is not a proper subset of any other subset of H that is connected, contains p, and has $f(x) = f(p)$ at each of its points.*

PROOF. Let p be a point of H and let T denote the set of all points of H for which $f(x) = f(p)$. If T is connected, it is the desired set. Since T is bounded, we may find a hypercube C_1 of edge d containing T on its interior. Let $C_1, C_2, \cdots$ denote the set of hypercubes obtained by dividing C_1 into 2^n hypercubes $C_2, C_3, \cdots, C_{2^n+1}$ of edge $d/2$, then dividing each of these into hypercubes of side $d/4$ and continuing this subdivision indefinitely. We suppose these arranged so that for every i, C_i has its edge greater than or equal to the edge of C_{i+1}. Let C_{n_1} be the cube of lowest subscript that contains a point of T and is such that T can be expressed as the sum of two mutually exclusive closed sets P_1 and Q_1 such that P_1 contains p but no point of C'_{n_1}. Certainly such a cube exists since, by hypothesis, T is not connected and hence can be expressed as the sum of two mutually exclusive closed sets S_1 and S_2. There exists a positive number ϵ such that the sets S_1 and S_2 are at a distance apart greater than ϵ and hence any cube of the set $C_1, C_2, \cdots$, that has its edge of length less than $\epsilon/2$ and contains a point of the set which fails to contain p may be chosen. If P_1 is not connected, we let C_{n_2} be the hypercube of the set $C_1, C_2, \cdots$, of lowest subscript which contains points of P_1 and has the property that P_1 can be expressed as the sum of two mutually exclusive sets P_2 and Q_2 such that P_2 contains p but contains no point of C'_{n_2}.

This process may be continued indefinitely unless one of the P's, P_j, is connected. If P_j is connected, it is the desired set since $T-P_j=Q_1+Q_2+\cdots+Q_j$ and hence P_j cannot be a proper subset of a connected subset of T. If the process continues indefinitely, then the sequence $P_1, P_2, \cdots$, has at least one common point, namely, p. Let M denote the totality of such common points. M must be connected, since otherwise it would be the sum of two mutually exclusive closed sets (since M is closed). These sets would be at a distance apart greater than a definite positive constant and hence we could find a hypercube C_k of the set $C_1, \cdots$, containing points of that set which failed to contain p, but having no point of the other set in C_k'. This, however, is impossible since by our method of picking $C_{n_1}, \cdots, C_k$ would have been chosen at at least the kth stage. Since $T=M+\sum_{i=1}^{\infty} Q_i$, it follows that any point q of $T-M$ belongs to one of the Q's, Q_q, and since M belongs to P_q, it follows that q could not belong to a connected subset of T that contains M. Hence M is the desired set $M(H, p)$.

DEFINITION. *By the critical sets of $f(x)$ in R we mean the sets $M(H, p)$ of Theorem* 1.

THEOREM 2. *Let $K_1, K_2, \cdots, K_m$, where m is a positive integer, be subsets of H and let each K_i have the property that any two of its points can be joined in K_i by a rectifiable Jordan arc* of finite length. If $k=K_1'+\cdots+K_m'$ is connected, $f(x_1, \cdots, x_n)$ is constant on k.*

PROOF. A theorem on rectifiable curves in the plane and in three-space can readily be extended to hold in n-space.† This theorem states that if $x=f_1(t)$, $y=f_2(t)$, $z=f_3(t)$ is a rectifiable curve of finite length, then df_1/dt, df_2/dt, df_3/dt exist and are finite at every point of the t interval with the possible exception of a set of points of measure zero.

* The definition of such an arc is an obvious extension of the two-dimensional definition.

† Hobson, *Theory of Functions of a Real Variable*, Cambridge, 1921, vol. I, pp. 320 and 376.

Let p_1 and p_2 be any two points of K_i and let $x_j = f_j(t)$, $j = 1, \cdots, n$, where t denotes the length of arc from p_1, be a Jordan arc of the type whose existence is conditioned in the theorem. We have $df = \sum_{j=1}^{n} f_j dx_j$ *almost everywhere* on the t interval and since the arc is a subset of H, we have $df = 0$ *almost everywhere* on this arc. An application of Lebesgue integration yields $f(p_1) = f(p_2)$. Hence $f(x)$ is constant on K_i. From continuity, it is therefore constant on K_i' and from the connectedness of k it follows that $f(x)$ has a fixed constant value on this set of points.

DEFINITION. *A critical set $M(H, p)$ is said to be a minimal set of $f(x)$ provided it is true that for every point q of $M(H, p)$ there exists a positive number ϵ_q such that $f(x) \geqq f(q)$ for every point (x) whose distance from q is less than ϵ_q, the equality sign holding only for points of $M(H, p)$.*

THEOREM 3. *If $f(x_1, \cdots, x_n)$ has two distinct minimal sets M_1 and M_2 in R, there exists at least one critical point of $f(x_1, \cdots, x_n)$ in R that does not belong to $M_1 + M_2$.*

PROOF. Let the notation be chosen so that if m_1 and m are the values of $f(x)$ on M_1 and M_2, respectively, then $m \geqq m_1$. Let the notation $G(f \leqq h)$ mean the subset of R' on which $f \leqq h$, and let $G(f = h)$ mean the subset of R' on which $f = h$, while $G(f < h)$ means the subset of R' on which $f < h$. In $G(f \leqq m)$ there are two closed, connected, and mutually exclusive sets N_1 and N_2 such that N_1 contains M_1, $N_2 \equiv M_2$, and N_1 and N_2 are the maximal connected subsets* of $G(f \leqq m)$ that contain M_1 and M_2, respectively. That N_1 and N_2 are mutually exclusive follows when one notes that if they had a point q in common it would follow from the connectedness of N_1 and N_2 that this point would belong to M_2 and be a limit point of points of $N_1 - M_1$ (since M_1 is closed), that is, points

* By a maximal connected subset of a set T having certain properties we understand a subset which has the properties and is connected but which is not a proper subset of any other subset of T having these properties. The existence of such sets for each of our applications is established by a repetition of the arguments used in proving Theorem 1 with slight verbal modifications.

of $G(f<m)$. This is impossible since M_2 is a minimal set of $f(x)$.

Let h be any number such that $m \leqq h \leqq K$ and let N_{1h} and N_{2h} be the maximal connected subsets of $G(f \leqq h)$ that contain M_1 and M_2, respectively. For $h=m$, N_{1h} and N_{2h} have no point in common while for $h=K$, these two sets have a common point and are therefore identical. An application of the Dedekind-cut postulate yields a number k, $m \leqq k \leqq K$, such that k is either the greatest number for which N_{1k} and N_{2k} have no point in common or it is the least number for which $N_{1k} \equiv N_{2k}$. We show that k is the least number for which $N_{1k} \equiv N_{2k}$. Suppose k is the greatest number for which N_{1k} and N_{2k} are mutually exclusive. Noting that $R' = G(f \leqq K)$ is connected, we can and will choose a sequence $k_1 > k_2 > \cdots > k_i > \cdots$, so that $k < k_i < K$ and $\lim_{i \to \infty} k_i = k$. $N_i = N_{1k_i} = N_{2k_i}$, $i = 1, 2, \cdots$, is a sequence of closed, bounded, and connected sets having the property that for each i, N_{i+1} is a subset of N_i. A direct generalization of the usual proof in the plane, making use of the Heine-Borel property and the set of hypercubes introduced in the proof of Theorem 1, shows that the common part, $N_{1k} + N_{2k}$, of this sequence is closed and connected. This, however, contradicts the hypothesis that k was a number for which the sets N_{1k} and N_{2k} have no common point and shows that k is the least number for which $N_{1k} \equiv N_{2k}$.

The set of points $U = G(f=k)$ contains no point of $M_1 + M_2$. Assume that U contains no critical point of $f(x)$. We show that this assumption leads to a contradiction.

LEMMA 1. *U consists of at most a finite number of maximal connected subsets.*

PROOF. Assume the contrary; then $U = u_1 + u_2 + \cdots$, where no two of the u_i's have a point in common, each u_i is closed, and the u_i's are chosen so that the sum of two or more of them is not connected. There is at least one point p of U that is a limit point of a sequence of points $p_1, p_2, \cdots$ of U and no two of the p_i's belong to the same u_i. Since one of the partial derivatives, f_j, is different from zero at p and

f_j is continuous in R, we may find a neighborhood D of p throughout which f_j is different from zero. Furthermore, by the classical implicit function theorem* there exists a sub-neighborhood E of D such that all points of U that lie in E belong to a single continuous function $x_j = F(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$ and hence the subset of U that lies in E is connected. This contradicts the hypothesis that p is a limit point of points of infinitely many distinct u_i's. Hence $U = U_1 + U_2 + \cdots + U_r$, where r is a positive integer and U_i, $i = 1, \cdots, r$, is closed and connected.

LEMMA 2. *U_i, $i = 1, \cdots, r$, is a closed manifold† in the sense of analysis situs and its connectivity R_{n-1} is given by $R_{n-1} = 2$.*

PROOF. S. S. Cairns‡ has recently shown that U_i is a complex in the sense of analysis situs. This is accomplished by breaking up U_i into $(n-1)$-cells. An application of the implicit function theorem used in proving Lemma 1 shows that U_i has each of its $(n-2)$-cells incident with exactly two $(n-1)$-cells and hence U_i has no boundary. Hence U_i is a closed manifold and since it consists of a single $(n-1)$-circuit and this is, of course, non-bounding, we have $R_{n-1} - 1 = 1$, or $R_{n-1} = 2$.

PROOF OF THEOREM 3. By Alexander's theorem§ $S - U_i$, $(i = 1, \cdots, r)$, consists of two mutually exclusive connected point sets C_i and D_i. More generally, this theorem shows that $S - \sum_{n=1}^{j} U_i$ consists of $j+1$ mutually exclusive connected sets $G_1, \cdots, G_{j+1}$. Since $S - \sum_{i=1}^{r} U_i$ consists of two mutually exclusive sets one of which contains M_1 and the other contains M_2, it follows as an immediate consequence of a theorem by Alexandroff‖ that at least one of the sets U_i, call it U_1, is such that $S - U_1 = C_1 + D_1$, where C_1 contains

* See Hobson, loc. cit., p. 410.

† See, J. W. Alexander, Transactions of this Society, vol. 23 (1922), pp. 333 ff.

‡ Unpublished work. An indication of the procedure is given by Morse, loc. cit., p. 355.

§ Loc. cit., p. 343, Theorem Y.

‖ Comptes Rendus, vol. 183, p. 723, Theorem II_n.

M_1 and D_1 contains M_2. Since C is connected, it is a subset of C_1 or D_1. A reapplication of Alexandroff's theorem yields U_2 such that $S-(U_1+U_2)=G_1+G_2+G_3$, where G_1 contains M_1, G_2 contains M_2, and G_3 contains C. For definiteness let D_1 contain M_2 and C. We distinguish between two cases: Case I, C_2 contains M_1, M_2, and U_1 while D_2 of necessity contains C; Case II, C_2 contains M_1, U_1, and C while D_2 contains M_2. No other cases can arise.

CASE 1. Since C_2 contains U_1 and since $f(x)$ is continuous over C_2+U_2, it follows that $f(x)$ assumes its maximum for this domain in at least one point of the domain. If this maximum is greater than k then the point is an interior point and a consideration of the difference quotients of which the partial derivatives of $f(x)$ are the limits shows immediately that this point is a critical point and thereby contradicts our hypothesis. If the maximum has the value k, a similar consideration of the difference quotients shows that every point of U_1 is a critical point and likewise contradicts our hypothesis.

CASE 2. In this case C_1+U_1 and D_2+U_2 are closed and mutually exclusive. $N_{1k}=[C_1+U_1]+[D_2+U_2]+Q$, where Q cannot be vacuous since N_{1k} is connected. If A is a point of Q, we find U_3 so that $S-U_3=C_3+D_3$, where C_3 contains A and D_3 contains C. If either M_1 or M_2 belongs to C_3, a repetition of the argument of Case 1 proves the existence of a critical point of the type desired for Theorem 3. Hence both C_1 and D_2 are subsets of D_3 and hence neither they nor their boundaries can contain limit points of the subset of Q that belongs to C_3. A repetition of this argument a finite number of times (at most r times) together with the observation that $\sum_{i=3}^{r} U_i$ is closed shows that $[C_1+U_1]$ and $[D_2+U_2]$ contain no limit points of Q. Hence N_{1k} cannot be connected, contrary to the manner in which it was constructed. This contradiction yields Theorem 3.

COROLLARY 1.* *If* $f(x_1, \cdots, x_n)$ *has two distinct minimal*

* This corollary follows directly from the proof of Theorem 3.

sets M_1 and M_2 in R, there exists at least one critical point of $f(x_1, \cdots, x_n)$ in R that does not belong to a minimal set of $f(x_1, \cdots, x_n)$.

COROLLARY 2. *If $f(x_1, \cdots, x_n)$ is restricted so that each of its minimal sets contains a single point, Theorem 3 becomes Bieberbach's form of the minimax principle.**

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SYMMETRIC FUNCTIONS OF n-IC RESIDUES (mod p)†

BY T. A. PIERCE

If p be an odd prime, q is said to be an n-ic residue of p if the congruence $x^n \equiv q \pmod{p}$ has solutions; otherwise q is an n-ic non-residue of p. A necessary and sufficient condition that q be an n-ic residue of p is that

$$q^{(p-1)/\delta} \equiv 1, \pmod{p}, \tag{1}$$

where $\delta = \text{g.c.d. } (p-1, n)$. The number‡ of n-ic residues of a given prime p is $(p-1)/\delta$.

It is with the symmetric functions of these n-ic residues that this paper deals.

By means of (1) we readily prove that the product of two n-ic residues is an n-ic residue and that the product of an n-ic residue by an n-ic non-residue is an n-ic non-residue.

Put $(p-1)/\delta = r$ and let $q_1, q_2, \cdots, q_r$ be the set of all distinct n-ic residues of p. Then $q_iq_1, q_iq_2, \cdots, q_iq_r$ is the same set in different order, for the assumption that two members of this last set are congruent leads to the conclusion that two members of the first set are not distinct.

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* Loc. cit., p. 140.

† Presented to the Society, March 30, 1929.

‡ Dirichlet-Dedekind, *Zahlentheorie*, 4th ed., 1894, p. 74.

Let $S_\nu(q_1, q_2, \cdots, q_r)$ be an integral homogeneous symmetric function of $q_1, q_2, \cdots, q_r$ of total degree ν. Assume that ν is not a multiple of r. Take q_i to be an n-ic residue belonging to the exponent r; such a q_i exists since $\phi(r) > 1$. Then by virtue of the homogeneity

$$S_\nu(q_iq_1, q_iq_2, \cdots, q_iq_r) = q_i^\nu S_\nu(q_1, q_2, \cdots, q_r).$$

Moreover

$$S_\nu(q_iq_1, q_iq_2, \cdots, q_iq_r) \equiv S_\nu(q_1, q_2, \cdots, q_r), \qquad (\text{mod } p).$$

Hence

$$(q_i^\nu - 1)S_\nu(q_1, q_2, \cdots, q_r) \equiv 0, \qquad (\text{mod } p).$$

Since q_i belongs to the exponent r and ν is not a multiple of r, the first factor is not divisible by p; therefore the second factor is divisible by p. Hence we may state the following theorem.

THEOREM 1. *An integral symmetric function of dimension ν of the n-ic residues of the odd prime p is divisible by p if ν is not a multiple of $(p-1)/\delta$.*

Any symmetric function of the rth powers of the q's may be evaluated by (1). Thus

$$\sum q_i^r \equiv 1 + 1 + \cdots = (p-1)/\delta \equiv -\delta^{p-2}, \qquad (\text{mod } p).$$

A general theory of symmetric functions whose dimensions are multiples of $(p-1)/\delta = r$ is a desideratum here as it is in the case of ordinary residues when the dimensions are multiples of $p-1$. The two simplest cases will be treated here.

First, since

$$x^{(p-1)/\delta} - 1 \equiv 0, \quad (\text{mod } p),$$

is satisfied by the $(p-1)/\delta$ distinct n-ic residues this congruence has its full complement of real roots and hence

$$q_1q_2 \cdots q_r \equiv \pm 1, \quad (\text{mod } p), \tag{2}$$

according as r is odd or even.

Next assume that $(p-1)/\delta$ is even and greater than 2 and let q belong to the exponent $r/2>1$. Such a q exists since $\phi(r/2)>1$. Then $(q^{r/2})^{\delta}\equiv q^{(p-1)/2}\equiv 1 \pmod{p}$ and q is a quadratic residue of p. Thus the congruence

$$x^2 - q \equiv 0, \pmod{p} \tag{3}$$

has a solution x. This solution x must be an n-ic residue, for from $x^2\equiv q \pmod{p}$ we have $x^r\equiv q^{r/2}\equiv 1 \pmod{p}$, and x satisfies (1).

The fact that (3) has a solution among the set of q's may be expressed

$$(q - q_1^2)(q - q_2^2)\cdots(q - q_r^2) \equiv 0, \pmod{p}.$$

Expanding the left member and omitting intermediate terms whose coefficients are multiples of p, by the theorem above we have

$$q^r + (-1)^{r/2}S_{r/2}(q_1^2, \cdots, q_r^2)q^{r/2} + (q_1\cdots q_r)^2 \equiv 0, \pmod{p}, \tag{4}$$

where $S_{r/2}(q_1^2, \cdots, q_r^2)$ is the sum of the products of $q_1^2, \cdots, q_r^2$ taken $r/2$ at a time. The dimension of $S_{r/2}$ is thus $(p-1)/\delta$. The last term of (4) is unity by (2) and since q belonged to the exponent $r/2$ we have

$$S_{r/2}(q_1^2, \cdots, q_r^2) \equiv (-1)^{r/2+1}\,2, \pmod{p}.$$

This last result corresponds to one due to Glaisher* for ordinary residues.

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* Glaisher, *Congruences relating to sums of products*, etc., Quarterly Journal of Mathematics, vol. 31 (1900), p. 34.

SINGULAR MANIFOLDS AMONG THOSE OF AN ANALYTIC FAMILY*

BY O. D. KELLOGG

1. *Exceptional Occurrence of Singular Manifolds.* This note is concerned with the following theorem.

Let R denote a closed region, consisting of an open continuum of the space of the n complex variables $z_1, z_2, \cdots, z_n$, together with its boundary points. Let the functions $F_i(z_1, z_2, \cdots, z_n)$, $i=1, 2, \cdots, m$, $m \leqq n$, be analytic at all points of R. Let M_k denote the matrix

$$(M_k) \qquad \left\| \frac{\partial F_i}{\partial z_j} \right\|, \qquad \begin{matrix} (i = 1,2, \cdots, k), \\ (j = 1,2, \cdots, n). \end{matrix}$$

We assume that M_m is of rank m at some point of R.

Consider the manifold defined by the equations

$$(A) \qquad F_1(z_1,z_2, \cdots, z_n) = c_1, \cdots, F_m(z_1,z_2, \cdots, z_n) = c_m,$$

where $c_1, c_2, \cdots, c_m$ are complex constants.

For all but a finite number of values of c_1 the manifold defined by the first equation (A) contains no points in R at which the rank of M_1 is less than 1.

If $c_1, c_2, \cdots, c_k$ have been chosen so that the matrix M_k is of rank k at every point in R on the manifold defined by the first k equations (A), then for all but a finite number of values of c_{k+1} the manifold defined by the first $k+1$ equations (A) contains no points in R at which the rank of the matrix M_{k+1} is less then $k+1$.

Thus, if $c_1, c_2, \cdots, c_m$ are chosen in order, each avoiding a certain finite set of values, the manifold defined by the equations (A) will have no singular points in R.†

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* Presented to the Society, March 29, 1929.

† As far as I know, a proof of this general theorem has not been published. Birkhoff and I gave it for the case in which the functions F_i are polynomials (Transactions of this Society, vol. 23 (1922), pp. 97–98), and

By a singular point of a manifold, we mean a point at which a unique tangent linear manifold of the same order does not exist. A sufficient condition that a point of the manifold defined by the equations (A) be not singular is that the rank of the matrix M_m is m at that point. That the condition is not necessary may be seen by the example

$$z_1^2 z_2^2 - z_3^3 = 0, \qquad z_1^2 z_2^2 + z_3^3 = 0.$$

The manifold defined by these equations consists of the two intersecting straight lines $z_1 = z_3 = 0$ and $z_2 = z_3 = 0$. Its only singular point is at the origin, yet the rank of M_2 is less than 2 at every point of the manifold.

It is not stated in the theorem that the values of the c_i characterizing singular manifolds are finite in number. In general, the value of c_{k+1} to be avoided, if the resulting manifold is to be non-singular, is a function of the c_i previously chosen. Thus in the case

$$z_1 - z_2 z_3 = c_1, \qquad z_1 = c_2,$$

the value of c_2 to be avoided is c_1. What is the case, is that among the ∞^{2m} members of the family defined by (A), there are at most ∞^{2m-2} singular manifolds, a precise interpretation of this statement being given in the theorem.

2. *Needed Results in the Theory of Analytic Functions.* We attach our considerations to the study of simultaneous equations in the second volume of Osgood's *Lehrbuch der Funktionentheorie*,* where we find a theorem, of which the part we need may be stated as follows:

Given a system of simultaneous equations

$$G_1(z_1, z_2, \cdots, z_n) = 0, \cdots, G_l(z_1, z_2, \cdots, z_n) = 0,$$

where $G_k(z_1, z_2, \cdots, z_n)$, $k = 1, 2, \cdots, l$, *is analytic at the point* $(a) = (a_1, a_2, \cdots, a_n)$ *and vanishes there, but does not*

Morse gives a proof for the case $m = 1$ in a paper in the April number of the American Journal of Mathematics, vol. 51 (1929). The theorem is apparently useful; in addition to its role in the two papers just cited it has served me at several points in a forthcoming book on Potential Theory.

* Leipzig, 1st ed., 1924, 2d ed., 1929, Chap. II, §17.

vanish identically. The solutions of this system in the neighborhood of (a) *consist either of the point* (a) *alone, or of one or more manifolds* g. The context makes it clear that the number of these manifolds is *finite.*

The essential property of the manifolds g for our purposes is that *any two points of any one of them can be connected by a curve* γ, *lying in the manifold, and analytic except at a finite number of points.*

We begin, in establishing this property, with a pseudo-algebraic manifold $\mathfrak{G}$ defined by the irreducible equation

$$(1)\quad H(w,u_1,u_2,\cdots,u_\rho) = w^\mu + E_1 w^{\mu-1} + \cdots + E_\mu = 0$$

and the inequalities $|u_i| < h$, where h is a suitably restricted positive number. The coefficients E_j are power series in the variables u_i convergent for $|u_i| < h'$, $h' > h$, vanishing at the origin $O(u_1 = u_2 = \cdots = u_\rho = 0)$.

If $P'(w', u_1', u_2', \cdots, u_\rho')$ is any point of $\mathfrak{G}$, and if in the equation (1) we set $u_1 = u_1' t$, $u_2 = u_2' t$, $\cdots$, $u_\rho = u_\rho' t$, the resulting equation defines w as an algebroid function of t. If the latter equation is reducible, we select the irreducible factor of the left hand member which vanishes for $w = w'$ when $t = 1$, and equate it to 0. The discriminant of the resulting equation vanishes at most a finite number of times on the real interval (0, 1), and hence from the branches of the function w of t there can be selected a (complex) single-valued function $w = f(t)$, which is continuous on the closed interval (0, 1), analytic except at a finite number of points, and which reduces to 0 for $t = 0$, and to w' for $t = 1$. Then the curve

$$(\gamma')\qquad w = f(t),\quad u_1 = u_1' t, \cdots, \quad u_\rho = u_\rho' t, \quad 0 \leqq t \leqq 1,$$

lies in $\mathfrak{G}$, connects P' with O, and is analytic except at a finite number of points. By means of a curve consisting of two such parts, any two points P' and P'' of $\mathfrak{G}$ can be connected.

We now consider a set of functions $w_1, w_2, \cdots, w_\sigma$ of

$u_1, u_2, \cdots, u_\rho$, belonging to the pseudo-algebraic manifold $\mathfrak{G}$, that is, having the following properties: they are one-valued and continuous on $\mathfrak{G}$, and analytic at every point of $\mathfrak{G}$ at which the discriminant of the equation (1) does not vanish; the function $\alpha_1 w_1 + \cdots + \alpha_\sigma w_\sigma$ of the u_i has, for non-specialized values of the parameters $\alpha_1, \cdots, \alpha_\sigma$, branches, corresponding to the branches of w, no two of which are identically equal. Then the set of points of the space of $(w_1, w_2, \cdots, w_\sigma, u_1, u_2, \cdots, u_\rho)$ which correspond to the points $(w, u_1, \cdots, u_\rho)$ of $\mathfrak{G}$ constitutes a manifold g. The most general manifold g is either one defined in this way, or else one derived from it by a non-singular linear transformation of the variables. As such a transformation does not change the essential character of the curve γ, we may confine ourselves to the manifolds g as first defined.

The functions $w_1, w_2, \cdots, w_\sigma$ are pseudo-algebraic, and as they are one-valued and continuous on $\mathfrak{G}$, they assume at O unique values $w_{10}, w_{20}, \cdots, w_{\sigma 0}$. If $p'(w_1', w_2', \cdots, w_\sigma', u_1', u_2', \cdots, u_\rho')$ is any point of g, there corresponds to it a unique point P' of $\mathfrak{G}$, which is connected with O by a curve γ'. Just as the function $w = f(t)$ was determined, we may determine from among the branches of w_1, a single-valued function $w_1 = f_1(t)$, continuous on the interval $(0, 1)$, analytic, except at a finite number of points, and reducing to w_{10} for $t = 0$ and to w_1' for $t = 1$. When this and similar functions have been determined for $w_1, \cdots, w_\sigma$, we shall have a curve

$$(\gamma)\quad w_1 = f_1(t), \cdots, w_\sigma = f_\sigma(t), u_1 = u_1' t, \cdots, u_\rho = u_\rho' t, \qquad 0 \leqq t \leqq 1,$$

from two of which can be constructed a curve of the required character, connecting any two given points of g.

3. *Proof of the Theorem.* Let (a) be any point of R. Then in the neighborhood of (a), the solutions of the equations

$$(2)\qquad \frac{\partial F_1}{\partial z_1} = 0, \quad \frac{\partial F_1}{\partial z_2} = 0, \cdots, \frac{\partial F_1}{\partial z_n} = 0,$$

(not all of which are identically satisfied, since M_m is of rank m at some point of R), if there are any, consist, by Osgood's theorem, either of the point (a) alone, or of a finite number of manifolds g. Along an analytic arc of a curve γ joining any two points of such a manifold, $dF_1 = 0$. It follows that F_1 is constant on each such manifold, so that the number of values of F_1 corresponding to solutions of the equations (2) in the neighborhood of (a), is finite. By the Heine-Borel theorem, the same is true for the whole region R. Hence for all but a finite number of values of c_1 the manifold defined by the first equation (A) contains no points in R at which M_1 is of rank less than 1. The first part of the theorem is thus established.

Suppose now that $c_1, c_2, \cdots, c_k$ have been chosen so that at every point in R of the manifold defined by the first k equations (A), the matrix M_k is of rank k. We consider the simultaneous system of equations consisting of the two sets

$$(3)\quad F_1(z_1, z_2, \cdots, z_n) - c_1 = 0, \cdots, F_k(z_1, z_2, \cdots, z_n) - c_k = 0,$$

and

$$(4)\quad D_1(z_1, z_2, \cdots, z_n) = 0, \cdots, D_r(z_1, z_2, \cdots, z_n) = 0,$$

where $D_1, \cdots, D_r$ are all the determinants of order $k+1$ formed from the matrix M_{k+1}, except such as may be identically 0. Not all vanish identically, since, by hypothesis, M_m is of rank m at some point of R.

If this system has any solutions in the neighborhood of a point (a) of R, they consist, by Osgood's theorem, of the point (a) alone, or of a finite number of manifolds g. Let g denote such a manifold. On it M_k and M_{k+1} are everywhere of rank k. Hence from the equations $dF_1 = dF_2 = \cdots = dF_k = 0$, which follow from (3), we infer that $dF_{k+1} = 0$ at all points of the curve γ connecting any two given points of g. By reasoning previously employed, it follows that for all but a finite number of values of c_{k+1}, the matrix M_{k+1} is of rank $k+1$ at all points of R on the manifold defined by the first $k+1$ equations (A). This proves the second part of the theorem.

The truth of the final statement of the theorem emerges when to k are assigned successively the values $1, 2, \cdots, m-1$, and when it is recalled that a sufficient condition that the manifold defined by the equations (A) has no singular points in R is that the matrix M_m is of rank m at all points of the manifold in R.

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ON THE FOUNDATIONS OF GENERAL INFINITESIMAL GEOMETRY*

BY HERMANN WEYL

In connection with a seminar on infinitesimal geometry in Princeton, in which I took part, it seemed desirable to clarify the relations between the work of the Princeton school and that of Cartan.

With a group $\mathfrak{G}$ of transformations in m variables ξ is associated, in accordance with Klein's Erlanger Program, a homogeneous or plane space $\mathfrak{R}$ of the kind $\mathfrak{G}$; a point of $\mathfrak{R}$ is represented by a set of values of the "coordinates" ξ^α and figures which go into each other on subjecting the coordinates to a transformation of $\mathfrak{G}$ are to be considered as fully equivalent. The transformations of $\mathfrak{G}$ give at the same time the transition between two allowable "normal" coordinate systems in $\mathfrak{R}$. If we have two spaces $\mathfrak{R}$, $\mathfrak{R}'$ of the kind $\mathfrak{G}$ and set up a definite normal coordinate system in each of them, then such a transformation can be interpreted as an isomorphic representation of $\mathfrak{R}$ on $\mathfrak{R}'$. $\mathfrak{G}$ is assumed to be transitive.

Cartan† developed a general scheme of infinitesimal geometry in which Klein's notions were applied to the tangent plane and not to the n-dimensional manifold M itself. The

* Presented to the Society, June 21, 1929

† E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée*, Annales de l'École Normal Supérieure, vol. 40 (1923), pp. 325–412; in particular, p. 383, etc.

object of his work can be briefly described as follows. There is associated with each point P of M an m-dimensional plane space of the kind $\mathfrak{G}$. To the transition from P to a neighboring point P', that is, to the line element PP', corresponds an isomorphic representation of T_P on $T_{P'}$, the "displacement" $T_P \rightarrow T_{P'}$ or briefly $\overline{PP'}$. On this basis the introduction of a general concept of curvature is possible: if we displace the tangent plane T_P in P along a curve L on M which leads back to P the tangent plane returns in a new position or orientation. The final position is obtained from the original by a certain isomorphic representation of T_P onto itself, and this we call the "curvature along L."

For the purpose of analytic formulation we refer M to coordinates x^i and introduce a normal coordinate system ξ^α in each tangent plane T_P. Let the components of $\overline{PP'}$ be dx^i. We assume that the ξ^α of all the T_P can be so chosen that the displacement PP' is an infinitesimal isomorphic representation of the same order of magnitude as the dx^i. A radical specialization is introduced by the further assumption that this displacement depends linearly on $\overline{PP'}$, i.e. that the consecutive application of the displacements PP' and $P'P''$ shall yield the displacement PP''. The displacement from P to all neighboring points P' is then expressed by the formula

$$d\xi^\alpha = -\sum_i u_i^\alpha(\xi) dx^i. \tag{1}$$

The curvature along the infinitesimal parallelogram formed by line elements dx and δx, which consequently has as components

$$(\Delta x)^{ik} = dx^i \delta x^k - \delta x^i dx^k,$$

is given by

$$\Delta\xi^\alpha = R_{ik}^{\alpha}(\xi) dx^i \delta x^k = \tfrac{1}{2} R_{ik}^{\alpha}(\xi)(\Delta x)^{ik},$$

$$R_{ik}^{\alpha} = \left(\frac{\partial u_k^\alpha}{\partial x^i} - \frac{\partial u_i^\alpha}{\partial x^k}\right) + \left(\frac{\partial u_k^\alpha}{\partial \xi^\rho} u_i^\rho - \frac{\partial u_i^\alpha}{\partial \xi^\rho} u_k^\rho\right).$$

In the foregoing the tangent plane is not tied up with the manifold; in order to justify this designation and hold to the

idea of a tangent plane we must now imbed it into the manifold. The first step in this process consists in taking a definite point O of T_P as center which shall, by definition, cover the point P on M (imbedment of 0th order). This leads to a restriction in the choice of a normal coordinate system ξ on T_P; because of the transitivity of $\mathfrak{G}$ it can and shall be so chosen that the coordinates ξ vanish in the center. The group $\mathfrak{G}$ is restricted to the subgroup $\mathfrak{G}_0$ of all representations of $\mathfrak{G}$ which leave the center O invariant. On displacing the tangent plane along a closed curve L, O goes over into a point O^* whose deviation from O characterizes the "torsion along L." $R_{ik}^{\alpha}(0)=0$ is the necessary and sufficient condition that M be without torsion.

The idea of tangent plane further requires that the line elements of T_P issuing from O shall "coincide" with the line elements of M issuing from P; this correspondence must be a one-to-one affine representation. But having already required imbedment of 0th order the method of accomplishing this imbedment of 1st order is fixed. The center O' of $T_{P'}$ arises by the displacement PP' from a definite point O_1 of T_P, and we let the line element $\overline{OO_1}$ on T_P correspond to the line element $\overline{PP'}$ on M. For purposes of calculation it is, however, more convenient to consider the line element $\overline{O_1'O'}$ on $T_{P'}$ which arises from $\overline{OO_1}$ by the displacement PP'. The ξ-coordinates of O_1' on $T_{P'}$ are $d\xi^{\alpha}=-u_i{}^{\alpha}(0)dx^i$, and consequently

$$d\xi^{\alpha} = u_i{}^{\alpha}(0)dx^i \tag{2}$$

are the components of the line element $\overline{O_1'O_1}'$ on $T_{P'}$ or $\overline{OO_1}$ on T_P. The condition that this linear relation between (dx) and $(d\xi)$ be one-to-one reciprocal involves two requirements: (1) the dimensionality m of the tangent plane (which was until now arbitrary) must be the same as the dimensionality n of the manifold M, and (2) the determinant $|u_i{}^{\alpha}(0)|\neq 0$. If $\mathfrak{G}$ contains the affine group, and we shall henceforth assume that it does, the coordinate system ξ^{α} on T_P can be further adapted to the given coordinate system x^i on M in such a

way that corresponding line elements shall have the same components: $u_i^\alpha(0) = \delta_i^\alpha$.

If $\mathfrak{G}$ were the affine group the previous requirements would fully specify the normal coordinate system ξ^α on T_P in its dependence on the coordinates x^i chosen on M; but this is not the case if $\mathfrak{G}$ is a more extensive group. That is, the "tangent plane" T_P is not as yet uniquely determined by the nature of M, and so long as this is not accomplished we can not say that Cartan's theory deals only with the manifold M. Conversely, the tangent plane in P in the ordinary sense, that is, the linear manifold of line elements in P, is a centered affine space; its group $\mathfrak{G}$ is not a matter of convention. This has always appeared to me to be a deficiency of the theory; I consider that above all, the infinitesimal-geometric researches of Eisenhart, Veblen, T. Y. Thomas, and others in Princeton† have remedied this blemish for projective and conformal geometry.

The connection between ξ and x, although not yet uniquely determined by the previous postulates, allows us to conclude the following: In the development

$$R_{ik}^{\alpha}(\xi) = R_{ik}^{\alpha}(0) + R_{\beta ik}^{\alpha}\xi^\beta + \cdots,$$

the quantities $R_{ik}^{\alpha}(0)$ and $R_{\beta ik}^{\alpha}$ in the point P are determined by the coordinates x^i alone and transform on transformation of coordinates as tensors of order 3 and 4, of the kind indicated by the position of the indices. It is therefore an invariantive restriction to require that our manifold be such that (1) it is without torsion and (2) $\sum_\alpha R_{\alpha ik}^{\alpha}$ vanish; we call such a manifold "special."

Let $\mathfrak{G}$ be the projective group. We must then proceed to imbedment of second order in order that T_P be completely

† L. P. Eisenhart, *Non-Riemannian Geometry*, 1927. O. Veblen, *Projective tensors and connections*, Proceedings of the National Academy, vol. 14 (1928), p. 154; *Conformal tensors and connections*, ibid., vol. 14 (1928), p. 735. T. Y. Thomas, *A projective theory of affinely connected manifolds*, Mathematische Zeitschrift, vol. 25 (1926), p. 723.

determined by M. We consider, as an analog, the contact of two surfaces

$$y = f(x^1, \cdots, x^n), \quad y = f'(x^1, \cdots, x^n)$$

in $(n+1)$-dimensional space. Let

$$f' - f = a + a_i x^i + \tfrac{1}{2} a_{ik} x^i x^k + \cdots$$

in the neighborhood of the origin. There is contact of 0th order (intersection) if $a = 0$, of 1st order (tangency) if in addition the linear terms are not present, $a_i = 0$, and finally contact of second order (osculation) if further all a_{ik} vanish. I refer to the two surfaces as semi-osculating if in addition to a and a_i the sum $\sum_{(i)} a_{ii}$, the spur of the quadratic terms, vanishes. Analogously we demand that T_P not only be tangent to the manifold but further that it be semi-osculating. The name tangent plane is then misleading, but we shall use it instead of the more correct "projective semi-osculating plane" for the sake of brevity. The exact definition is as follows. Given an infinitesimal volume element V in P, say a parallelepipedon obtained from line elements in P which shall be infinitely small in comparison with $\overline{PP'} = (dx^i)$. Let V' be the "same" volume element in P', that is, it shall be generated from the line elements with the same components in P'; naturally this construction is dependent on the particular coordinate system employed. Because of the imbedment of order V coincides with a volume element V in O on T_P and V' with one such in O' on $T_{P'}$; this latter is obtained from an element V_1 in O_1 on T_P by the displacement PP'. We now require that V and V_1 have the same volume, measured in the coordinates ξ on T_P.

It is again more convenient for the calculation to take V and V_1 over into V_1', V' on $T_{P'}$ by means of the displacement PP'. The isomorphic representation which carries V' into V_1' is by definition simply (1)—taken for ξ's which are infinitely small compared to the dx^i. Consequently we write

$$u_i^{\alpha}(\xi) = \delta_i^{\alpha} + \Gamma_{\beta i}^{\alpha} \xi^{\beta},$$

and on introducing

$$\overset{\alpha}{\Gamma}_{\beta i} dx^i = d\overset{\alpha}{\gamma}_{\beta},$$

we find

$$\log (V'/V_1') = \log (V_1/V) = \sum_{\rho} d\gamma_{\rho}^{\rho}.$$

Our condition is that this trace shall vanish, and we assert that it can be fulfilled by appropriate choice of the projective coordinates ξ^α on T_P. The previous requirements determine the ξ except for a projective transformation of the type

$$\bar{\xi}^i = \frac{\xi^i}{1 + \sum \alpha_k \xi^k},$$

which leaves the center unaltered and is the identity to terms of first order in the neighborhood of the center. The ratio of the measures of volume elements V and $\overline{V}$ in ξ and $\bar{\xi}$, situated at (ξ^α), is given by the functional determinant

$$\left| \frac{\partial \bar{\xi}^i}{\partial \xi^j} \right| = (\bar{\xi}\xi).$$

For infinitely small ξ this determinant is

$$(\bar{\xi}\xi) = 1 - (n+1) \sum \alpha_k \xi^k.$$

Our volume V is at the point $\xi = 0$, V_1 at $\xi^\alpha = dx^\alpha$, and consequently

$$\log (\overline{V}_1/\overline{V}) = \log (V_1/V) - (n+1) \sum \alpha_i dx^i$$

from which we see that in order that $\overline{V}_1 = \overline{V}$ the α_i can be chosen in one and only one way; $\alpha_i = \Gamma_{\rho i}^{\rho}$.

The projective coordinate system ξ on T_P is now completely specified by the coordinates x^i on M. If we refer M to a new coordinate system $\bar{x}^i$ we shall have a new projective coordinate system ξ on T_P. The projective transformation $\xi \to \bar{\xi}$ can be described by the facts (1) that it agrees in the neighborhood of the origin with the transformation $x \to \bar{x}$ in terms of first order about P and (2) that the functional determinant $(\bar{\xi}\xi)$ in 0 agrees with $(\bar{x}x)$ in terms of 1st as well as

0th, order. H. P. Robertson* pointed out in a short note that this relation is the decisive point in Veblen's transformation theory of projective space. What we do here, however, is not simply connect the *transformations* of the ξ with those of x but rather we associate a projective ξ *coordinate system* on T_P with an individual x system on M. This possibility arises from the fact that we begin with the projective connection and with its aid tie up the ξ with the x, i.e. accomplish the complete imbedment of the tangent plane. But on the other hand the transformation of the ξ is determined by the transformation of the x, as described above, without taking the given projective connection into account. Veblen's procedure corresponds to this method: this relation between the two transformations is first obtained and the corresponding invariant theory of possible projective connections then developed.

The introduction of $n+1$ homogeneous projective coordinates η by means of the equation $\xi^i = \eta^i/\eta^0$ is for the present purely a matter of convenience. The formulas for the displacement expressed in terms of them have the form

$$d\eta^\alpha = -\, d\gamma_\beta^\alpha \eta^\beta, \quad d\gamma_\beta^\alpha = \Gamma_{\beta i}^\alpha dx^i. \tag{3}$$

(From now on Greek indices shall run from 0 to n and Latin from 1 to n.) Since only the ratios of the η are to be considered we can and shall introduce the normalizing condition $d\gamma_0^0 = 0$. We then have $\Gamma_{\rho i}^\rho = \delta_i^\alpha$ and $\Gamma_{\rho i}^\rho = 0$. In the case of a special manifold (see above) we have furthermore the symmetry condition $\Gamma_{ki}^\alpha = \Gamma_{ik}^\alpha$ and the Γ with only Latin indices determine the remaining components. This leads to the theorem: *If we allow only special manifolds the projective connection is determined uniquely by the geodesics.*†

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* H. P. Robertson, *Note on projective coordinates*, Proceedings of the National Academy, vol. 14 (1928), p. 153.

† See J. A. Schouten, Rendiconti di Palermo, vol. 50 (1926), pp. 142–169, in particular p. 158. I do not find that this work, which is closely related to our process, gives a clear account of the fact that the coordinates ξ must be tied up with the x as described above.

For the complete development of projective infinitesimal geometry we must, in my estimation, add three independent ideas to Cartan's scheme; the first and most important of these consists in connecting the coordinates ξ with the x by the requirement of "semi-osculation," the second answers the question to what extent the geodesics determine the projective connection by the invariantive "specialization." The third idea, which I now consider, is due to T. Y. Thomas: it is possible to give the variables η themselves, and not only their ratios, a geometrical interpretation. The analytic expression in coordinates η for any projective mapping which leaves O invariant,

$$\bar{\eta}^i = \sum_k a_k{}^i \eta^k, \quad \bar{\eta}^0 = \eta^0 + \sum_k \alpha_k \eta^k, \tag{4}$$

can be so normalized that the coefficient $a_0{}^0 = 1$. This normalization is useful because it is not destroyed by composition: the group $\mathfrak{G}_0$ is replaced by the isomorphic group of affine transformations in $n+1$ dimensions of the form (4). If the transition $\eta \to \bar{\eta}$ on T_P corresponds to the transition $x \to \bar{x}$ on M the transformation (4) can further be described as agreeing with the transformation of the differentials $dx^0, dx^1, \cdots, dx^n$ in P when the additional coordinate x^0 transforms in accordance with the law

$$\bar{x}^0 = x^0 - \frac{1}{n+1} \log (\bar{x}x). \tag{5}$$

We now have an $(n+1)$-dimensional manifold M^* instead of M; each point of M is replaced by a filament of M^* along which $x^1, \cdots, x^n$ are constant and only x^0 varies. By (5) the distance between points on the same filament, i.e. the difference of their x^0 coordinates, as well as the filaments themselves, have an invariantive significance. An $(n+1)$-dimensional affine tangent plane, the domain of the variables η, is associated with each point of M^*. The transformation of the η, which is related to a transformation of the x on M, is the same for all points on the same filament. Extending the Γ by adding $\Gamma_{\beta 0}^{\alpha} = \delta_{\beta}^{\alpha}$, this means that the ratios $\eta^0 : \eta^1 : \cdots : \eta^n$

of a point on the tangent plane are unaltered by displacement along the filament. The n-dimensional projective displacement on M^* defined by

$$d\eta^\alpha = -d\gamma_\beta^\alpha \eta^\beta, \quad d\gamma_\beta^\alpha = \Gamma_{\beta\rho}^{\alpha} dx^\xi, \tag{6}$$

is consequently invariantively determined by the projective displacement on M.

We must next ask if this is also true for the $(n+1)$-dimensional affine displacement expressed by the same formulas; the answer is affirmative, because our normalization is so chosen that $\Gamma_{\beta\rho}^{\alpha}$ is symmetric in β and ρ. To show this, let $\overline{\Gamma}_{\beta\rho}^{\alpha}$ be the projective connection of M evaluated in a new coordinate system $\bar{x}^i$ in the manner described above, and let $\tilde{\Gamma}_{\beta\rho}^{\alpha}$ be the components of the same affine connection on M^* expressed in terms of the coordinates $\bar{x}^i$ in the manner indicated by (6) in terms of the x^i. Then the corresponding equations (6) characterize the same projective connection, that is, $\overline{\Gamma}_{\beta\rho}^{\alpha}$ and $\tilde{\Gamma}_{\beta\rho}^{\alpha}$ can differ only by a term of the form $\delta_\rho^\alpha \lambda_\rho$. Now $\overline{\Gamma}$ as well as $\tilde{\Gamma}$ must be symmetric in the two lower indices and consequently

$$\delta_\beta^\alpha \lambda_\rho = \delta_\rho^\alpha \lambda_\beta,$$

from which we obtain by the contraction $\alpha = \rho$

$$\lambda_\beta = (n+1)\lambda_\beta, \quad \lambda_\beta = 0.$$

All that we have said here can be taken over mutatis mutandis to the conformal geometry. Here the equation

$$g_{ik} dx^i dx^k = 0 \tag{7}$$

on M is fundamental. If g_{ik} be the values of the coefficients in P then the conformal geometry on the homogeneous "tangent" plane T_P is described by the equation $g_{ik} d\xi^i d\xi^k = 0$ with constant coefficients g_{ik}; the group $\mathfrak{G}$ consists of all transformations of the ξ which leave this equation invariant and consequently depends on the point P in question. The conformal displacement PP' must be a transformation which takes the equation $g_{ik} d\xi^i d\xi^k = 0$ over into $g'_{ik} d\xi'^i d\xi'^k = 0$. Consequently this does not agree literally with the scheme de-

veloped above—furthermore $\mathfrak{G}$ does not contain the affine group (only the orthogonal one). We must expressly require that the coincidence of the line elements $\overline{PP'}$ on M and $\overline{OO_1'}$ on T_P is a conformal, and not merely an affine, relation. (1) The requirement of semi-osculation is also here sufficient to tie up the conformal coordinate system ξ uniquely with the x. (2) If the manifold is special the conformal connection is uniquely determined by equation (7). (3) The transition to homogeneous coordinates η, in which the conformal representation appears as a homogeneous linear transformation, is here accomplished, following Möbius, by

$$\eta^0 : \eta^1 : \cdots : \eta^n : \eta = 1 : \xi^1 : \cdots : \xi^n : -\tfrac{1}{2} g_{ik}\xi^i\xi^k .$$

These coordinates are subject to the relation

$$g_{ik}\eta^i\eta^k + 2\eta^0\eta = 0.$$

It is convenient for the purpose of calculation to normalize the coefficients g_{ik}, only the ratios of which are given, by the condition $|g_{ik}| = 1$. This third and last step, which was carried through by Veblen in a recent paper, proceeds as before, but the result is more complicated since we have $n+2$ variables η, whereas Thomas' extension gives but $n+1$ coordinates x. Consequently we do not arrive at an affine connection on an $(n+1)$- or $(n+2)$-dimensional manifold M^*, invariantly related to the conformal connection on M.

Princeton University

HENRY BURCHARD FINE—IN MEMORIAM

Dean Fine was one of the group of men who carried American mathematics forward from a state of approximate nullity to one verging on parity with the European nations. It already requires an effort of the imagination to realize the difficulties with which the men of his generation had to contend, the lack of encouragement, the lack of guidance, the lack of knowledge both of the problems and of the contemporary state of science, the overwhelming urge of environment in all other directions than the scientific one. But by comparing the present average state of affairs in this country with what can be seen in the most advanced parts of the world, and extrapolating backwards, we may reconstruct a picture which will help us to appreciate their qualities and achievements.

Henry Burchard Fine was born in Chambersburg, Pennsylvania, September 14th, 1858, the son of Lambert Suydam Fine and Mary Ely Burchard Fine. His father, a Presbyterian minister, died in 1869 leaving his widow with two sons and two daughters. Mrs. Fine lived with her children for a while at Ogdensburg, New York, and afterward at Winona. Minnesota, and in 1875 brought them to Princeton to complete their education. She was, by all accounts, a woman of great ability and force of character and launched all her children on honorable careers.

Thus Fine's early years were those of a country boy under pioneer conditions. He was always enthusiastic about the two great rivers, the St. Lawrence and the Mississippi, beside which he lived in those years and on which he learned to row. Rowing, by the way, was the only college sport in which he took an active part, though he was always keen on athletics and gave much time and energy to the direction of them up to the end of his life. During the years when I knew him he got much of his exercise on his bicycle, which he used both in going to and from his classes and for long rides in the country.

He finished his preparation for college during his first year in Princeton and entered the College of New Jersey, as it was then called, in 1876. Throughout his college course he was ranked at the head of his class and he graduated with first honors in 1880. As an undergraduate he specialized in the Classics, which he expected to teach later on, and he also began the study of Sanskrit. Besides this he found time for a normal undergraduate's interest in athletics, and played the flute in the college orchestra. He served for three years on the board of the college paper, the Princetonian, of which he was managing editor in his senior year. It was in this connection that he formed a life-long friendship with Woodrow Wilson.

The picture which one gets of his undergraduate years fits in completely with the impression which those of us got who knew him only in the last three decades of his life. We saw that he was widely read in literature and history, that he was fond of good music and active in bringing good concerts to Princeton, and that he took a keen interest in the games and the daily life of the undergraduates.

It seems that his adoption of mathematics as a career was largely due to George Bruce Halsted, that extraordinary enthusiast for non-euclidean geometry. Halsted had graduated from Princeton in 1875 and after studying under Sylvester at the Johns Hopkins, spent the years from 1878 to '81 as "instructor in post-graduate mathematics" in Princeton. Halsted communicated his enthusiasm to a little group of undergraduates and in particular to Fine, who had a natural bent for questions of a logical order.

On graduating, Fine was appointed Class of 1860 Fellow in Experimental Science. In this capacity he did a certain amount of experimental work and even published a paper, *On the shadows obtained in the glow discharge*, in the American Journal of Science, vol. 21 (1881) pp. 394–5, in collaboration with W. F. Magie who was subsequently Professor of Physics in Princeton. But Fine was never really interested in experimental work and in 1881 gladly accepted a position as Tutor in Mathematics, which he held until 1884.

In March 1884 he and Magie sailed together for Germany. According to his own account, he knew very little German and almost no mathematics. Nevertheless he developed so rapidly that he obtained the Ph.D. in Leipzig in May, 1885. He used to tell how Klein advised him, in spite of his ignorance, to come into the advanced lectures, and assured him that he would be learning mathematics even if he did not seem to understand everything as it went along. Fine did so and after a few months was surprised, on looking back at his early notes, to find that the whole story was clear to him. The thesis subject first proposed by Klein (on enumerative geometry) turned out to be unsuitable and, with Klein's consent, he changed to another subject suggested by Study. The latter, also a student of Klein's, has remained one of Fine's closest friends. After taking his degree, Fine spent the summer semester of 1885 in Berlin where he listened to Kronecker's lectures on the theory of eliminations and was profoundly influenced by them.

After this brief contact with the main current of mathematics, Fine returned to Princeton as an assistant professor, in which position he remained until 1889. During this period he published four papers in the American Journal of Mathematics. The first was his thesis, *On the singularities of curves of double curvature*, vol. 8 (1886), p. 156–177. The second was an extension of the results of this paper to n dimensions, vol. 9, pp. 180–184. The other two were: *On the functions defined by differential equations with an extension of the Puiseux polygon construction to these equations*, vol. 11 (1889), pp. 317–328, and *Singular solutions of ordinary differential equations*, vol. 12 (1890) pp. 295–322.

He was married September 6, 1888 to Miss Philena Fobes of Syracuse, N. Y. In 1889 he was promoted to a full professorship and in 1898 was appointed Dod professor of mathematics.

In the summer of 1891 he had another short period of study in Germany during which he became better acquainted with Kronecker. There is a reference to this in his charmingly written obituary notice of the latter, *Kronecker and his arithmetical theory of the algebraic equation*, Bulletin of the New York Mathematical Society, vol. 1 (1892), pp. 173–184. Fine remained

an admirer and student of Kronecker's work throughout his life, as is witnessed by his important paper, *An unpublished theorem of Kronecker respecting numerical equations*, this Bulletin, vol. 20 (1914), pp. 339–358. This was Fine's retiring address as President of the American Mathematical Society.

From this time onward Fine's mathematical activities were devoted more and more to teaching and to the logical exposition of elementary mathematics, though he always kept up an active interest in higher analysis, and indeed published a paper of some importance, *On Newton's method of approximation* (Proceedings of the National Academy of Sciences, vol. 2, pp. 546–552) as late as 1917.

His first book, *The Number System of Algebra treated Theoretically and Historically* (Boston and New York, Leach Shewell and Sanborn, 1891), was devoted in a single-minded way to the exposition of the logical foundations of analysis. It is still a very useful reference book on this subject. The subsequent books were all intended as text-books for undergraduates. They were *A College Algebra* (1905), *Coordinate Geometry* (with Henry Dallas Thompson) (1909) and *Calculus* (1927). Of these, the first and the last are the most successful and probably are not excelled by any other books in their respective fields from the point of view of accuracy of statement and the adequacy with which they represent the subject. They are admirably concise and free from unnecessary verbiage. In every case they had been printed or mimeographed in a preliminary form and used with classes for many years, and modified time and again as a result of this experience, before they were published.

In 1903 shortly after Woodrow Wilson became President of Princeton, Fine was made Dean of the Faculty and his energies were devoted chiefly to university administration. When Wilson resigned to become Governor of New Jersey in 1910, Fine carried the chief burden of the university administration for two years until another president was appointed. He then resigned as Dean of the Faculty but was induced to continue as Dean of the Departments of Science, which he did till his death.

His firmness of character and wisdom in counsel made him the preeminent figure among his colleagues of the Princeton faculty up to the very end. He was the man to whom they turned more than to anyone else for leadership or advice, both in public and private affairs. These qualities were widely recognized outside Princeton, for he had more than one opportunity to become a University President, and President Wilson made several efforts to induce him to take important posts in his administration. But as he said quite simply, he preferred to continue as a professor of mathematics.

In 1905, when the Princeton faculty was being enlarged because of the institution of the preceptorial system, Fine was in a position to determine the character of the new personnel in the scientific departments. He used the opportunity so well as to bring about a development in these departments which has had a vital effect on the scientific history of the whole country. To mention only a few instances in other departments than his own, which show the style of his choices, Fine was instrumental in calling or

in promoting Richardson in Physics, Russell in Astronomy, Hulitt in Chemistry, and Conklin in Biology.

In his own department, the expansion of 1905 meant the promotion of Eisenhart and Gillespie to preceptorships and the calling in of Bliss, Veblen, and Young to the same rank, and Jeans to a professorship. When Bliss and Young were called away Birkhoff and Wedderburn were called in, and later Boutroux in succession to Birkhoff. Without cataloguing more names it can be said that Fine's department has had the services of a considerable proportion of the best mathematicians of America during this period. In every case these men were called in before they became well known, at the time when this recognition meant the most to them. Though many of them stayed only a few years their contact with Fine and his group was important both to them and in the continuing growth of this group as an organism. No doubt this organism has had a greater influence on the mathematical enterprise in general because its constituents have changed rather rapidly.

In Fine's last years he was able to seize another opportunity for the advancement of science. Largely because of the confidence which he inspired, the General Education Board offered Princeton University a million dollars for research in pure science on condition that the University raise two additional millions for the same purpose. The fund was completed in 1928, chiefly by the contributions of Mr. Thomas D. Jones, a life-long friend of Fine's. Mr. Jones graduated from Princeton in 1876 before Fine entered, but they became acquainted and learned each others qualities during Wilson's administration as President of Princeton while Mr. Jones was one of the Trustees. This friendship is commemorated by the Henry B. Fine Professorship of Mathematics founded by Mr. Jones, and the Thomas D. Jones Professorship of Mathematical Physics founded by Miss Gwethalyn Jones. As a further memorial, Mr. Jones and his niece are giving a building for the department of mathematics which is to bear the name of Fine.

Fine was one of the founders of the American Mathematical Society. In January 1891, Professor T. S. Fiske as secretary of the New York Mathematical Society wrote to six prominent mathematicians asking their cooperation in enlarging the scope of this society and thus enabling it to publish a Bulletin. Of these six, Fine and Woolsey Johnson promised unqualified support. With this, and the somewhat less definite backing of the others, Fiske and his colleagues went ahead with another appeal to a wider circle of mathematicians. The first number of the Bulletin of the New York Mathematical Society appeared before the end of the same year. It records (p. 54) that Fine was appointed to fill a vacancy on the Council of the Society at the October meeting in 1891. The name of the Society was changed to the American Mathematical Society in 1894 and the first number of the present Bulletin was issued in the Autumn of that year. Fine was elected a member of the Council of the reorganized society at its first Annual Meeting in December 1894. He was President in 1911 and 1912.

In the latter part of his life Fine had to suffer the loss of two of his children and of his wife, who died in April, 1928. His own death was the

result of an accident. In the evening of December 21, 1928, he was riding his bicycle on a road in the outskirts of Princeton and was struck from behind by an automobile, the driver of which failed to see in the uncertain evening light that he was starting to make a left turn. He died early the next morning without having recovered consciousness.

The last time I saw him was in Princeton in August 1928 just after his return from a trip to Europe in which he had revisited old scenes and old friends and had recovered to some extent, in the distractions of travel, from his personal sorrows. He looked strong and well and seemed to have many years of activity before him. He spoke with humorous appreciation of the change he had observed in the attitude of European mathematicians toward their American colleagues and with pride of the esteem in which he had found his own department to be held. But most of our talk was of the future.

Shortly after this I was myself in Europe and met some of those who had seen and conversed with him on his last trip, most of them younger men who had met him for the first time. They all spoke of his charm of manner and the impression they had gained of a man of serene strength and poise and wisdom. It was a striking testimony to the way in which a man's character can be written into his appearance and manner by a long and strenuous life.

OSWALD VEBLEN

DE VRIES ON FOUR DIMENSIONS

Die vierte Dimension, eine Einführung in das Vergleichende Studium der Verschiedenen Geometrien. By Dr. H. K. de Vries. Translated into German from the second Dutch edition by Frau Dr. Ruth Struik. Wissenschaft und Hypothese, XXIX, Leipzig and Berlin, Teubner, 1926. ix+167 pp.

Although there is no mention of non-euclidean geometry in the title, nor even on the title-page, nearly half of the book is devoted to it. In fact, we have here two books put together under the title of the first. The books are independent: either could have been put first or published alone. Only in one place in the second (pages 139–141) is there introduced a little of the geometry of four dimensions, and at the very end there is an outline of the rise of this geometry, a little more than two pages, that might have been put at the end of the first part.

The treatment of the book is not strictly mathematical. We have rather a popular exposition, although considerable mathematical detail is suggested in the explanations, giving some idea of the way in which these subjects have been developed as well as the mere results of their development.

A small book, not too condensed, must leave out many things. Thus we find nothing about hyperprisms, hyperpyramids, hypercylinders, and hypercones, and nothing about hypervolume. Some might have chosen these and given only a paragraph to the regular polytopes or hypersolids. Many details of the non-euclidean geometry are presented very briefly. But the choice of material is in general excellent, and the book is well planned to impart some knowledge of these two branches of mathematics.

In both parts of the book much is said about the historical development of the subjects considered, about their usefulness, and about the frame of mind in which the reader should study them, and many interesting remarks are thrown in. Thus of the value of extending the chain of dimensions beyond the third we are told (page 22) that the notion of value is in high degree relative, and what is of inestimable value for one man leaves another entirely cold. But for the mathematician and for everyone who has an interest in scientific thought, it is of great value to know that a four or five or n dimensional geometry is possible, and the mathematician will not be satisfied to know that it is possible, but will want to know more exactly about it, as always when a new field of investigation appears.

At the beginning the relations of the simpler elements, lines, planes, and hyperplanes, their determinations and intersections, parallelism and perpendicularity, and the various kinds of angles formed by them, are gone into very thoroughly and carefully, and this study is the best sort of preparation even for a superficial understanding of the more complicated figures and relations of hyperspace.

The notion of point-value of a space is most helpful, for the dimension-number d is 1 less than w, the number of points necessary to determine a given space, and some formulas are obtained more easily by consider-

ing w. Thus two spaces, R_{d_1} and R_{d_2}, having in common a space $R_{d_{12}}$, determine a space R_d, the point-values of these spaces being connected by the relation $w = w_1 + w_2 - w_{12}$ and therefore their dimension-numbers by the relation $d = d_1 + d_2 - d_{12}$, which may also be used to determine the dimension of the intersection of two given spaces lying in and determining another given space. After this formula has been applied to spaces in R_3 and in R_4, it is suggested that the reader experiment with R_5 and R_6, and then we are told (page 35) that he will not be able to suppress the feeling of having extended very widely his horizon, and that this is the reward for the striving necessary at the beginning.

Turning to the second part, the non-euclidean geometry, we find the quadrilateral of Saccheri and his three hypotheses taken as the starting-point, with a brief indication of the results to which these three hypotheses lead. There is also an account of the way in which the hyperbolic and elliptic geometries are represented as the geometries on the pseudo-sphere and sphere, and by projection on a plane. The *boundary-curve* or *horocycle*, and the *equidistant-curve* are described (page 150) as new figures of hyperbolic geometry, and the polar relation of lines and points and of planes and points in the elliptic geometry (page 130), while the two kinds of elliptic geometry, the double and simple elliptic geometries, are mentioned (page 133). But the treatment of all of these details is very brief.

We close with an outline of the contents of this book.

After an introduction of about three pages we have Part First, euclidean geometry of several dimensions [chiefly of four dimensions], pages 4–92, and Part Second, non-euclidean geometry, pages 92–165.

The first part may be divided roughly as follows:

Points, lines, planes, and spaces, their intersections, point-values, etc., pages 4–35.

Infinitely distant elements and parallelism, pages 35–45.

Perpendicularity and rotation, the hypersphere, pages 45–59.

Angles, the number of angles between two spaces, the two angles between two planes, pages 60–77.

The regular polytopes, pages 77–92.

An outline of the second part, with some changes in order, will be:

The postulates of Euclid and the three hypotheses of Saccheri, pages 92–112.

The sum of the angles of a triangle and its area, pages 112–116, 153–161.

Euclidean geometry, pages 116–120.

Hyperbolic geometry, pages 120–125, 145–153.

Elliptic geometry, pages 125–134.

Geometrical representations, pages 134–145, 161–163.

And finally, a historical paragraph in regard to geometry of n dimensions, pages 163–165.

H. P. MANNING

PICARD ON PARTIAL DIFFERENTIAL EQUATIONS

Leçons sur quelques Types Simples d'Equations aux Dérivées Partielles avec des Applications à la Physique Mathématique. By Émile Picard. Paris, Gauthier-Villars, 1927. i+214 pp.

This book forms the first volume of a new mathematical series, *Cahiers Scientifiques*, edited by G. Julia. From the preface to volume 3 of the series, one learns with regret that Picard has abandoned the project of completing volume 4 of his *Traité d'Analyse*, which was to give the theory of partial differential equations. Instead, the subject matter of this volume will be presented in several volumes of *Cahiers Scientifiques*. The book under review contains lectures given at the Sorbonne in 1907 and repeated, with some additions, in 1925.

The first four lectures deal with the simplest and best known of the equations of parabolic type, namely Fourier's equation

$$\partial u/\partial x = \partial^2 u/\partial y^2.$$

Lecture 1 begins with the existence theorem, shows that non-analytic solutions are possible for analytic initial data, and derives Fredholm's example of a power series whose circle of convergence is a natural boundary. Lecture 2 derives the fundamental solution, shows how this is used to solve the boundary problem $u=f(x)$ for $t=0$, and as an application gives Weierstrass' original proof of his theorem on the approximation to a continuous function by polynomials. Lecture 3 applies the preceding results to various boundary problems in heat conduction, and Lecture 4 gives an account of Lord Kelvin's investigation of the propagation of an electric current along a cable.

Lectures 5 to 11 deal with various aspects of the theory of integral equations. In Lecture 5, a brief account of the Fourier integral is given and applied to an integral equation which was set up in Lecture 3. Lecture 6 deals with the properties of the integrals

$$\int_0^\infty \cos xy\phi(y)dy \text{ and } \int_0^\infty e^{-xy}\phi(y)dy$$

considered as functions of x. In Lecture 7, an integral equation of the first kind is set up for the solution of Dirichlet's problem (in two dimensions) by a potential of a single layer on the boundary curve, and the main results of Fredholm on integral equations of the second kind are enumerated.

In Lecture 8, the singular integral equations of the second kind

$$f(x) + \lambda \int_0^\infty \cos x\xi f(\xi)d\xi = \phi(x)$$

and

$$f(x) + \lambda \int_{-\infty}^\infty e^{-|x-\xi|} f(\xi)d\xi = \phi(x)$$

are briefly considered. Lecture 9 deals with the integral

$$\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-x},$$

where $f(z)$ is defined only on the closed curve C (so that $f(z)$ is not necessarily analytic), and the principal value of the integral, when x approaches C, is defined. Lecture 10 raises the question of the necessary and sufficient condition to be imposed on $f(z)$ on C in order that the above integral shall be an analytic function (which is then $=f(x)$ by Cauchy's formula). The condition is

$$f(x) = \frac{1}{\pi i} \text{ Princ. val. } \int_C \frac{f(z)}{z-x}dz$$

for every x on C. Lecture 11 applies the principal values to the solution of Riemann's problem of finding two analytic functions one regular inside C and the other outside, such that their ratio shall take prescribed values on C. Hilbert's problem of determining an analytic $f(z)=u(x, y)+i\,v(x, y)$ such that $au+bv+c=0$ on C, where a, b, and c are functions of position on C, is also considered, and the lecture closes with a discussion of the tangential derivative of a single layer potential. With Lecture 12, the author returns to partial differential equations and gives a brief survey of Cauchy's problem and the theory of characteristics. Lectures 13 to 16 deal mainly with linear equations of the hyperbolic type, and the method of successive approximations is used to solve the following boundary problems: (1) the values of u are given on two characteristics; (2) u and one of its first derivatives are given along a curve which is not a characteristic (Cauchy's problem); (3) u is given on a characteristic and also on a non-characteristic; (4) u given on two non-characteristics. In Lectures 17 and 18, the problems of Lectures 13 to 16 are treated by Riemann's integration method. This method is applied to the equation of a vibrating string and to the telegraph equation in Lecture 19. Lecture 20 introduces a method due to Poincaré for the telegraph equation, and this gives occasion for some remarks on integral equations of the first kind, particularly that of Abel.

The author now passes to linear equations of the second order and elliptic type. Lecture 21 gives the proof of the analyticity of the solutions of the general second-order equation of the elliptic type, and Lecture 22 presents some results on the rather delicate question of the existence of the normal derivative of the solution. Lecture 23, which is rather loosely connected with the preceding ones, finally gives some properties of the spherical potential of Boussinesq.

Many results and still more proofs in this book are new, and the exposition has all the qualities of lucidity and elegance which one is accustomed to expect from its distinguished author.

T. H. GRONWALL

SHORTER NOTICES

Applications of the Theory of Elliptic Functions to the Theory of Numbers. By P. S. Nazimoff. Translated from the Russian by Arnold E. Ross. Mimeographed by O. E. Brown. Evanston, 1928. 146 pp.

The original of this remarkable book was presented as a magister thesis in Russia and was printed in Russian at Moscow in 1884. Abstracts of certain parts of it were published shortly after its Russian appearance, both in the Fortschritte and in a prominent French journal. Neither abstract did the book justice. That this assertion of fact is so can be seen from the following. Had an adequate account of the contents of this book appeared at the time of its publication, much later work in the theory of numbers of representations of integers in special quadratic forms need never have been published. For example, in 1914, Bulyguin, using the methods of the first few chapters of Nazimoff's book, gave a complete solution, by comparatively elementary means, of the determination of the numbers of representations of integers as sums of any even number of squares. This work itself, like the neglected Nazimoff's, seems to have escaped the notice of the professional abstractors. As late as 1919, G. Humbert, in a letter to the present reviewer, inquired what had become of this most remarkable work of Bulyguin; it was already in print. Episodes such as this but emphasize the lamentable ignorance under which most of us labor concerning Russian work in the theory of numbers. Reverting for a moment to the question of representations as a sum of an even number of squares, we may state that many recent and elaborate investigations were out of date before they were printed, on account of the work of the neglected Russian school.

In the present translation only those parts of Nazimoff's book have been presented that contain matter which was original at the time of publication and a great part of which, in the light of the above remarks, is still as fresh as it was when published. The title sufficiently describes the contents of the work. Nazimoff made the first systematic attempt to obtain everything arithmetical that is implicit in the theory of elliptic functions. That such a vast undertaking appears only rough hewn in the final product, is not to the author's discredit. Precisely where his work is least finished is the place where it is most suggestive. The problem of discovering everything that is implied for the theory of numbers by the multiply periodic functions is a well-put and solvable one; this book took the first significant step toward a solution. It is worthy of the attention of all those interested in the algebraic applications of analysis to the theory of numbers—which is quite a different thing from the analytic theory of numbers.

For those interested, the general problem which will be suggested by a reading of pages 138–144 of the present translation, is worthy of the closest attention. A complete solution of the obviously implied problem, extended to functions of any number of variables, contains the solution of any

arithmetical problem whatever that is solvable by the application of algebraic functions of elliptic theta functions.

The translation was made from what is apparently the unique copy of this book in the United States. The translators wish it to be stated that copies can be purchased only at the University of Chicago Book Store, 5802 Ellis Ave., Chicago, and that the price is $3.00. The 146 pages of mimeographing are, on the whole, clearly done; the book is substantially bound in a stiff paper cover and, as such work goes, is of a very high quality. It is to be hoped that this translation will at last make Nazimoff's penetrating ideas familiar to those who either do not read Russian or who are content to accept second hand abstracts from those who also do not read Russian.

E. T. Bell

The new Quantum Mechanics. By G. Birtwistle. Cambridge University Press, 1928. xiii+290 pp.

This book contains an exposition of the theories of Heisenberg, Schrödinger and Dirac, with applications mainly to the theory of spectra. There are also chapters on de Broglie's particle waves, and the Bose-Einstein and Fermi-Dirac statistics. The closing chapter deals with Heisenberg's indetermination relations and the formulation given to them by Bohr in his article in Nature, April 1928 (and communicated to the author before publication; the author's preface is dated Copenhagen, 1 Oct. 1927).

The book contains a great deal of information on the main developments in quantum theory up to the latter part of 1927, and is convenient for reference; on the other hand, owing perhaps to lack of time, the presentation is uncritical, so that the book cannot replace the study of the original memoirs.

T. H. Gronwall

Traité d'Analyse, volume 3, third edition. By Émile Picard. Paris, Gauthier-Villars, 1928. ix+660 pp.

This is a reprint of the second edition, with additional notes on pp. 601–656. Although no review of previous editions has appeared in the Bulletin, the *Traité* is so well known and so highly appreciated by the mathematical public, that it seems superfluous to describe the contents of the main body of the volume. The additions to the new edition are as follows: (1) a note on Sundman's work on the problem of three bodies (reprinted from Bulletin des Sciences Mathématiques, 1913, pp. 313–320); (2) a lecture on integral invariants and Poisson stability (from the author's course at the Sorbonne in 1914, printed here for the first time); (3) remarks on some of Poincaré's results in analytical mechanics (Bulletin des Sciences Mathématiques, 1914, pp. 320–236); (4) on the solution of $\Delta u = e^u$ on a closed Riemann surface (Journal für Mathematik, vol. 150); (5) on linear partial differential equations and the generalization of Dirichlet's problem (proof of the analyticity of the solution for a second-order equation of the elliptic type, reprinted from Acta Mathematica, vol. 25 (1901), pp. 121–137).

T. H. Gronwall

Vorlesungen über Darstellende Geometrie. By Dr. Emil Müller. II. Band: *Die Zyklographie.* Edited from the manuscript by Dr. Josef Leopold Krames. Leipzig and Vienna, Franz Deuticke, 1929. ix+476 pp.

The third edition of the second volume of Dr. Müller's text and the first volume of his lectures were reviewed by the present reviewer in this Bulletin in 1923 (vol. 29, pp. 478–479). The first volume of the lectures was in the name of Dr. Müller and Dr. Erwin Kruppa.

Dr. Krames states in the introduction that the late Dr. Müller entrusted to him the editing of the manuscript for this second volume, only part of which was then in form for publication.

This book contains a more extensive use of the orientation of circles than had previously been made by Fiedler. In fact, the authors' fundamental operations are with orientated circles, (Zykeln), orientated straight lines (Speeren), and orientated curves (Richtungskurven). Their use of these terms is in harmony with that of Ed. Laguerre.

The ten chapters deal with the "zyklographic" representation of points, straight lines, planes, curves, and nets of curves, and surfaces and with contact transformations, etc. Special attention is directed to the chapter on "*C*-Geometrie" (parabolic pseudo geometry).

We are endebted to Dr. Krames for his skill in editing this unfinished manuscript—the last contribution of a talented and industrious worker in the field of descriptive geometry.

E. B. Cowley

Repertorium der Höheren Mathematik. By E. Pascal. Second edition: first volume, *Analysis*; second section. Edited by E. Salkowski. Leipzig, B. G. Teubner, 1927. Pp. v–xii and 529–1023.

Parts of the second edition of the Pascal Repertorium have been reviewed, upon their appearance, in this Bulletin, and the outstanding features of the work have been fully discussed. It is therefore unnecessary to make very detailed comments on the present section. Reference is made to the Bulletin: vol. 19, pp. 372–374, and vol. 29, p. 373.

Approximately two-thirds of this second section of the first volume is devoted to the theory of analytic functions in its classical ramifications, and a good exposition is given of both the Weierstrassian and Riemannian points of view. A relatively large portion of the space thus reserved for function theory is assigned to the automorphic functions and the elliptic modular functions. The remainder of the book offers a brief treatment of fundamental concepts and methods in (1) the theory of differential and difference equations, and that of differential forms, (2) the theory of continuous transformation groups, and of contact transformations, and (3) the calculus of variations. Contributors to the section are: A. Guldberg of Oslo, Ernesto Pascal of Naples, F. Engel of Giessen, Hans Hahn of Vienna, Gustav Doetsch of Stuttgart, E. Jahnke and A. Barneck of Berlin, H. W. E. Jung of Halle, and R. Fricke of Braunschweig.

In conformity with the general plan of the undertaking, the present section, like its predecessors in the second edition, attains a definitely

higher degree of readability than one has hitherto been accustomed to expect in professedly encyclopedic summaries of mathematical science. Although proofs are, of necessity, frequently omitted, theorems are stated with clearness and accuracy, and in well-arranged sequence. References to the literature, while by no means exhaustive, will in most cases provide ample orientation for those who wish to undertake specialized studies.

By reason of the arrangement of the material and the style of exposition, it is probable that most mathematicians will find in this section of the Repertorium not only appreciable stimulation but also a measure of downright enjoyment.

L. S. Hill

Tables of Damped Vibrations. By W. E. Milne. University of Oregon Publication. Mathematics Series. Vol. 1, No. 1, March 1929. 39 pp.

In an earlier paper entitled *Damped vibrations. General theory together with solutions of important special cases*, dated August, 1923, the author gave fairly extensive tables covering the case in which the damping is proportional to the square of the velocity, and short tables applicable to the more general situation in which the resistance function is of the form $R=v(B\pm Dv)$, where v is the velocity and B and D are positive constants. The sign to be taken is the same as that of v.

The objects of the present paper are (a) to furnish more extensive tables for the solution of the problem of oscillations with a resistance function of the type cited and (b) to supply additional tables applicable to the case where the term involving v^2 does not change sign when the sense of the motion reverses. A concrete application of case (b) is afforded by the oscillation of water in the hydraulic surge chamber.

The fourteen tables,—all of four decimal places,—are preceded by useful preparatory paragraphs on the "Scope of Application, The Hydraulic Surge Chamber, Interpolation and Method of Computation." In the first of these attention is called to the fact that, in many cases, the binomial resistance function quoted above may represent experimental data as satisfactorily as the conventional relation $R=Cv^m$. In particular, when m lies between 1 and 2, (as is often the case in practice), it is shown that it is possible to determine B and D in such a manner as to make the error incurred by using the easily calculable binomial in place of the mth power monomial resistance formula of no greater importance than the other uncertainties inherent in the problem. For illustration, the "standard error" attains its greatest magnitude of about 0.0082 when 1.4 is the value of m. Since the tables furnish solutions of differential equations which cannot be solved in terms of known functions they will be welcomed by mathematicians, physicists, and engineers when problems of damped oscillations arise in their investigations and demand numerical solution. The computation of the tables was aided materially by a liberal grant of funds from the National Academy of Sciences.

H. S. Uhler

Exercices d'Analyse, vol. 1. By Gaston Julia (edited by René Harmegnies and Roger Julia). Paris, Gauthier-Villars, 1928. vii+454 pp.

This book is a worthy descendent of a long line of French *Exercices sur le calcul infinitésimal*. Such collections of problems are intended primarily for the students who prepare themselves for the licence or the agrégation and contain problems of the type set in these examinations. A thorough knowledge of the theory is expected as well as skill in calculation and the training is directed towards developing both qualities in the students.

The present book contains a small number of carefully chosen problems, each problem followed by one or more complete solutions. About two-thirds of the first volume is devoted to the applications of analysis to geometry. An admirable account of the theory of Fourier series (pp. 120–190) is eminently suitable as outside reading for first year graduate students. This part of the book will probably be found the most useful one to the general mathematical public outside of France. The printing is good, but when will MM. Gauthier-Villars return to the use of good paper in their books?

EINAR HILLE

A History of Physics. Revised and enlarged edition. By Florian Cajori. New York, Macmillan, 1929.

The first edition of this book gave in 285 pages a concise yet detailed account of the important phases of the development of physics from ancient times to the time of publication in 1899. While many additions and alterations have been made, the new edition is substantially the same as the old so far as this part of the work is concerned. The important change in the second edition is due mainly to the excellent summary, covering about 100 pages, of the development of physics during the first twenty-eight years of the twentieth century.

The method of presentation adopted in the first edition, that is, the arrangement of the material by periods, and within a period according to the well known sub-divisions of physics, has been retained. In this connection it is interesting to note the space allotted to different periods. Thus ancient civilizations are given 22 pages; the Middle Ages and the Renaissance 32 pages; the seventeenth and eighteenth centuries 83 pages; the nineteenth century 147 pages; and the first quarter of the twentieth century, 116 pages. Taking this distribution as a fair measure of the progress of physics during these periods and taking into account the length of each period, we find that the increase of our knowledge in physics has followed practically an exponential curve, indicating that our knowledge of the physical world is not only not tending to a saturation value, but its rate of increase is rising with time.

One of the commendable features of the book consists in the numerous references to original sources.

Professor Cajori has earned the gratitude of scientists by his works on the history of science.

H. M. DADOURIAN

NOTES

By invitation of the Program Committees, several addresses will be delivered at meetings of the Society during the Fall, as follows:

At New York City, October 26, 1929, by Professor Tomlinson Fort, on *The general theory of factorial series.*

At Ann Arbor, November 29, 1929, by Professor Tomlinson Fort, on *The general theory of factorial series;* and by Professor L. M. Graves, on *Discontinuous solutions in the calculus of variations.*

At Pasadena, November 29, 1929, by Professor M. W. Haskell, on *Autopolar configurations in two and in three dimensions.*

At the Annual Meeting of the Society, to be held at Lehigh University, Bethlehem, Pennsylvania, on December 27–28, 1929, both the morning and the afternoon of Saturday will be devoted to a symposium on the general topic, *The differential equations of engineering.* By invitation of the Program Committee, addresses will be given by Professor H. W. March, The University of Wisconsin, on *The problem of diffusion;* by Professor Vannevar Bush, The Massachusetts Institute of Technology, on *Integraphs for differential equations;* by Dr. T. H. Gronwall, Columbia University, on *Numerical integration of differential equations;* by Dr. Ing. A. Nadai, The Westinghouse Electric and Manufacturing Company (formerly Professor of Applied Mathematics, University of Goettingen), on *Plasticity and related problems of non-rigid bodies;* by Mr. R. H. Park, Engineering Department, The General Electric Company, on *Analytic determination of flux plots;* by Professor S. Timoshenko, The University of Michigan (formerly with the Westinghouse Company), on *Problems in elasticity.* Professor Tomlinson Fort is chairman of the Committee on Arrangements.

At the meeting of the Society at Des Moines, Iowa, in conjunction with the meetings of the American Association for the Advancement of Science and of the Mathematical Association of America, on December 30–31, 1929, Professor L. L. Dines will give an address on *Linear inequalities and some related properties of functions* before the joint meeting of the Society and the Mathematical Association; and Professor O. D. Kellogg will give an address on *An unsolved problem of uniqueness in potential theory.* The annual Josiah Willard Gibbs address by Professor Irving Fisher on *Applications of mathematics to the social sciences,* will be held on Tuesday, December 31. A special announcement concerning this meeting will be issued.

At the meeting in New York City, February 21, 1930, by invitation of the Program Committee, Mr. E. C. Molina will give an address on *The theory of probabilities.*

The Colloquium Editorial Committee wishes to call attention to the fact that in addition to the Colloquium Lectures delivered before the Society, it is in a position to print a few other volumes. Authors are invited

to submit manuscripts. For further information, see page 100 of volume 32 of this Bulletin.

A new volume of the Colloquium Series, Coble's *Algebraic Geometry and Theta Functions*, has appeared and may be obtained through the New York Office of the Society. The edition of Veblen's *Analysis Situs* (Cambridge Colloquium, Part II) being exhausted, a new edition will be prepared and will probably appear in 1930.

The tercentenary of the birth of Christiaan Huygens was celebrated at Leiden in April, 1929, under the auspices of the Royal Academy of Sciences and the Senate of the University of Leiden.

On June 29, 1929, the Berlin Physical Society celebrated the fortieth anniversary of Professor Max Planck's entrance into scientific work. That Society has created a medal in his honor, to be awarded for distinguished work in mathematical physics (see this Bulletin, vol. 34, p. 792). On this occasion Professor Planck presented the first cast of this medal to Professor Albert Einstein, emphasizing particularly his influence on the development of the quantum theory.

The Alfred Ackermann-Teubner memorial prize for 1928 has been awarded to Professor Albert Defant, director of the Institut für Meereskunde, Berlin, for his work collected and published in *Das Zeitenproblem des Meeres in Landesnähe*, Hamburg, 1925.

The Adams prize of Cambridge University for 1927–28 has been awarded to Professor Sydney Chapman, of the Imperial College of Science Technology, London, for a memoir on the announced subject *The variations in the earth's magnetic field in relation to electric phenomena in the upper atmosphere and on the earth.* The Sudbury Hardyman prize of Emmanuel College, Cambridge, has been awarded to A. H. Wilson, for a dissertation on *Quantum mechanics.*

The medal of the class of 1889, Columbia University, has been awarded to Dr. Irving Langmuir, of the General Electric Company, for "distinguished achievement in improving the incandescent electric light."

Professor Frank Schlesinger, of Yale University, has been awarded the Bruce medal of the Astronomical Society of the Pacific.

On the occasion of the Abel centennary celebration, the University of Oslo conferred honorary degrees on the following mathematicians, in addition to those listed in the last issue of this Bulletin (pp. 583–584): Professors L. E. J. Brouwer of Amsterdam, Friedrich Engel of Giessen, Rudolph Fueter of Zürich, Kurt Hensel of Marburg, Christian Juel of Copenhagen, Edmund Landau of Göttingen, E. L. Lindelöf of Helsingfors, L. E. Phragmén of Stockholm, Salvatore Pincherle of Bologna, Teiji Takagi of Tokyo, Charles de la Vallée-Poussin of Louvain, Oswald Veblen of Princeton, and Hermann Weyl of Zürich.

Professor Jules Drach, of the Sorbonne, has been elected a member of the Paris Academy of Sciences in the section of mechanics, as successor to the late Professor J. Boussinesq.

The Royal Academy dei Lincei has elected Professor G. Peano (Turin) national member; Professors G. D. Birkhoff (Cambridge, Mass.), C. Carathéodory (Munich), A. Sommerfeld (Munich) foreign associates.

The Accademia Pontificia in Rome has elected Father G. Boccardi (retired professor of the University of Turin) and Professor E. Pistolesi, of the University of Pisa, ordinary members, and Professor T. Levi-Civita, of the University of Rome, corresponding member.

Associate Professors L. Brusotti and F. Cecioni, of the University of Pisa, G. Giorgi, of the University of Cagliari, G. Marletta, of the University of Catania, and E. G. Togliatti, of the University of Genoa, have been promoted to professorships.

The University of Paris has conferred an honorary doctorate on Professor Albert Einstein.

Professor Leon Lichtenstein, of the University of Leipzig, has been elected a member of the Polish Academy of Sciences, Cracow.

Professor I. Schur, of the University of Berlin, has been elected a corresponding member of the Leningrad Academy of Sciences.

The University of Glasgow has conferred an honorary doctorate on Professor H. S. Carslaw, of the University of Sydney and Emmanuel College, Cambridge.

Cambridge University has conferred honorary doctorates on Professor Paul Langevin and Sir Frank Dyson.

Dr. Irving Langmuir, of the General Electric Company, has received an honorary doctorate from the Charlottenburg Technical School.

Among the recent elections to the National Academy of Sciences are the following: Dr. C. J. Davisson, of the Bell Telephone Laboratories; Professor Bergen Davis, of Columbia University; Professor William Hovgaard, of the Massachusetts Institute of Technology.

The following have been elected to the American Academy of Arts and Sciences: fellows, Professors Marston Morse and J. L. Walsh, of Harvard University; foreign honorary members, Professors V. F. K. Bjerknes, of the University of Oslo, Ludwig Prandtl, of the University of Göttingen, Willem de Sitter, of the University of Leiden, and Hermann Weyl, of the Zurich Technical School.

Honorary doctorates have been conferred on Professor A. H. Compton, of the University of Chicago, by Yale University, and by Ohio State University.

The University of Michigan has conferred the honorary degree of doctor of laws on Professor R. A. Millikan, of the California Institute of Technology.

Columbia University has conferred an honorary degree on Dr. G. A. Plimpton, trustee of Barnard College, known to mathematicians for his collections of mediaeval mathematical works.

Harvard University has conferred an honorary doctorate on Professor H. N. Russell, of Princeton.

Swarthmore College has conferred an honorary doctorate on Dr. W. F. G. Swann, director of the Bartol Foundation of the Franklin Institute.

Guggenheim fellowships for 1929 have been awarded to the following persons: Dr. Dinsmore Alter, professor of astronomy, University of Kansas, for research on the application of statistical methods to meteorological and astronomical problems; Dr. O. S. Duffenbach, assistant professor of physics, University of Michigan, for an investigation of molecular impact phenomena in gases and the relation of these to the excited states of molecules and the chemical binding of their atoms; Dr. Olive C. Hazlett, assistant professor of mathematics, University of Illinois, for the continuation of studies of the arithmetics of linear associative algebras; Dr. R. J. Kennedy, research associate, California Institute of Technology, for the continuation of researches in the theory of radiation; Dr. W. M. Latimer, associate professor of chemistry, University of California, for the study of the thermodynamic treatment of solutions with a view to extending work on the entropy of aqueous ions; Dr. R. S. Mulliken, associate professor of physics, University of Chicago, for work on theoretical problems connected with the subject of the formation and dissociation of molecules and the assignment of quantum numbers for electrons in molecules, based on the study of band spectra; Dr. J. C. Slater, assistant professor of physics, Harvard University, for research in quantum mechanics; Dr. C. H. Smiley, instructor in mathematics, University of Illinois, to investigate methods of determining orbits of comets and asteroids; Dr. J. H. Van Vleck, professor of theoretical physics, University of Wisconsin, for research in quantum mechanics; Dr. G. T. Whyburn, adjunct professor of mathematics, University of Texas, for research in point set theory, with particular emphasis on the structure of continua and of continuous curves. The above list includes renewals.

Professor A. Fraenkel, of the University of Kiel, has declined a permanent professorship at the University of Jerusalem, but has accepted an appointment for the academic years 1929–30, 1930–31. Dr. R. Schmidt will take his place at the University of Kiel during this interval.

Dr. Vojtěch Iarník, of the Czech University at Prague, has been promoted to an associate professorship of mathematics.

Dr. Heinrich Ott has been promoted to an associate professorship of theoretical physics at the University of Würzburg.

Associate Professor Bernhard Schilling, of the Dresden Technical School, has been appointed professor of mathematics at the University of Santiago, Chile.

Associate Professor Víktor Trkal, of the Czech University at Prague has been promoted to a professorship of theoretical physics.

Dr. W. Hurewicz has been admitted as privat docent at the University of Amsterdam.

Professor J. Sun, of Yenching University, Pekin, has been appointed head of the department of astronomy at North-Eastern University, Mukden, China.

At Cambridge University, F. C. Burkill, of Peterhouse, P. A. M. Dirac, of St. John's, and F. P. Ramsey, of King's, have been appointed university lecturers in mathematics. Dr. W. M. Smart, of Trinity College, chief assistant at the Observatory, has been appointed to the John Couch Adams astronomership.

Mr. E. C. Titchmarsh, of the University of London, has been appointed professor of pure mathematics at the University of Liverpool.

Associate Professor B. A. Bernstein, of the University of California, has been promoted to a professorship of mathematics.

Professor Florian Cajori, of the University of California, has retired, as professor emeritus of the history of mathematics.

Associate Professor C. C. Camp, of the University of Nebraska, has been promoted to a professorship.

Assistant Professor E. F. A. Carey, of the University of Montana, has been promoted to an associate professorship.

Assistant Professor E. U. Condon, of Princeton, has been appointed professor of theoretical physics at the University of Minnesota.

Dr. C. M. Cramlet, National Research Fellow, has been appointed assistant professor of mathematics at the University of Washington.

Associate Professor J. C. Fitterer, of the Colorado School of Mines, has been promoted to a professorship.

Dr. William Gafafer has been appointed research associate at Johns Hopkins University.

Mr. W. H. Gage, of Victoria College, British Columbia, has been promoted to an assistant professorship of mathematics.

Associate Professor C. A. Garabedian, of the University of Cincinnati, has been appointed associate professor of mathematics at St. Stephen's College, Columbia University.

Professor W. V. N. Garretson, of Ouachita College, Arkadelphia, has been appointed associate professor of mathematics at the Oklahoma Agricultural and Mechanical College.

Assistant Professor H. C. Hicks, of the University of Oregon, has been appointed professor of mathematics and aeronautics at Texas Technological College.

Assistant Professor J. W. Hurst, of Montana State College, has been promoted to an associate professorship.

Dr. C. G. Jaeger, of the University of Missouri, has been appointed to an assistant professorship at Tulane University.

Assistant Professor R. A. Johnson, of Hunter College, has been promoted to an associate professorship.

Miss Elizabeth Knight, of the Milwaukee State Normal School, has been promoted to an assistant professorship.

Miss Edna Kramer has been appointed assistant professor of mathematics at the New Jersey State Teachers College, Upper Montclair.

Assistant Professor C. H. Langford, of the University of Washington, has been appointed associate professor of philosophy at the University of Michigan.

Mr. T. R. Long has been promoted to an assistant professorship at the University of Rochester.

Associate Professor C. E. Love has been promoted to a professorship at the University of Michigan.

Mr. C. I. Lubin, of the University of Cincinnati, has been promoted to an assistant professorship.

Mr. C. T. Male, of Union College, has been promoted to an assistant professorship.

Mr. A. E. Meder has been promoted to an assistant professorship at the New Jersey College for Women, Rutgers University.

Dr. T. W. Moore, National Research Fellow, has been appointed research associate at Princeton University.

Dr. G. W. Myers, professor of the teaching of mathematics and astronomy in the school of education at the University of Chicago, has retired.

Mr. G. A. Parkinson, of the University of Wisconsin, has been promoted to an assistant professorship.

Mr. C. S. Porter, of Amherst College, has been promoted to an associate professorship.

Associate Professor B. P. Reinsch, of Southern Methodist University, has been promoted to a professorship of mathematics.

Assistant Professor H. P. Robertson, of the California Institute of Technology, has been appointed assistant professor of mathematical physics at Princeton University.

Dr. I. M. Sheffer, National Research Fellow, has been appointed assistant professor of mathematics at Pennsylvania State College.

Assistant Professor J. H. Taylor, of the University of Wisconsin, has been appointed professor of mathematics at George Washington University.

Mr. G. C. Vedova has been promoted to an assistant professorship at St. John's College, Annapolis.

Assistant Professor R. L. Wilder, of the University of Michigan, has been promoted to an associate professorship.

Mr. F. G. Williams, of Cornell University, has been appointed professor of mathematics at Susquehanna University.

The following appointments to instructorships are announced:

Columbia University, Dr. A. A. Albert.

College of the City of Detroit, Dr. K. W. Folley, Dr. D. C. Morrow.

Harvard University, Mr. A. E. Andersen, Mr. A. E. Currier, Mr. G. A. Hedlund, Mr. S. H. Kimball, Mr. G. B. Morrey, Mr. S. B. Myers, Mr. G. Sauté, Mr. G. B. Van Schaack, Mr. C. Wexler.

Long Island University, Mr. E. A. Knobelauch.

Purdue University, Mr. E. L. Klinger, Mr. J. M. Thompson.

Rice Institute, Dr. L. M. Blumenthal.

Worcester Polytechnic Institute, Mr. S. A. Lepeshkin.

Professor Henri Andoyer, of the Sorbonne, member of the Paris Academy of Sciences in the section of astronomy, died June 12, 1929, at the age of sixty-six.

Dr. Karl Heun, professor of theoretical mechanics at the Karlsruhe Technical School, died January 10, 1929.

Professor E. Hoppe, of Göttingen, known for his work in the history of mathematics, died August 12, 1928.

George Birtwistle, fellow of Pembroke College, Cambridge, died May 19, 1929, at the age of fifty-two.

Sir George Knibbs, of Melbourne, Australia, formerly director of the Commonwealth Institute of Science and Industry, and author of *The Mathematical Theory of Population*, died March 30, 1929, at the age of seventy.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

APPELL (P.) Sur la décomposition d'une fonction méromorphe en éléments simples. (Mémorial des Sciences Mathématiques, No. 36.) Paris, Gauthier-Villars, 1929.

BOREL (E.). Leçons sur les fonctions de variables réelles et les développements en séries de polynomes. 2e édition. Paris, Gauthier-Villars, 1928. 12+158 pp.

BRODETSKY (S.). The meaning of mathematics. London, Benn, 1929. 80 pp.

CARMICHAEL (R. D.). Théorie des nombres. Paris, Les Presses Universitaires de France, 1929. 90 pp.

CARNAP (R.). Abriss der Logistik. (Schriften zur Wissenschaftlichen Weltauffassung, Band 2.) Wien, Springer, 1929. 113 pp.

CASSINA (U.). Calcolo numerico con numerosi esempi e note storiche originali. Bologna, Zanichelli, 1928. 19+453 pp.

CERF (G.). Transformations de contact et problèmes de Pfaff. (Mémorial des Sciences Mathématiques, No. 37.) Paris, Gauthier-Villars, 1929. 58 pp.

CURRIER (C. H.) and WATSON (E.E.). A course in general mathematics. New York, Macmillan, 1929. 8+413 pp.

DEVAUX (P.). See RUSSELL (B.).

EDDINGTON (A. S.). Science and the unseen world. London, Allen and Unwin, 1929. 56 pp.

FORSYTH (A. R.). Mathematics in life and thought. Cardiff, University of Wales Press Board, 1929. 34 pp.

GAUSS (C. F.). Werke, herausgegeben von der Gesellschaft der Wissenschaften zu Göttingen. 11ten Bandes 2te Abteilung, Abhandlung 2: Über Gauss' physikalische Arbeiten (Magnetismus, Elektrodynamik, Optik), von C. Schaefer. Band 12: Varia. Kleinere Notizen verschiedenen Inhalts. Berlin, Springer, 1929. 217+415 pp.

GORDIN (J.). Untersuchungen zur Theorie des unendlichen Urteils. Berlin, Akademie-Verlag, 1929. 6+168 pp.

GRAU (J. G.). Grundriss der Logik. 3te durchgesehene und erweiterte Auflage. Leipzig, Teubner, 1929. 6+162 pp.

GRELLING (K.). See RUSSELL (B.).

GRIFFITH (C. L. T.). Logarithms of feet, inches and fractions for direct calculation in British units of length. London, Spon, 1929. 34 pp.

HJELMSLEV (J.). See MOHR (G.).

JAVELOT (—.). Exercices d'algèbre. Paris, Vuibert, 1929. 200 pp.

KEELING (S. V.). Logic and reasoning. London, Benn, 1929. 80 pp.

KEYNES (J. N.). Studies and exercises in formal logic including a generalization of logical processes in their application to complex in-

ferences. 4th edition, rewritten and enlarged. London, Macmillan, 1928. 24+548 pp.

LAMPE (E.). Mathematik und Sport. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1929. 53 pp.

LATTA (R.) and MACBEATH (A.). The elements of logic. London, Macmillan, 1929. 8+393 pp.

LÖTZBEYER (P.). Vierstellige Tafeln zum logarithmischen und Zahlenrechnen. Graphische Rechentafeln. Ausgabe B. 3te Auflage. Leipzig, Teubner, 1928.

LORIA (G.). Histoire des sciences mathématiques dans l'antiquité hellénique. Paris, Gauthier-Villars, 1929. 215 pp.

MACBEATH (A.). See LATTA (R.).

MOHR (G.). Euclides Danicus Amsterdam 1672. Mit einem Vorwart von J. Hjelmslev und einer deutschen Übersetzung von J. Pál. Udgivet af det Kgl. Danske Videnskabernes Selskab. København, Høst, 1928. 8+36+41 pp.

NEUGEBAUER (O.), STENZEL (J.), und TOEPLITZ (O.), herausgegeben von. Quellen und Studien zur Geschichte der Mathematik. Abteilung B: Studien. Band 1, Heft 1. Berlin, Springer, 1929. 112 pp.

NEVANLINNA (R.). Le théorème de Picard-Borel et la théorie des fonctions méromorphes. Paris, Gauthier-Villars, 1929. 7+174 pp.

NOEL (E.) et PRÉVOST (J.). Deux heures de mathématiques. Paris, Editions Kra, 1929.

NÖRLUND (N. E.). Leçons sur les équations linéaires aux différences finies. (Borel Monograph.) Paris, Gauthier-Villars, 1929. 6+153 pp.

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PASCAL (E.). Repertorium der höheren Mathematik. 2te Auflage, herausgegeben von E. Salkowski und H. E. Timerding. Band 1: Analysis. 3ter Teilband. Leipzig, Teubner, 1929. 12+574 pp.

PEIRCE (B. O.). A short table of integrals. 3d revised edition. Boston, Ginn, 1929. 156 pp.

PRABODH CHANDRA SENGUPTA. The Aryabhatiyam. Translation. Calcutta, University Press, 1927. 56 pp.

PRÉVOST (J.). See NOEL (E.).

RUSSELL (B.). Méthode scientifique en philosophie. Traduit de l'anglais par P. Devaus. Paris, Vrin, 1929. 24+196 pp.

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—— Sceptical essays. 2d edition. London, Allen and Unwin, 1929. 251 pp.

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SÁNCHEZ PÉREZ (J. A.). Las matemáticas en la Biblioteca del Escorial. Madrid, Real Academia de Ciencias, 1929.

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SPERNER (J.). Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes. (Abhandlungen aus dem mathematischen Seminar der Hamburger Universität, Einzelschriften, Heft 7.) Leipzig, Teubner, 1928. 88 pp.

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Timerding (H. E.). See Pascal (E.).

Toeplitz (O.). See Neugebauer (O.).

Tortorici (P.). Esercitazioni matematiche. Parte I. Palermo, Tipografia Fratelli Vena, 1929.

Watson (E. E.). See Currier (C.H.).

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Wieleitner (H.). Mathematische Quellenbücher. Band 4: Infinitesimalrechnung. Berlin, Salle, 1929. 8+160 pp.

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PART II. APPLIED MATHEMATICS

Arrhenius (S.). See van 't Hoff (J. H.).

Auerbach (F.) und Hort (W.). Handbuch der physikalischen und technischen Mechanik. Band IV, Lieferung 1; Band VI, Lieferung 2; Band VII, Lieferung 2. Leipzig, Barth, 1929, 1928, 1929. 6+198 +18+458+6+252 pp.

Banister (H.). Elementary applications of statistical method. London, Blackie, 1929. 4+57 pp.

Becker (A.). See Wien (W.).

Boffito (G.). Gli strumenti della scienza e la scienza degli strumenti. Firenze, Libreria Internazionale Seeber, 1929. 16+217 pp.

Boll (M.). L'électron et les applications de l'électricité. Paris, Albin Michel, 1929. 403 pp.

Bouligand (G.). Compléments et exercices sur la mécanique des solides. Paris, Vuibert, 1929. 132 pp.

Cajori (F.). The chequered career of Ferdinand Rudolph Hassler, first superintendent of the United States Coast Survey. A chapter in the history of science in America. Boston, Christopher, 1929. 245 pp.

Carson (J. R.). Elektrische Ausgleichungsvorgänge und Operatorenrechnung. Erweiterte deutsche Bearbeitung von F. Ollendorff und K. Pohlhausen. Berlin, Springer, 1929. 9+186 pp.

Chapel (—.). Aperçu sur le rôle des astéroïdes dans la physique du globe. Volume 1. Paris, Berger-Levrault, 1928. 16+164 pp.

Darrow (K. K.). Elementare Einführung in die Schrödingersche Wellenmechanik. Aus dem Englischen übersetzt und ergänzt durch E. Rabinowitsch. Leipzig, Hirzel, 1929. 120 pp.

Debye (P. D.). Polar molecules. New York, Chemical Catalog Company, 1929. 172 pp.

Eisner (F.). Widerstandsmessungen an umströmten Zylindern von Kreis und Brückenpfeilerquerschnitt. Berlin, Springer, 1929. 6+98 pp.

Emeléus (K. G.). The conduction of electricity through gases. London, Methuen, 1929. 10+94 pp.

Eucken (A.). See Müller (J.), Wien (W.).

Försterling (K.). Lehrbuch der Optik. Leipzig, Hirzel, 1928. 610 pp.

Fortrat (R.). Introduction à l'étude de la physique théorique. Fascicule 4: Electricité et magnétisme. Paris, Hermann, 1929. 180 pp.

Fric (R.). See Thomson (J. J.).

Haas (A.). Atomtheorie. 2te, völlig umgearbeitete und wesentlich vermehrte Auflage. Berlin, de Gruyter, 1929. 258 pp.

Harms (F.). See Wien (W.).

Herold (S. C.). Analytical principles of the production of oil, gas and water from wells. A treatise based upon a system of fluid mechanics particularly adapted to the study of the performance of natural reservoirs. With a foreword by C. F. Tolman and a final summary by E. K. Parks. Stanford University, Stanford University Press, 1928. 19+659 pp.

van 't Hoff (J. H.) and Arrhenius (S.). The foundations of the theory of dilute solutions. Papers on osmotic pressure, by J. H. van 't Hoff, and on electrolytic dissociation, by S. Arrhenius. Edinburgh, Oliver and Boyd, 1929. 67 pp.

Hort (W.). See Auerbach (F.)

Inchley (W.). The theory of heat engines. Revised by A. Morley. 3d edition. London and New York, Longmans, 1929. 12+504 pp.

Jeans (J. H.). Astronomy and cosmogony. 2d edition. Cambridge, University Press, 1929. 10+428 pp.

Jeffreys (H.). The earth. Its origin, history and physical constitution. Cambridge, University Press, 1929. 346 pp.

Joos (G.). Atomphysik und Sternphysik. Antrittsvorlesung. Jena, Fischer, 1929. 15 pp.

Kammüller (K.). Die Theorie der Gewichtsstaumauern unter Rücksicht auf die neueren Ergebnisse der Festigkeitslehre. Berlin, Springer, 1929. 7+60 pp.

Kirchberger (P.). Die Entwicklung der Atomtheorie. 2te verbesserte und erweiterte Auflage. Karlsruhe-Baden, C. F. Müller, 1929. 294 pp.

Klemin (A.) and Titterton (G. F.). Airplane stress analysis. An introductory treatise. New York, Ronald Press, 1929. 278 pp.

Knoblauch (O.). See Wien (W.).

von Laue (M.). See Voigt (W.).

Lawrence (A. S. C.). Soap films. A study of molecular individuality. London, Bell, 1929. 11+141 pp.

Lenz (H.). See Wien (W.).

Levinson (H. C.) and Zeisler (E. B.). The law of gravitation in relativity. Chicago, University of Chicago Press, 1929. 116 pp.

Lummer (O.). See Müller (J.).

McFarland (E.). Textbook of ordnance and gunnery. New York, Wiley, 1929. 10+625 pp.

Magnus (A.). Lehrbuch der Thermodynamik für Studierende der Chemie und verwandter Wissenschaften. Leipzig, Akademische Verlagsgesellschaft, 1929. 12+288 pp.

Malsch (F.). Grundrisse der darstellende Geometrie. Leipzig, Quelle und Meyer, 1928. 113 pp.

MAYES (C.). Electricity and magnetism. London, Dent, 1929. 7+203 pp.

MORLEY (A.). See INCHLEY (W.).

MÜLLER (C.). See WIEN (W.).

MÜLLER (J.) und POUILLET (C. S. M.). Lehrbuch der Physik. 11te Auflage, herausgegeben von A. Eucken, O. Lummer, und E. Waetzmann. In 5 Bänden. Band I: Mechanik und Akustik. 1ter, 2ter und 3ter Teil, herausgegeben von E. Waetzmann. Braunschweig, Vieweg, 1929. 16+848+8+410+12+484 pp.

MUNK (M. M.). Fundamentals of fluid dynamics for aircraft designers. New York, Ronald Press, 1929. 198 pp.

NEWTON (I.). Optics; or, a treatise of the reflections, inflections and colours of light. The third edition, corrected. In Russian, translated by S. I. Vavilof. Moscow and Leningrad, Gosudarstvennia Izdatelstvo, 1927. 373 pp.

OLLENDORFF (F.). See CARSON (J. R.).

PARKS (E. K.). See HEFOLD (S. C.).

PLUMMER (H. C.). The principles of mechanics. An elementary course. London, Bell, 1929. 12+307 pp.

PÖSCHL (T.). Methoden und Probleme der Elastizitätstheorie. Braunschweig, Vieweg, 1929. 23 pp.

POHL (R. W.). Einführung in die Elektrizitätslehre. 2te, verbesserte Auflage. Berlin, Springer, 1929. 7+259 pp.

POHLHAUSEN (K.). See CARSON (J. R.).

POUILLET (C. S. M.). See MÜLLER (J.).

RABINOWITSCH (E.). See DARROW (K. K.).

RAMSEY (A. S.). Dynamics. A text-book for the use of the higher divisions in schools and for the first year students at the universities. Cambridge, University Press, 1929. 12+259 pp.

REIHER (H.). See WIEN (W.).

REINICKE (R.). Über die Veranschaulichungsmöglichkeit des gesamten Spektralgebietes durch ein gemeinsames kubisches Elektronengitter. München-Schwabing, Selbstverlag, 1928. 16 pp.

ROTHE (H.). See WIEN (W.).

ROY (L.). Problèmes de statique graphique et de résistance des matériaux. Paris, Gauthier-Villars, 1929. 8+119 pp.

SANDGATHE (F.). Die absolute Zeit in der Relativitätstheorie. Ein raumzeitlicher Umbau der Relativitätstheorie. Berlin, Heymann, 1928. 132 pp.

SCHAEFER (C.). Einführung in die theoretische Physik. 3te, verbesserte und vermehrte Auflage. Band 1. Berlin, de Gruyter, 1929. 12+991 pp.

SCHOTTKY (W.). Thermodynamik. Die Lehre von den Kreisprozessen der physikalischen und chemischen Veränderungen und Gleichgewichten. Eine Einführung zu den thermodynamischen Problemen unserer Kraft- und Stoffwirtschaft. Berlin, Springer, 1929. 25+619 pp.

—— See WIEN (W.).

SCHWEIDLER (E.). See WIEN (W.).

SERVIEN (P.). Introduction à une connaissance scientifique des faits musicaux. Paris, Blanchard, 1929. 55 pp.

Simon (H.). See Wien (W.).

Stracke (G.). Bahnbestimmung der Planeten und Kometen. Berlin, Springer, 1929. 8+365 pp.

Thomson (J. J.). La structure de la lumière. Traduit par R. Fric. Paris, Blanchard, 1929. 50 pp.

Titterton (G. F.). See Klemin (A.).

Tolman (C. F.). See Herold (S. C.).

Vahlen (T.). Deviation und Kompensation. Neue Grundlegung der Theorie, neue Anwendung auf die Praxis. Braunschweig, Vieweg, 1929. 8+188 pp.

Vavilof (S. I.). See Newton (I.).

van Veen (H. J.). Leerbook der beschrijvende meetkunde. Deel II. Groningen, Noordhoff, 1929.

Voigt (W.). Lehrbuch der Kristallphysik. Nachdruck der ersten Auflage ergänzt durch eine spätere Arbeit des Verfassers und mit einem Geleitwort von M. von Laue. Leipzig, Teubner, 1928. 26+978 pp.

Waetzmann (E.). See Müller (J.).

von Wartenberg (H.). See Wien (W.).

Wien (W.) und Harms (F.), herausgegeben von. Handbuch der Experimentalphysik. Band 8, Teil 1: Energie und Wärmeinhalt, bearbeitet von A. Eucken. Band 9, Teil 1: Hohe und tiefe Temperaturen, von H. von Wartenberg; Gasverflüssigung und ihre thermodynamischen Grundlagen, von H. Lenz; Wärmeleitung, von O. Knoblauch und H. Reiher; Wärmestrahlung, von W. Wien und C. Müller. Band 13, Teil 1: Ionenleitung in Gasen, von E. Schweidler; Die elektrischen Eigenschaften der Flamme, von A. Becker. Band 13, Teil 2: Physik der Glühelektroden, von W. Schottky und H. Rothe; Herstellung der Glühelektroden, von H. Simon; Technische Elektronenröhren und ihre Verwendung, von H. Rothe. Leipzig, Akademische Verlagsgesellschaft, 1929, 1929, 1929, 1928. 15+736+14+484+8+314+492 pp.

Wintner (A.). Spektraltheorie der unendlichen Matrizen. Einführung in den analytischen Apparat der Quantenmechanik. Leipzig, Hirzel, 1929. 8+270 pp.

Zander (W.). Der Einfluss von Oberflächenbeschädigungen auf die Biegungsschwingungsfestigkeit. Berlin, Nem-Verlag, 1929. 3+65 pp.

Zeisler (E. B.). See Levinson (H. C.).

THE THIRTY-FIFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The thirty-fifth summer meeting and thirteenth colloquium of the Society were held at the University of Colorado, Boulder, Colorado, from Tuesday to Friday, August 27–30, 1929, preceded by the summer meeting of the Mathematical Association of America.

The four colloquium lectures by Professor R. L. Moore, each of an hour and one-quarter, were delivered on Tuesday and Thursday afternoons and Thursday and Friday mornings, with ninety-one members in attendance; a report of these lectures, by Professor G. T. Whyburn, appears in the present issue of this Bulletin. On Thursday morning the Society held sectional sessions for the reading of papers, a Section of Analysis, and a Section of Algebra. General sessions were held on Tuesday afternoon and Friday morning. The retiring address of President Snyder, entitled *The problem of the cubic variety* in S_4, which has already appeared on pages 607–642 of the September-October issue of this Bulletin, was delivered on Thursday afternoon. In place of the joint dinner of the Society and the Association, there was an excursion to University Camp, where a dinner was served, after which there were informal speeches, Professor A. J. Kempner presiding.

This is the farthest west a summer meeting has been held since 1915, and in every particular it justified the high hopes of those interested in the expansion of the area of the Society's activities. Boulder is picturesquely situated, the University's ample facilities were placed unreservedly and efficiently at the disposal of the visitors, the attendance was good, and the scientific standard of the meetings was high. The scientists came from twenty-eight states and Canada, representing nearly all sections of the country. Especially memorable among mathematical outings were the dinner at University

Camp in the mountains on Tuesday evening and the day trip on Wednesday to the continental divide at Fall River Pass, nearly twelve thousand feet above sea level.

At the concluding session, by a rising vote, resolutions of thanks were passed to the President and Regents of the University, to the members of the department of mathematics, to those who furnished transportation into the mountains, and to the Boulder Country Club.

The college fraternity houses were opened for lodging the visitors, and meals were served at a cafeteria near the campus, all at modest rates.

The attendance included the following one hundred nine members of the Society:

Bacon, S. R. Benedict, William James Berry, Brink, Bruce, H. E. Buchanan, Burnam, Bussey, Cairns, C. C. Camp, A. G. Clark, Abraham Cohen, Teresa Cohen, Colyer, L. P. Copeland, C. F. Craig, Curtiss, D. R. Davis, H. T. Davis, Dean, Denton, Dines, Dodd, Edmonson, Epsteen, Erikson, H. P. Evans, Fitterer, L. R. Ford, W. B. Ford, Fry, Gaba, J. L. Gibson, Gouwens, Griffin, Griffiths, W. L. Hart, Hazard, E. R. Hedrick, Hennel, Herr, Hicks, Hildebrandt, J. C. Hughes, Hurst, C. A. Hutchinson, Ingraham, Dunham Jackson, Kempner, Kendall, Kindle, Kuhn, Latimer, D. N. Lehmer, Lester, A. J. Lewis, Lubben, McFarland, McMaster, Mickelson, W. E. Milne, U. G. Mitchell, Mossman, R. L. Moore, W. K. Nelson, Newton, Nyswander, H. L. Olson, F. W. Owens, H. B. Owens, Pierce, Pierpont, M. B. Porter, Rambo, Raynor, Rawles, Rechard, Remick, R. G. D. Richardson, Richert, H. L. Rietz, J. H. Roberts, T. G. Rodgers, Roever, Rutt, Shewhart, Sisam, Skinner, Smail, C. E. Smith, E. R. Smith, G. W. Smith, R. G. Smith, Virgil Snyder, Spooner, Eugene Stephens, Stouffer, Tappan, Tripp, Tyler, L. E. Ward, Weida, Weiss, Wester, J. J. Wheeler, M. B. White, G. T. Whyburn, Winger, Zippin.

The secretary announced the entrance of the following eighty persons to membership in the Society:

Mr. James Edward Barker, University of California;
Mr. Amos Hale Black, West Virginia University;
Miss Dorothy Stahl Brady, New York University;
Mr. Frederick James Brand, University of Washington;
Mr. Joseph Hobart Bushey, University of Michigan;
Mr. Lloyd Cline Christianson, Texas Technological College;
Miss Nancy Cole, Radcliffe College;
Mr. Hollis R. Cooley; New York University;
Mr. Prescott Durand Crout, Milwaukee, Wisconsin;
Mr. Albert Eldred Currier, Harvard University;
Mr. Edward Hutchins Cutler, Harvard University;

Dr. Dewey Colfax Duncan, University of California;
Mr. John Dyer Elder, California Institute of Technology;
Mr. Orin James Farrell, Pennsylvania State College;
Mr. Arne Fisher, Western Union Telephone and Telegraph Company, New York City;
Miss Claire Beatrice Fisher, New York University;
Miss Elizabeth Agnes Flood, University of Montana;
Mr. David Herman Frank, George Washington High School, New York City;
Mr. Kenneth Gary Fuller, Northwestern University;
Mr. Michael Goldberg, Bureau of Ordnance, United States Navy;
Mr. Jacob Goodman, Rutgers University; Mr. Edward Kenneth Haviland, Harvard University;
Mr. William Jonathan Hazard, University of Colorado;
Mr. Charles Harold Helliwell, New York University;
Mr. Magnus Rudolph Hestenes, University of Nebraska;
Mr. John James Luett Hinrichsen, Iowa State College;
Miss Catherine Hood, Colorado College;
Professor Borden P. Hoover, Carnegie Institute of Technology;
Professor Frederick W. John, New York University;
Professor Francis Edgar Johnston, George Washington University;
Dr. Franklin Wesley Kokomoor, University of Florida;
Professor Michael Kovalenko, Swarthmore College;
Mr. Lonnie Langston, Texas Technological College;
Professor Victor F. Lenzen, University of California;
Mr. Arthur James Lewis, University of Denver;
Mr. Chin-Nien Liu, Nan Kai University, Tienstin, China;
Professor Hilton Frank Lusk, College of the Pacific;
Dr. Albert Joseph McConnell, Trinity College, Dublin, Ireland;
Miss Dorothy McCoy, State University of Iowa;
Mr William Milton Maiden, New York University;
Mr. Monroe Harnish Martin, Johns Hopkins University;
Professor Wendell Earle Mason, University of California at Los Angeles;
Mr. Frederic Howell Miller, Columbia University;
Dr. Alfred King Mitchell, Johns Hopkins University;
Mr. Morris Monsky, Mutual Life Insurance Company, New York City;
Dr. Walter Lee Moore, University of Illinois;
Mr. William Richard Morgans, California Institute of Technology;
Dr. William Kelso Morrill, Johns Hopkins University;
Mr. David Moskovitz, Carnegie Institute of Technology;
Mr. J. Herbert Moss, New York University;
Professor Walter Kenneth Nelson, University of Colorado;
Mr. James Edward O'Connell, Morris Preparatory School, Little Rock, Arkansas;
Professor Edwin Glenn Olds, Carnegie Institute of Technology;
Miss Emma Julia Olson, Northwestern University;
Mr. Charles Kenneth Payne, New York University;
Dr. Joseph William Peters, University of Illinois;

Professor Saul Pollock, Coast Artillery Corps, Fort Collins, Colorado;
Mr. Griffith Baley Price, Harvard University;
Professor A. W. Richeson, University of Maryland;
Professor Ramon Salas-Edwards, University of Chile, Santiago, Chile;
Mr. William Francis Shaw, vice-president, American Rio Grande Land and Irrigation Company;
Miss C. Grace Shover, Ohio State University;
Mr. Clifford Edward Smith, Fresno State College;
Professor Turner L. Smith, Carnegie Institute of Technology;
Mr. Clyde Allen Spicer, Johns Hopkins University;
Dr. Lee Houghton Swinford, University of California;
Professor Herbert Tate, McGill University;
Mr. Henry Peter Thielman, Ohio State University;
Mr. William Rae Thompson, Yale University;
Professor Arthur Tilley, New York University;
Professor Louis Russell Weber, Friend's University, Wichita;
Mr. Edward Henry Wells, Princeton University;
Professor George Warren White, College of the Pacific, Stockton, California;
Professor Edwin A. Whitman, Carnegie Institute of Technology;
Mr. Hassler Whitney, Yale University;
Mr. Charles Robert Wilson, Rutgers University;
Mr. Herbert Paul Wirth, College of the City of New York;
Mr. Kamcheung Woo, University of California;
Professor Leon P. Woods, Northeastern State Teachers College, Tahlequah, Oklahoma;
Dr. Minoru Yoshida, Sendai, Japan.

Eight applications for membership were received.

The Secretary also announced the following election to sustaining membership:

Members of the Department of Mathematics in Brown University.

The following appointments were announced: to represent the Society at the inauguration of President Clarence A. Barbour at Brown University, Professor G. D. Birkhoff; to represent the Society at the fiftieth anniversary celebration of the American Society of Mechanical Engineers, Dr. G. A. Campbell and Professor E. V. Huntington; as Committee on Arrangements for the summer meeting of 1930, at Brown University, Professors C. R. Adams (chairman), J. L. Coolidge, C. H. Currier, R. G. D. Richardson and H. W. Tyler.

It was announced that for 1929–30 the National Academy of Sciences had allotted to the Society from funds donated

by the General Education Board the sum of $4500 to be distributed as follows: Bulletin, $500; Transactions, $1500; American Journal of Mathematics, $1500; Colloquium, $1000.

The list of nominations for officers and other members of the Council was adopted and ordered printed on the ballot.

Titles and abstracts of the papers read at this meeting follow below. The papers numbered 1 to 7 were read at the general session on Tuesday afternoon, President Hedrick presiding; the papers numbered 8 to 23 were read before the Section of Analysis, ex-President Snyder and Professor Curtiss presiding; those numbered 24 to 43 before the Section of Algebra, Vice-President Rietz and Professor Cohen presiding; those numbered 44 to 60 at the general session on Friday morning, President Hedrick and Professor Tyler presiding. Dr. W. D. Baten was introduced by Professor J. A. Shohat and Professor Uspensky by the Secretary. The papers of Ayres, Campbell, Carson, Garver, Hill, Kempner, Langer, LaPaz, Lehmer, Lubben, McDonnell, Marden, Poritsky, Rainich, Robinson, Roth, Rowe, Rutledge, Simmons, Thielman, Vandiver, Ward (second paper), Wilder and Wilson were read by title.

1. Professor J. A. Nyswander: *On the solutions of linear differential equations.*

This paper gives a simple and direct derivation of the form of the solution of a general system of linear differential equations in the vicinity of a point $t=0$ which is an isolated uniform singularity for the coefficients of the system of differential equations. Solutions of the system in the vicinity of a regular point $t=t_0$ are determined so that after a circuit about the point $t=0$, certain specified changes occur in the solutions. In order that these changes shall result from analytic continuation about the singular point $t=0$ certain relations among the coefficients of the solutions must exist. That these relations can be satisfied is proved by differentiating a set of identities obtained from the characteristic determinant. The form of the solutions is established under entirely general hypotheses regarding the roots of the characteristic equation, without recourse to the theory of transformations.

2. Professor C. H. Sisam: *On three-dimensional varieties with six generations by right lines.*

In this paper, the author shows that the varieties mentioned in the title are cubic hypersurfaces in space of four dimensions.

3. Professor G. Y. Rainich: *Decomposition of a symmetric tensor in space-time.*

It is proved in this paper that in four-dimensional pseudo-euclidean space a non-singular symmetric tensor of rank two can be split up in only one way into two parts which possess respectively the formal properties of the hydrodynamical tensor (tensor of matter) and the electromagnetic stress-energy tensor. Applying this result and results previously established to the contracted Riemann tensor, it follows that curvature of space-time determines uniquely the field of matter as well as the electromagnetic field. In the singular case the decomposition is not unique: one of the tensors (each of which represents in a sense radiation) can be chosen to a certain degree arbitrarily.

4. Professor L. L. Dines: *A mininum problem from physiology.*

The sensitive portion of the retina consists of a collection of circular surfaces ("cone" surfaces) of equal known radii, uniformly distributed, the intervening area being non-sensitive. Assuming known the size of the smallest circular image which can be seen, it is desired to know what is the minimum area of cone surface which such an image can occupy. This question, proposed by a colleague of the author, is considered in the present note.

5. Professor E. L. Dodd: *Probability as expressed by asymptotic limits of pencils of sequences.*

In order to deal with probability without recourse to the much criticized notion of "cases equally likely," a set of sequences is introduced, called a *pencil of sequences*, inasmuch as they radiate (figuratively speaking) from a common origin. If H means "heads," and T means "tails," an individual sequence, $HHTH \cdots$ is not conceived as giving rise to any limit. But for a pencil an actual limit is involved when only an evanescent percentage of the sequences fail to conform to certain inequalities. An exact and rather simple formulation—avoiding all non-mathematical phrases and conceptions, indeed, *avoiding the word "probability" itself*—is given to the Bernoulli theorem, its extensions, and its inverse. This involves only the counting of sequences, and the use of ordinary limits. This may give the student of probability a clearer conception of the Bernoulli inverse than he gets from the following statement: With arbitrary *a priori* probabilities, there is ultimately a very high *a posteriori* probability that the *empirical* probability will closely approximate that *unknown* probability that gives rise to the *productive* probability. References are made to the work of Cantelli, von Mises, Borel, Lomnicki, and Steinhaus.

6. Professor W. E. Milne: *On the degree of convergence of expansions on an infinite interval.*

The author studies the rate of convergence of expansions in terms of the characteristic functions which are associated with the differential equation $d^2u/dx^2+(\lambda-q(x))u=0$ and which are also subject to the conditions $\lim_{x=\pm\infty}u=\lim_{x=\pm\infty}u'=0$. With appropriate restrictions of the function $q(x)$ an approximate formula is obtained relating the nth characteristic number λ_n with n. In particular it is found that the order of magnitude of λ_n is always less than n^2. It is shown that if $f(x)$ has a continuous kth derivative of bounded variation in the infinite interval, and satisfies certain conditions at infinity, then the remainder after n terms of the expansion of $f(x)$ is $O(\lambda^{-k/2})$ in any fixed finite interval.

7. Professor H. T. Davis: *Properties of the Laurent expansion of differential operators of infinite order with constant coefficients.*

This paper exhibits some simple properties of the development of the symbolic operator $G(z)=1/F(z)$, in which we abbreviate $F(z)=a_0+a_1z+a_2z^2+\cdots$, where $z=d/dx$, in annuli formed by concentric circles about the origin. The following theorem is proved: If $H(z)$ designates a Laurent expansion of $G(z)$ in an annulus formed by two concentric circles about the origin and if $G(z)$ is any other expansion about the origin, then the function $U(x)=\{H(z)-G(z)\}\rightarrow f(x)$, where $f(x)$ is arbitrary to within the limits of the existence of the right hand member, is a solution of the homogeneous equation, $F(z)\rightarrow u(x)=0$. The case where $G(z)$ is a rational function is shown to yield the fundamental operator of the Heaviside operational calculus when the expansion is that in the region exterior to all the poles of the function.

8. Dr. W. D. Baten: *The evaluation of certain definite integrals by the use of probability functions.*

The object of this paper is to present three methods of evaluating certain definite integrals by using probability functions. The first method consists in finding the probability law or function for the sum of n independent variables, which are each subject to given probability laws, by Mayr's method and comparing this with the probability for the same sum obtained by Dodd's method. Method two consists in finding the probability law for n independent variables, which is expressed as an integral, and then allowing n to be equal to 1. Method three is founded on setting up a probability function containing the integral to be evaluated as a factor, determining a second probability function defined over the same interval, and then finding a third probability function for a chosen relation of the variables under consideration. The integral need not be a factor of the first function, but should arise somewhere in the process. The first two methods, as far as the writer knows, seem to be new.

9. Professor D. R. Davis: *Integrals whose extremals are a $2n$-parameter family of curves.*

One considers here a given $2n$-parameter family of arcs of the form

$y_i = y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n)$ $(i = 1, \cdots, n)$, and seeks the properties which are sufficient to ensure that they are the extremal arcs of an integral of the form $I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$. These properties are determined under the hypothesis that the differential equations $y_i'' = F_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$, found by eliminating the $2n$ parameters $a_1, \cdots, a_n, b_1, \cdots, b_n$, are the solutions for y_i'' of a system of equations of the form $H_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0$, whose equations of variation are self-adjoint. The general type of integral whose extremals are the arcs $y_i = a_i x + b_i$ $(i = 1, \cdots, n)$ is completely determined.

10. Professor D. R. Davis: *Determining integrals whose extremals are parabolas.*

In this paper the inverse problem for the trajectory of a particle under the action of gravity is considered. Properties of its characteristic differential equations are developed and a general type of solution is obtained. From the latter the integrals desired are completely determined.

11. Professor W. L. Hart: *Infinite systems of non-linear equations and related integral equations.*

Let x and y represent points in real Hilbert space of infinitely many dimensions. Without employing convergent infinite determinants, the author establishes the unique existence of a solution $x(y)$ of the following infinite system of equations (1): $f_1(x, y) = 0$ $[x \equiv (x_1, x_2, \cdots); y \equiv (y_1, y_2, \cdots); i_1 1, 2, \cdots]$ in a neighborhood of a solution $(x = a, y = b)$. The most important of the hypotheses used are that the matrix $F \equiv (\partial f(x, y)/\partial x)$ is limited in the Hilbert sense, that $F(a, b)$ possesses a unique limited reciprocal, and that $[F(x, y) - F(a, b)]$ possesses a bound $g(x, y)$, continuous at (a, b), where $g(a, b) = 0$. By use of the properties of limited multilinear forms, it is demonstrated that the results for (1) can be applied in the case of various infinite systems of non-linear equations in infinitely many unknowns. Each non-linear system considered can be used to establish the existence of continuous solutions of a certain non-linear integral equation. The transition from an integral equation to a system of the form (1) is made by the usual method which introduces the Fourier constants of the continuous functions involved relative to some complete orthogonal system.

12. Mr. C. M. Erikson: *A direct solution of systems of homogeneous linear difference equations having constant coefficients.*

The paper gives in detail the method and theory of obtaining the solutions of simultaneous systems of linear difference equations with constant coefficients. The first case treated is the simple one in which the characteristic equation, $D(\lambda) = 0$, has distinct roots. The solutions are expressed in terms of first minors of the characteristic determinant $D(\lambda_i)$. The next case considered is the one in which $D(\lambda) = 0$ has a triple root. The most general case is then treated, under hypotheses that include every

special case that could arise. In this case the solutions are obtained corresponding to a multiple root $\lambda=\lambda_i$ of the characteristic equation $D(\lambda)=0$, where entirely general assumptions are made regarding the vanishing of minors of different orders in the characteristic determinant $D(\lambda_i)$. The solutions thus obtained are shown to form a fundamental set. The problem is treated without making use of the theory of transformations.

13. Professor C. C. Camp: *On multiparameter expansions associated with a differential system and auxiliary conditions at several points in each variable.*

For $p=2$ the system $dX_i/dx_i+\sum_{j=1}^{p}\lambda_j a_{ij}(x_i)X_i=0$, $(i=1, 2, \cdots, p)$, and boundary conditions of the Wilder type in each lead to suitable characteristic solutions provided the a_{ij} are properly restricted. It is sufficient if $a_{ij}(x_i)$ is a real, integrable function, either identically zero or of constant sign, which maintains its mean value in each segment of the boundary interval $a_i \leqq x_i \leqq b_i$, and the determinant $\left|\int_{a_i}^{b_i} a_{ij}dx_i\right| \neq 0$. If the k_i segments have lengths commensurable with b_i-a_i and are the same for $i=1, 2$, the factors containing one parameter may be eliminated by Sylvester's method, so that the characteristic values of the other will be zeros of an entire function. For $k_i>3$ the restrictions on $a_{ij}(x_i)$ may be lightened in the interior subintervals.

14. Dr. Lincoln LaPaz: *Problems of the calculus of variations with prescribed transversality conditions.*

The transversality condition for a problem of minimizing the integral

$$I=\int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y'_{1n}, \cdots, y'_n)dx \tag{1}$$

is

$$(f-y'_\mu\partial f/\partial y'_\mu)dx+\partial f/\partial y'_\mu dy_\mu=0 \tag{2}$$

(μ is umbral, ranging from 1 to n). In this paper it is shown that a relation

$$dx+T_\mu\,(x, y_1, \cdots, y_n, y'_1, \cdots, y'_n)\,dy'_\mu=0, \quad 1+y'_\mu T_\mu\neq 0, \tag{3}$$

can be the transversality relation (2) of a problem (1) only in case the T_i are such that a certain system of n linear homogeneous partial differential equations is a complete system. Necessary and sufficient conditions for this system to be complete are that

$$y'_\mu(T_m\partial T_k/\partial y'_\mu-T_k\partial T_m/\partial y'_\mu)+(\partial T_k/\partial y'_m-\partial T_m/\partial y'_k)=0, \tag{4}$$

$(m, k=1, \cdots, n, m<k)$. For n functions T_i satisfying (4) the most general integral (1) for which (3) is the transversality condition is determined. The results obtained supplement those found by a different method by Rawles (Transactions of this Society, vol. 30 (1928), pp. 765–784).

15. Dr. Lincoln LaPaz: *An inverse problem of the calculus of variations for multiple integrals.*

The Lagrange equation of a problem of minimizing an n-tuple integral $J=\int_{(n)} f(x_1, \cdots, x_n, z, p_1, \cdots, p_n)dx_1 \cdots dx_n$, $(p_i=\partial z/\partial x_i)$, must be of the

form $A_{\mu\nu}(x_1, \cdots, x_n, z, p_1, \cdots, p_n)\, p_{\mu\nu} + B(x_1, \cdots, x_n, z, p_1, \cdots, p_n) = 0$, $(p_{ij} = \partial^2 z/\partial x_i \partial x_j,\ A_{ij} = A_{ji},\ i, j, \mu, \nu = 1, \cdots, n,\ \mu$ and ν umbral), with an equation of variation which is self-adjoint on every hypersurface $z = z(x_1, \cdots, x_n)$. It is shown that if an equation $F(x_1, \cdots, x_n, z, p_1, \cdots, p_n, p_{11}, \cdots, p_{nn}) = 0$ has the above properties, there is always an integral J having $F = 0$ as its Lagrange equation. The most general such integral has an integrand function of the form $f = C(x_1, \cdots, x_n, z, p_1, \cdots, p_n) + D(x_1, \cdots, x_n, z) + E_\mu(x_1, \cdots, x_n, z)p_\mu + \partial w_\mu/\partial x_\mu$, where C is a particular solution of the system $\partial^2 C/\partial p_i \partial p_j = A_{ij}$, and (D, E_i) is a particular solution of $\partial D/\partial z - \partial E_\mu/\partial x_\mu = \partial^2 C/\partial p_\mu \partial x_\mu + (\partial^2 C/\partial p_\mu \partial z)p_\mu - \partial C/\partial z - B$, and where the w_i occurring in the sum of total derivatives with respect to x_i are arbitrary functions of $x_1, \cdots, x_n, z$ alone. The method of proof is similar to that used by Davis (Transactions of this Society, vol. 30 (1928), pp. 710–736) in connection with an analogous question for simple integrals $I = \int_{x_1}^{x_2} f(x, y, z, y', z')dx$.

16. Mr. H. P. Thielman: *The application of fractional operations to a class of integral equations occurring in physics.*

This paper is concerned with the theory of a class of Volterra integral equations of the first kind occurring in physics. The methods used are those of fractional operators and Volterra's calculus of composition. The theorems and results of Wrinch and Nicholson (Philosophical Magazine, (7), vol. 4(1927), pp. 531–560) are obtained as special cases of the author's more general and elegant theory.

17. Mr. L. B. Robinson: *On a lacuna in the theory of systems of partial differential equations.*

In his book *Les Systèmes d'Equations aux Dérivées Partielles*, Riquier examines in detail the passivity conditions of such systems, but does not consider the case where the passivity conditions entrain relations among the independent variables. The author has partly succeeded in filling in this lacuna. Consider the three equations

(1) $\quad \partial u/\partial x = f(x, y, z, u),\ \partial u/\partial y = \phi(x, y, z, u),\ \partial u/\partial z = \psi(x, y, z, u).$

This case illustrates that for n equations. Supposing certain relations between the independent variables

$$(2)\qquad \begin{aligned} &\partial u/\partial x = f(x,\ \eta(x),\ \zeta(x);\ u),\\ &\partial u/\partial y = \phi(\eta_{-1}(y),\ y,\ \zeta[\eta_{-1}(y)];\ u),\\ &\partial u/\partial z = \psi(\zeta_{-1}(z),\ \eta[\zeta_{-1}(z)],\ z;\ u), \end{aligned}$$

where η and η_{-1} and ζ and ζ_{-1} are pairs of inverse functions such that

$$(A)\qquad \begin{aligned} &\eta_{-1}(y)dy = x\eta'(x)dx,\ \zeta_{-1}(z)dz = x\,\zeta'(x)dx,\\ &\eta[\zeta_{-1}(z)]dz = \eta(x)\zeta'(x)dx,\quad \zeta[\eta_{-1}(y)]dy = \zeta(x)\eta'(x)dx, \end{aligned}$$

u can be computed in terms of x and definite integrals involving $\eta(x)$ and $\zeta(x)$, or in terms of y and definite integrals involving $\eta_{-1}(y)$ and $\zeta[\eta_{-1}(y)]$, or in terms of z and definite integrals involving $\zeta_{-1}(z)$ and $\eta[\zeta_{-1}(z)]$, the definite integrals arising in the process of solving system (2) by successive approximations. Now keeping in mind the equalities (A), we find $\eta(x)$

and $\zeta(x)$ by forming two integro-differential equations which can easily be solved, as the kernals do not depend on the upper limits of integration. This method can be extended to systems of equations of the second order.

18. Professor George Rutledge: *The basic power series of interpolation.*

The coefficients of the polynomial $P^{(2n)}(x)$, determined by the values $f(-n), \cdots, f(0), \cdots, f(n)$, are expressed in the form of series the terms of which are independent of n. Conditions for convergence of $P^{(2n)}(x)$ are readily derived from this form of expansion of the coefficients. This paper is a direct continuation of the paper of similar title published in the Transactions of this Society (vol. 26 (1924), p. 113).

19. Dr. Hillel Poritsky (National Research Fellow): *On certain polynomial approximations to analytic functions of a complex variable.*

This paper deals with the convergence to $f(z)$ of certain polynomials $P_{m,n}(z)$ as n becomes infinite, where $P_{m,n}(z)$ is of degree $mn-1$ at most and is uniquely determined by mn conditions requiring that its derivatives of order $0, m, \cdots, m(n-1)$ at m given fixed points $a_1, a_2, \cdots, a_m$, no two of which are alike, be equal respectively to the same order derivatives of $f(z)$ at the same points. The case $m=1$ leads essentially to Taylor's expansion of $f(z)$ about $z=a_1$, but is not typical of the general case $m>1$. Thus, mere analyticity of $f(z)$ in any region will not suffice to ensure the convergence of $P_{2,n}(z)$ to $f(z)$ as n becomes infinite. This may be seen by putting $a_1=0$, $a_2=\pi$, and choosing $f(z)=\sin z$; $P_{2,n}(z)$ will then identically vanish. Sufficient conditions for convergence of $P_{m,n}(z)$ to $f(z)$ as n becomes infinite, it is shown, are that $f(z)$, in addition to being an integral function of z, satisfy an inequality $|f(z)|<Ce^{k|z|}, k<K$, where C, k are constants, and K is a constant determined by $a_1, a_2, \cdots, a_m$; the convergence then takes place for all z. For $m=2$, $K=\pi/|a_1-a_2|$. In the case $m=2$ necessary conditions that are nearly sufficient are also found.

20. Dr. Morris Marden (National Research Fellow): *On the zeros of solutions of a generalized Laplacian differential equation.*

This article aims to furnish specific means for describing qualitatively the distribution of the zeros of any solution of the equation

$$d^2U/dz^2+(a+b/z)\,(dU/dz)+(c+d/z)+f/z^2)\,U=0, \tag{1}$$

where z is a complex variable, but a, b, c, d, f are real constants. These means are constructed as follows: Through the transformation

$$U=z^{-b/2}e^{-az/2}w,$$

equation (1) is converted into one of the form

$$d^2w/dz^2+(A+B/z+C/z^2)w=0. \tag{2}$$

Every solution of the new equation has, with the possible exception of the

point $z=0$, the same zeros as the corresponding solution of (2). Now by use of Hille's method of "zero-free regions" (Arkiv för Matematik, vol. 18 (1924), pp. 1–56) information can be gathered which, with a few moments' additional work, enables one to locate regions where neither any particular solution of (2) nor its derivative vanishes. The efficacy of the so-constructed means is illustrated by the study of solutions of (2) with the property $W(0)=0$ or $W(+\infty)=0$, where $W(z)=\overline{w(z)}\,w'(z)$.

21. Professor R. E. Langer: *A class of irregular boundary problems of the second order.*

The characteristic parameter values for the boundary problem of an ordinary linear differential equation are determined as the roots of a relation (1): $\sum[\alpha_i]e^{\rho\beta_i}=0$, in which ρ is the parameter, the β_i are complex constants, and $[\alpha_i]=\alpha_i+\alpha_{i1}/\rho+\cdots$. With this transcendental equation there is associated in the complex plane the convex polygon which has vertices only at points of the set β_i and which contains all the points β_i in its interior or on its perimeter. The boundary problem is called "regular" if in equation (1) $\alpha_i\neq 0$ for those values of i for which β_i is a vertex of the polygon. Only for particular differential equations has the irregular boundary problem been studied. The present paper considers the problem of the general differential equation of the second order for the irregular cases in which the quantities $[\alpha_i]$ vanish to various degrees in ρ but not identically. It is shown that the associated expansions of arbitrary functions are in these cases summable by suitable means under hypotheses similar to those which ensure convergence of the expansions in the regular cases.

22. Professor L. E. Ward: *On the differentiation and integration of certain formal series.*

The series considered are those associated with the differential system $u'''+\rho^3u=0$, $u(0)=u'(0)=u(\pi)=0$. Necessary and sufficient conditions that the third derived series of the formal series for a function capable of being expanded in a series of the characteristic functions of this differential system should be the formal series for the third derivative of the function are obtained. It is proved that the threefold term-by-term indefinite integration of the formal series for a function yields the formal series for a certain function, and a formula for the latter is found.

23. Professor L. E. Ward: *On the determination of the formal coefficients in a certain expansion problem.*

If the functions $u_k(x)$ are the characteristic functions of the differential system $u'''+\rho^3u=0$, $u(0)=u'(0)=u(\pi)=0$, and $f(x)$ is a function capable of expansion in a series of these characteristic functions, and if $R_n(x)=f(x)-\sum_{k=1}^{n}a_ku_k(x)$, it is shown that $\int_0^\pi R_n(x)R_n(\pi-x)dx$ has an extremum when the a_k's are the coefficients of the formal series for $f(x)$.

24. Professor J. V. Uspensky: *On a new method for finding units in cubic rings of a negative discriminant.*

The basic idea of this new method can be best explained for the simplest

case of a cubic ring $[1, \theta, \theta^2]$ where θ denotes a real root of the equation $\theta^3=a$. Set $\phi=x+y\theta+z\theta^2$, $\phi=\phi'\phi''$, ϕ' and ϕ'' being conjugate forms to ϕ, and similarly $\omega=Z+Y\theta+Z\theta^2, \Omega=\omega'\omega''$. Following Hermite's suggestion we introduce the ternary quadratic form $f=2\phi+\phi^2/\Delta$ containing a variable positive parameter Δ, whose adjoint form differs from $F=\omega^2+2\Omega/\Delta$ by an irrelevant numerical factor. But instead of considering successive minima of the form f for a continuously increasing parameter, as Hermite suggests, it seems more advantageous to consider systems of integers x, y, z; X, Y, Z satisfying for every Δ the conditions $f\leqq 4(a^2/\Delta)^{1/3}$, $F\leqq 4(a^2/\Delta^2)^{1/3}$, $Xx+Yy+Zz=0$. It can be shown that, starting with $x=1, y=0, z=0$; $X=0, Y=0, Z=1$ for $\Delta=1$ we can reach systems x, y, z; X, Y, Z such that either $\phi=x+y\theta+z\theta^2$ or $\omega=Z+Y\theta+X\theta^2$ will be a fundamental unit. When applied to numerical examples the new method seems to be quite satisfactory and not inferior to any known method that can be used for the same purpose.

25. Professor H. S. Vandiver: *On power characters of singular integers in a properly irregular cyclotomic field.*

A properly irregular cyclotomic field is a field defined by $\zeta=e^{2\pi i/l}$, l an odd prime, which is irregular and such that certain units E_n, $n=1, 2, \cdots, (l-3)/2$ (defined as by the writer, Proceedings of the National Academy of Sciences, vol. 15 (1929), p. 202) are not lth powers of units in $k(\zeta)$. In the present paper the author shows how to determine in algebraic form the lth power characters of any singular integer in a properly irregular cyclotomic field $k(\zeta)$, with respect to any ideal in this field prime to l. Heretofore this determination has only been made for the cases where the ideals belonged to an exponent prime to l. The paper will appear in the Transactions of this Society.

26. Professor H. S. Vandiver: *On the composition of the class-group of ideals in an irregular cyclotomic field.*

If the class number h of an irregular cyclotomic field is expressed in the form $l^a g$, where g is prime to l and the cylotomic field is defined by $e^{2\pi i/l}$, with l an odd prime, then the gth powers of all ideals in the field may be separated into classes which form an abelian group under multiplication. In the present paper the invariants of this group of classes are shown to depend on the divisibility of certain Bernoulli numbers by powers of l, in the case where the cyclotomic field is properly irregular. Properties of a basis for this group are also given. The investigation gives a number of other results concerning the class-field of a cyclotomic field and a basis for the singular primary numbers in the field. The paper will appear in the Transactions of this Society.

27. Professor H. S. Vandiver: *Some properties of a certain system of independent units in a cyclotomic field.*

Pollaczek (Mathematische Zeitschrift, vol. 21 (1924), pp. 1–38) proved

that in the field $k(\zeta), \zeta = e^{2\pi i/l}$, l an odd prime, there exists a system of fundamental real units with the following property:

$$\eta_i^{s-r^{2i}} = (\eta')_i^l, \qquad (i = 1, 2, \cdots, (l-3)/2),$$

using the Kronecker-Hilbert symbolic powers. In the present note it is shown that certain units γ_i, $i = 1, 2, \cdots, (l-3)/2$, which are analogous to those defined by Hilbert (*Bericht*, p. 432), may be expressed in forms involving the η's. This paper will appear in the Annals of Mathematics.

28. Professor H. S. Vandiver: *Some theorems concerning properly irregular cyclotomic fields.*

This article contains a number of theorems concerning properly irregular cyclotomic fields. Statements of these theorems with proofs indicated appeared in Proceedings of the National Academy of Sciences, vol. 15 (1929), p. 202–207. Full proofs will appear later in the Transactions of this Society.

29. Professor H. S. Vandiver: *Summary of results and proofs on Fermat's last theorem.* Fourth Paper.

This note contains a number of results concerning Fermat's last theorem. Statements of the theorems without proofs appeared in Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 108–109. The proofs of the results will appear in the Transactions of this Society.

30. Professor Lennie P. Copeland: *Matrix conditions that the ternary cubic locus may have multiple points.*

It is the purpose of the present paper first to determine a minimum set of necessary and sufficient conditions that the general ternary cubic locus (degenerate) may have exactly two double points, and second to show that there is a locus analogous to the ternary Hessian in the case of the degenerate polar cubic, since the locus of points whose polar cubic with respect to a given quartic break up into three straight lines is expressible as the three necessary and sufficient conditions that a cubic be factorable into linear factors.

31. Professor C. G. Latimer: *On the class number of the cubic cyclotomic field.*

In a former paper, the writer determined the prime ideals in the cubic Galois, or cyclotomic, field. Employing this result, the author determines the class number of the field in terms of the regulator. The method is similar to the well known method of treating the same problem for the quadratic field.

32. Professor G. W. Smith: *Some results obtained by generalizing the analytical definition of a conic.*

The ordinary analytical definition of a conic is generalized by considering the eccentricity as a function of either or both of the coordinates of the generating point. Some general properties of the loci whose equations are

thus determined are given. Certain special functions for the eccentricity lead to interesting results in both rectangular and polar coordinates. It is shown that any curve whose equation is $f(x, y)=0$ may be considered as having a directrix and a focus, the eccentricity being variable.

33. Mr. R. G. Smith and Professor E. B. Stouffer: *A proof of a theorem of Halphen and its extension to n dimensions.*

If two curves in ordinary space have contact of order m at a point, there exists a unique plane from any point of which one can project the two curves by means of two cones with contact of order $m+1$. This theorem was proved by Halphen in 1880 (Journal de l'Ecole Polytechnique, vol. 47, p. 25). In the present paper a simpler proof of the same theorem is given, and the theorem is also extended to curves in contact in space of n dimensions.

34. Dr. Marie J. Weiss (National Research Fellow): *The limit of transitivity of a substitution group.*

In this paper inequalities giving the relation between n, the degree of a t-ply transitive group, and t, the transitivity of the group, are found. The following theorems are proved: (1) Let the order of the subgroup G_t that fixes t letters of a t-ply transitive group of degree n be divisible by an odd prime. Then if $t\geqq 8$, $(n-2t+1)/2\geqq t!/\{\delta!(t-\delta)!\}$, where δ is the greatest integer less than the quantity $t-(t-k+1)\ (\log 2)/(k+\log 2)$, k an integer such that $5\leqq k\leqq t-1$; or $(n-2t+1)/2\geqq t2^{\alpha}$, α an integer $\geqq 4$ and $\geqq k-3-\log k/\log 2$; or $(n-2t+1)/2\geqq 2^{\gamma}$, γ an integer $\geqq 4$ and $\geqq t-3-\log t/\log 2$. (2) Let G_t be of order 2^{μ}. Then if $t\geqq 8$, $(n-t)/2\geqq t!/\{\beta!(t-\beta)!\}$, where β is an integer chosen such that $5\leqq\beta\leqq t-3$, or chosen to be one unit less than the least value of β which satisfies the inequality

$$[(\beta+1)/2]!\geqq 2t!/\{\beta!(t-\beta)!\};$$

$$\text{or } n-t\geqq[(t+1)/2]!;\ \text{or } n-t\geqq t!/[t/2]!^2.$$

The symbol $[s]$ denotes the integral value of s.

35. Dr. L. W. Griffiths: *A theorem on extended polygonal numbers.*

If m is a positive integer, and $x=1, 0, -1, -2, -3, \cdots$, the integers $e(x)=x+m(x^2-x)/2$ are all the extended polygonal numbers of order $m+2$ and the first polygonal number, $e(1)=1$. Dickson has proved that every positive integer is the sum of m values of $e(x)$ if $m\geqq 6$, but of $m+1$ values if $m=3, 4, 5$. A generalization of this theorem is suggested by the following restatement. Let $n, a_1, \cdots, a_n$ be positive integers, and $e_1, \cdots, e_n$ be values of $e(x)$; then the function $f=a_1e_1+\cdots+a_ne_n$ is universal when $n=m\geqq 6$, or $n=m+1$ if $m=3, 4, 5$, and $a_1=\cdots=a_n=1$. In $+a_n\leqq m$, and also for $m=3, 4, 5$, and $a_1+\cdots+a_n\leqq m+1$. For $m=1, 2$ the problem reduces to that of polygonal numbers.

36. Professor L. S. Hill: *Concerning linear transformations whose coefficients are matrices.*

The term "ring" being used as specified in Hasse, *Höhere Algebra*, part 1, pp. 7–9, any finite or infinite ring containing a unit element (which is necessarily unique) is here called a "scale." In particular, every field, and every "Integritätsbereich," is a scale. The set of all square matrices, of any assigned order, in which the elements are elements of the scale S is called a "range," Γ. The ordinary type of matrix algebra is set up in Γ, and attention is directed to the transformation

$$T \qquad y_1 = a_{11}x_1 + \cdots + a_{1n}x + a_1, \cdots, \; y_n = a_{n1}x + \cdots + a_{nn}x_n + a_n,$$

defined by n linear equations in which the x_i, y_i, a_{ij}, a_i denote matrices of Γ. Conditions for the existence of a unique inverse transformation T^{-1}, and means to determine its equations, are given. A facile construction is presented for transformations T, of any desired complexity, which are involutory (that is, of period 2). All results are valid in ranges of matrices whose elements are in turn matrices, the elements of these elements being perhaps again matrices, etc. An application is made in the preparation of cipher transformations offering extraordinary resistance to cryptanalysis.

37. Professor A. D. Campbell: *Note on linear transformations of n-ics in m variables.*

This paper has appeared in the September-October number of this Bulletin.

38. Mr. D. H. Lehmer: *Numerical integration as applied to Dirichlet series.*

In the literature on the analytic theory of numbers, there are a large number of "identities" between certain Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ and simple expressions involving Riemann's ζ function. The methods used in proving these identities vary considerably; for the most part, however, use is made of the infinite product definition of $\zeta(s)$. The purpose of this note is to indicate a principle of unification by means of which all these identities fall into classes such that all identities of the same class may be obtained simultaneously. This unifying principle is based on Bougaieff's concept of numerical integration of a function $f(n)$ defined by the relation $F(n) = \sum_{\delta | n} f(\delta)$. We consider a sequence $f_r(n)$ defined for all positive and negative integer values of r by the relation $f_{r+1}(n) = \sum_{\delta | n} f_r(\delta)$, the fundamental function $f_0(n)$ being specified. In this way the function $f_0(n)$ defines a class of Dirichlet series $\sum_{n=1}^{\infty} f_r(n) n^{-s}$. Most of the known identities belong to three classes defined by $f_0(n) = n^k$, $\log n$, $\lambda(n)$, the latter being Liouville's function. By introducing new functions f_0 one obtains the rest of the known identities as well as many new ones.

39. Professor Raymond Garver: *Quartic equations with the alternating group.*

Baker and Seidelmann have given expressions for the most general quartic equation whose Galois group is the alternating group. The present paper shows that Baker's form actually does not represent all such quartics, while Seidelmanns' form does. The verification of this last fact

is entirely independent of Seidelmann's work, and requires only rather elementary algebraic manipulations.

40. Professor Raymond Garver: *Transformations on cubic equations.*

This paper, which will be published in the American Mathematical Monthly, studies exceptional Tschirnhaus transformations on cubic equations which are not equivalent to linear fractional transformations. A number of statements in the literature deny the existence of such transformations, and they are perhaps worth mentioning on that account as well as because they lead to certain rather interesting results.

41. Professor W. E. Roth: *On the unilateral equation in matrices.*

Let A_i, $i=0, 1, \cdots, p$, be rectangular matrices of m rows and n columns and let X be an unknown square matrix of order n; then the equation $\sum_{i=0}^{p} A_i X^{p-i}=0$ is said to be unilateral (Sylvester, Philosophical Magazine, vol. 17 (1884), p. 392). The present paper develops the necessary conditions for the existence of X satisfying this equation and in particular extends a theorem due to Sylvester regarding the latent (characteristic) roots of the unknown matrix (Comptes Rendus, vol. 99 (1884), pp. 621–631). This theorem for our more general equation may be stated thus: If $\sum_{i=0}^{p} A_i \lambda^{p-i}$ is the matrix obtained from the left member of the equation above when X is replaced by the scalar variable λ, then the latent roots of X must be values of λ for which the rank of $\sum_{i=0}^{p} A_i \lambda^{p-i}$ is less than n. The converse is not true. The results apply equally well when the unknown matrix of order m occurs on the left of all terms of the given equation.

42. Mr. John McDonnell: *Note on Fermat's last theorem.*

The author proves that if $x^p+y^p+z^p=0$, then (1) if $yz+zx+xy \not\equiv 0 \pmod{p}$, and r be any factor of x^2-y^2, then $r^{p-1} \equiv 1 \pmod{p^2}$, and (2) if $x(y-z)(x^2+y^2) \not\equiv 0 \pmod{p}$, and r be any factor of x^2+yz, then $r^{p-1} \equiv 1 \pmod{p^2}$. The proof resembles a proof of Furtwängler's for a similar result.

43. Professor H. A. Simmons: *Maximum numbers associated with a certain diophantine equation.*

This paper presents for the equation

$$\sum (1/(x_1 x_2 \cdots x_{n-1})) = b/[(m+1)b-1]$$

a solution in positive integers which has three special properties, namely: it contains the largest number x that can appear in any solution in which $x_1, x_2, \cdots, x_{n-1}$ are positive integers, with x_n only restricted to be a real number $\geqq x_i$, $i=1, 2, \cdots, n-1$; and, with the same hypotheses on the x's, the solution found maximizes the sum $x_1+x_2+\cdots+x_n$ and the product $x_1 x_2 \cdots x_n$. The final section of the paper exhibits for the equation $\sum (1/(x_1 x_2 \cdots x_r)) = b/[(m+1)b-1]$, r being less than n, the

number of variables in the symmetric function, a solution in positive integers which includes as a special case the solution whose properties were mentioned above.

44. Mr. J. H. Roberts: *A non-dense continuum.*

The author has shown that if M is a square plus its interior then there exists an upper semi-continuous collection G of mutually exclusive continua such that (1) every point of M belongs to some continuum of G, (2) G is topologically equivalent to M, and (3) each continuum of G contains more than one point. This question arises: "Is it necessary that a given continuum M contain a domain in order that such a set G exist?" In the present paper an example is given which shows that the answer to the above question is in the negative.

45. Dr. Leo Zippin: *Continuous curves and the Jordan curve theorem.*

It is shown that if a continuous curve C, is such that (1) every simple closed curve of C disconnects C, and (2) no arc of C disconnects C, then C satisfies the Jordan Curve-Theorem and is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected point set (possibly vacuous, if C is compact); these surfaces the author has previously termed cylinder-trees. It is proved that a necessary and sufficient condition that a continuous curve C containing at least one simple closed curve be a cylinder-tree, is that if K is any simple closed curve of C, then $K \subset \overline{C-K}$ and $C-K$ is the sum of two components. The author shows that the following postulate-set characterizes the number-plane: Axioms 1–5 of R. L. Moore's $\Sigma-1$, and Axiom 8′: if K is a simple closed curve of the space S, then $K \subset \overline{S-K}$, and $S-K$ is the sum of two components at least one of which is compact. Further, if in this system Axiom 2 be replaced by Axiom η': a region is a connected set of points not a single point, Axiom 3 may be replaced by 3′: if x is a point of S, $S-x$ is connected, or else by 3″: if m is an arc of S, there is a point x of m such that $S-x$ is connected.

46. Professor G.T.Whyburn: *Concerning continuous curves and denodular sets.*

A connected and locally connected set is denodular if all its nodular elements (that is, elements corresponding to the cyclic elements of a continuous curve) are points. It has previously been shown by the author that a denodular set is highly similar to a connected subset of an acyclic continuous curve. In this paper a study is made of the conditions under which a continuous curve may be expressed as the sum of a denodular set and another set H of simple structure. For example, H may be finite or may be null-dimensional. In this way characterizations of various types of continuous curves are obtained. For example, it is shown that a continuous curve is one-dimensional if and only if it is the sum of a denodular set and a null-dimensional set.

47. Professor R. L. Wilder: *Extension of a theorem of Mazurkiewicz.*

Mazurkiewicz has recently shown (Fundamenta Mathematicae, vol. 13, pp. 146–150), in answer to a question proposed by Knaster, that if A is a closed set in E_n which is homeomorphic with a subset of E_{n-1}, then all points of A are accessible from $E_n - A$. It is shown in the present paper that the points of A are regularly accessible from $E_n - A$. In the proof the following fundamental lemmas are established: (1) If a is a point of A, and S is a spherical neighborhood of a, then if the component C of $S \cdot A$ which contains a separates S, the set C is homeomorphic with a connected open subset of E_{n-1}. (2) If A separates two points P and Q in some spherical neighborhood S, there exists a component of $A \cdot S$ which separates P and Q in S.

48. Professor R. L. Wilder: *Concerning perfect continuous curves.*

The following results hold in any locally compact metric space: (1) Every compact connected subset of a perfect continuous curve has property S. (2) Every component of a quasi-open subset of a perfect continuous curve is arcwise connected im kleinen. (3) In order that a continuum M should be a perfect continuous curve, it is necessary and sufficient that the boundary points of the components of quasi-open subsets of M should be regularly accessible. (4) If K is a closed and bounded set lying in a component Q of a quasi-open subset of a perfect continuous curve, then K is a subset of a continuous curve which itself lies in Q.

49. Professor R. L. Wilder: *A converse of the Jordan-Brouwer separation theorem in three dimensions.*

This paper presents a converse, in E_3, of the Jordan-Brouwer separation theorem. (See J. R. Kline, this Bulletin, vol. 34, pp. 156–159.) The result may be stated as follows: Let K be a closed and bounded set in E_3 such that (1) the Betti numbers (mod 2) $R^0(E_3 - K)$ and $R^1(E_3 - K)$ are respectively equal to 2 and 1, (2) if D is a component of $E_3 - K$, the 0-chains in D are uniformly homologous to zero in D*, and every point of K is a limit point of D. Then K is a simple closed surface. Another statement of the result, not limiting the number of complementary domains, but strengthening the accessibility conditions, is also given.

50. Dr. W. L. Ayres (National Research Fellow): *On the density of the cut points and end points of a continuous curve.*

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* That is, for any number $e > 0$, there exists a number d such that if L^0 is a 0-chain of D of diameter less than d, L^0 bounds a chain L^1 in D of diameter less than e. This property is equivalent to uniform connectedness im kleinen here, and does not imply that a set is connected.

The author proves the following relations between the density of the end points and that of the cut points of a continuous curve M: the set K of all cut points of M is dense in M if, and only if, the end points of M are dense in $M-K$. In order that the end points of M be dense in M it is necessary and sufficient that the set of all cut points of Menger-Urysohn order $\geqq 3$ be dense in M. When the end points are dense in M, the set of all cut points of order 2 is zero-dimensional. And conversely, when the set of cut points of order 2 is zero-dimensional and the set of all cut points is dense in M, then the end points are dense in M.

51. Dr. W. L. Ayres: *On joining subsets of a continuous curve by arcs and simple closed curves.*

It is shown that if K is a finite subset of m points of a continuous curve M and no two points of K may be separated in M by a set of $m-1$ points of M, then there is a simple closed curve of M containing K. If no two points of K are separated by $m-2$ points, there is an arc of M containing K. Examples are given to show that the numbers $m-1$ and $m-2$ cannot be reduced, but for the special case of a plane continuous curve M the following sharper theorem is true: If M remains connected on the omission of any pair of points, then every finite subset lies on a simple closed curve of M.

52. Dr. W. L. Ayres: *On continuous images of a compact metric space.*

In this note it is shown that if M is a set of points and for each sequence $[q_i]$ of M there exists a "limit" function such that (1) limit $[q_i]$ is a unique subset of M (which may be vacuous), (2) if limit $[q_i] \neq 0$ and $[q_{n_i}]$ is a subsequence of $[q_i]$ then limit $[q_{n_i}] = \text{limit}\,[q_i]$, and further (3) if M is the continuous image of a compact metric space, then M is compact and metric. The result is a generalization of a theorem of Alexandroff (Mathematische Annalen, vol. 96, p. 562).

53. Professor W. A. Wilson: *A property of continua equivalent to local connectivity.*

This note proves the following theorem: If M is a metric space which is compact, separable, and connected, the necessary and sufficient condition for M to be locally connected at every point is that, for every pair of subcontinua A and B of M, there is a decomposition of M into two continua P and Q such that $P \cdot B = Q \cdot A = 0$. The same conclusion is valid if M is a bounded or unbounded continuum lying in a euclidean space.

54. Professor L. L. Dines: *Independent postulates for a generalized euclidean algorithm.*

In a Königsberg dissertation Plaut developed an algorithm for finding the highest common factor of a finite set of expressions of a general nature, including algebraic polynomials and linear differential and difference expressions as special cases. He based the development upon twelve postu-

lates, the mutual independence of which he said he had been unable to establish. In the present note it is shown that these postulates are not independent, and an equivalent set of ten independent postulates is presented.

55. Professor A. J. Kempner: *On the roots of algebraic equations and derived equations.*

This paper contains miscellaneous results of the following character: (1) A uniform method of derivation of a large number of known theorems such as *If* $f=0$ (*real coefficients*) *has all roots real and distinct, then* $f-kf'=0$ *has only real roots*; $f'\cdot f'-f\cdot f''=0$ *has only complex roots*; $f^2+k^2f'^2=0$ *has only complex roots*; $f^2-k^2f'^2=0$ *has only real roots*, and new results of related type. (2) For the Legendre polynomials P_n the equation $P_n'\cdot P_n'-P_n\cdot P_n''=0$ has all roots imaginary and of absolute value <1. A general theorem is given concerning the location of all roots of $f'\cdot f'-f\cdot f''=0$, f a polynomial. (3) A more precise form of Jensen's theorem concerning roots of $f'=0$, f a polynomial with real coefficients.

56. Professor H. E. Buchanan: *On a certain function of the masses in the three-body problem.*

In a recent paper (American Journal, vol. 50 (1928), No. 4) on the three-body problem, the writer encountered the following function of the masses: $f(m_1, m_2, m_3)\equiv(m_1+m_2+m_3)^2-27(m_1m_2+m_1m_3+m_2m_3)$. The present paper discusses this function in detail. If $f(m_1, m_2, m_3)>0$ the solutions of the differential equations for small oscillations contain only periodic terms and terms linear in t. On the other hand if $f(m_1, m_2, m_3)\leqq 0$, the positions are unstable due to the presence of terms of the form e^{at} as well as terms linear in t. If Gyldén (Bulletin Astronomique, vol. 1) and Moulton (Astronomical Journal, No. 483) are correct in their explanation of the Gegenschein it seems that there ought to be a much larger accumulation of matter near the equilateral triangle points than near the straight line positions, for the latter positions are always unstable with the exponential terms present. This paper has been accepted by the American Journal and will appear in an early number.

57. Professor H. L. Olson: *Doubly divisible quaternions.*

This paper derives a necessary and sufficient condition that a rational integral quaternion a be both a left and a right factor of a rational integral quaternion x.

58. Mr. J. R. Carson: *The asymptotic solution of an operational equation.*

This paper deals with the asymptotic solution of a specific operational equation with particular reference to the order of magnitude of the error committed by stopping with the numerically smallest term of the series, and the effect on numerical accuracy of adding an additional exponential term. The paper closes with a brief discussion of the general operational equation

and the conditions under which its divergent solution, where such exists, conforms to Poincaré's definition of an asymptotic series.

59. Professor C. H. Rowe: *A kinematical treatment of some theorems on normal rectilinear congruences.*

The author considers a two-parameter motion of a straight line l and a two-parameter motion of a rectangular trihedral, or observer, T, which is in one-to-one correspondence with that of l. He studies the relationship between the *actual* congruence and the *apparent* congruence which l describes, that is the congruence actually described by l and the congruence which l appears to describe to the observer T, and establishes a sufficient condition that the actual congruence be normal when the apparent congruence is normal. By means of this condition, he proves a number of known theorems and obtains extensions of certain of them.

60. Professor R. G. Lubben: *Concerning limiting sets.*

In this paper the following theorems are proved: (1) In a space L which has the distributive property, every uncountable point set has a limit point. (2) In a space S which has the distributive property, every uncountable point set *contains* a limit point of itself. (3) In order that a metric space have the distributive property, it is necessary and sufficient that it be locally compact and separable.

R. G. D. Richardson,
Secretary

THE BOULDER COLLOQUIUM

The lectures delivered by Professor R. L. Moore were remarkably well attended, there being a total of 91 persons registered for the series. Since it was thought advisable to dispense with the Friday afternoon session, four lectures of one and one-fourth hours each were given instead of the five one-hour lectures as announced in the program. The subject of the Colloquium, *Point set theory*, being one with which a considerable number of American mathematicians were not familiar, advantage was taken by many of these of this opportunity to gain an insight into some of the underlying concepts of this most fascinating and fundamental branch of mathematics.

Inasmuch as it is expected that the lectures will appear in full in the Society's Colloquium series within approximately twelve months, only a brief sketch of the material covered is here given. A good idea of their organization and content can be obtained from the synopsis furnished by Professor Moore, although some deviations from the synopsis were made. The treatment of the subject, based on a set of axioms, which was given, is to a large extent original with the lecturer, a most noteworthy contribution of this type having been published by him in the Transactions of this Society in 1916. In addition to his own work, however, the material covered embraced that of a number of other mathematicians both in this country and in Europe.

On the basis of a system of seven* axioms stated in terms of the undefined concepts *point* and *region*, it is first shown

* Although nine axioms are stated in the synopsis, it was pointed out by the lecturer that Axioms 0 and 6 are superfluous and that they are included because of their usefulness in studying spaces in which a part, but not all, of the axioms of the system are satisfied. That Axiom 8 also is redundant and is to be dropped from the system was communicated to me by Professor Moore a few days after the lectures were concluded.

that any space satisfying these axioms is homeomorphic with the surface of a sphere or with the euclidean plane according as it is or is not compact. Some of the main features of the treatment up to this point are the facts, 1) that at no point, up until the very last when coordinates are introduced, is it necessary or apparently even of any advantage to introduce or make use of the notion of distance, and, 2) that a very large body of propositions follow from very small sub-systems of the whole system of axioms, such as from Axioms 0 and 1 or from Axioms 1 and 2, for example. A surprisingly large number of very fundamental theorems can be proved on the basis of Axioms 0 and 1 alone; and although a space satisfying these axioms is necessarily metric,* the use of the notion of distance, which is foreign to the system of axioms as a whole, seems to add little if anything to the simplicity of the treatment. These propositions hold true in all spaces satisfying the reduced set of axioms on the basis of which they were proved; and since the class of such spaces obviously is larger (granting the irredundancy of the whole system, of course) than that of all spaces satisfying the whole system, the usefulness of the axiomatic method of procedure is undeniable. This fact was brought out very strikingly by the lecturer in the following manner. It was shown that on the basis of Axioms 1 and 2, it follows that every region is arcwise connected, and that the whole space, if connected at all, is arcwise connected. It was then pointed out that in any locally connected inner limiting set (that is, a set of points which is the common part of a family of open sets—frequently called a G_δ set) in a euclidean space of any number of dimensions, or indeed in much more general space, *region* may be defined in such a way that Axioms 1 and 2 are satisfied. Accordingly, it follows that *any connected and locally connected inner limiting set (or G_δ set)*

* This fact, which was unknown to the lecturer at the time of the lectures, was communicated to me by him less than a week after their conclusion.

is arcwise connected. This is a very interesting new result. The axiomatic method thus elucidates its usefulness not only as a very direct and easy approach to the subject, which it certainly is, but also in the approach to new problems through the elimination of extraneous material which adds merely to the confusion of the worker.

The lecturer then pointed out that the whole system of axioms is satisfied in a space whose elements are the elements of any upper semi-continuous collections of mutually exclusive compact continua filling up the surface of a sphere or the plane and none separating the sphere or plane. Accordingly, such a space is equivalent to the sphere or plane respectively. Applications of this extremely useful idea were discussed, as were also applications of upper semi-continuous collections and upper semi-continuous decompositions in the study of the structure of continuous curves and continua in general.

G. T. Whyburn

AN INTRODUCTION TO THE THEORY OF NUMBERS*

BY G. H. HARDY

PART I

1. *Farey Series.* The theory of numbers has always occupied a peculiar position among the purely mathematical sciences. It has the reputation of great difficulty and mystery among many who should be competent to judge; I suppose that there is no mathematical theory of which so many well-qualified mathematicians are so much afraid. At the same time it is unique among mathematical theories in its appeal to the uninstructed imagination and in its fascination for the amateur. It would hardly be possible in any other subject to write books like Landau's *Vorlesungen* or Dickson's *History*, six great volumes of overwhelming erudition, better than the football reports for light breakfast table reading.

The excursions of amateur mathematicians into mathematics do not usually produce interesting results. I wish to draw your attention for a moment to one very singular exception. Mr. John Farey, Sen., who lived in the Napoleonic era, has a notice of twenty lines in the *Dictionary of National Biography*, where he is described as a geologist. He received as a boy "a good mathematical training". He was at one time agent to the Duke of Bedford, but afterwards came to London, where he acquired an extensive practice as a consulting surveyor, which led him to travel much about the country and "collect minerals and rocks". His principal work was a geological survey of Derbyshire, undertaken for the Board of Agriculture, but he also wrote papers in the *Philosophical Magazine*, on geology and on many other subjects, such as

* The sixth Josiah Willard Gibbs Lecture, read at New York City, December 28, 1928, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

music, sound, comets, carriage wheels and decimal coinage. As a geologist, Farey is apparently forgotten, and, if that were all there were to say about him, I doubt that he would find his way into the *Dictionary of National Biography* today.

It is really very astonishing that Farey's official biographer should be so completely unaware of his subject's one real title to fame. For, in spite of the *Dictionary of National Biography*, Farey is immortal; his name stands prominently in Dickson's *History* and in the German encyclopaedia of mathematics, and there is no number-theorist who has not heard of "Farey's series". Just once in his life Mr. Farey rose above mediocrity and made an original observation. He did not understand very well what he was doing, and he was too weak a mathematician to prove the quite simple theorem he had discovered. It is evident also that he did not consider his discovery, which is stated in a letter of about half a page, at all important; the editor of the Philosophical Magazine printed a very stupid criticism in the next volume, and Farey, usually a rather acrid controversialist, ignored it completely. He had obviously no idea that this casual letter was the one event of real importance in his life. We may be tempted to think that Farey was very lucky; but a man who has made an observation that has escaped Fermat and Euler deserves any luck that comes his way.*

Farey's observation was this. The *Farey series of order* n is the series, in order of magnitude, of the irreducible rational fractions between 0 and 1 whose denominators do not exceed n. Thus

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2},$$

$$\frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

* It should be added that Farey's discovery had been anticipated 14 years before by C. Haros: see Dickson's *History*, vol. 1, p. 156. Cauchy happened to see Farey's note and attributed the theorem to him, and everyone else has followed Cauchy's example.

is the Farey series of order 7. There are two simple theorems about Farey series; (1) if p/q and p'/q' are two consecutive terms, then

$$p'q - pq' = 1,$$

and (ii) if p/q, p'/q', p''/q'' are three consecutive terms, then

$$\frac{p'}{q'} = \frac{p + p''}{q + q''}.$$

The second theorem (which is that actually stated by Farey) is an immediate consequence of the first, as we see by solving the equations

$$p'q - pq' = 1, \quad p''q' - p'q'' = 1,$$

for p' and q'.

The theorems are not of absolutely first class importance, but they are not trivial, and all of the many proofs have some feature of real interest. One of the simplest uses the language of elementary geometry. We consider the *lattice* or *Gitter* L in a plane formed by drawing parallels to the axes at unit distance from each other; the intersections, the points (x, y) with integral coordinates, are called the *points of the lattice*. It is obvious that the properties of the lattice are independent of the particular lattice point O selected as origin and symmetrical about any origin. The lattice is transformed into itself by the linear substitution

$$x' = \alpha x + \beta y, \quad y' = \gamma x + \delta y,$$

where α, β, γ, δ are integers and $\Delta = \alpha\delta - \beta\gamma = 1$, since then there is a pair x, y which give any assigned integral values for x', y'.

The area of the parallelogram P based on the origin and two lattice points (x_1, y_1), and (x_2, y_2), not collinear with O, is

$$\delta = \pm (x_1 y_2 - x_2 y_1).$$

We can construct a lattice L' (an oblique lattice) by producing and drawing parallels to the sides of P. A necessary and sufficient condition that L' should be equivalent to L, that

is, that they should contain the same lattice points, is that $\delta = 1$, that is, that δ should have its smallest possible value. It is clear that this is also a necessary and sufficient condition that *there should be no lattice point inside* P, and it is easy to see that if there is such a point inside P, there is one inside, or on the boundary of, the triangular half of P nearer to O.

We may call the lattice point (q,p) which corresponds to a fraction p/q in its lowest terms a *visible* lattice point; there is no other lattice point which obscures the view of it from O. Let us consider all the visible lattice points which lie inside, or on the boundary of, the triangle bounded by the lines $y=0$, $x=n$, $y=x$. It is plain that these points correspond one by one to the fractions of the Farey series of order n. When the ray R from O to (q, p) rotates from the x-axis to the line $y=x$, it passes through each of these points in turn. If we take two consecutive positions of R, corresponding to the points (q, p), (q', p'), the parallelogram based on these two points contains no lattice point inside it, since otherwise there would be a lattice point inside its nearer triangle, and therefore a Farey fraction between p/q and p'/q'. It follows that

$$\delta = p'q - pq' = 1,$$

which proves Farey's theorem.

2. *Purpose of this Lecture.* So much then for Farey's discovery; it is a curious theorem, and its history is still more curious; but I have no doubt allowed myself to dwell upon it a little longer than its intrinsic importance deserves. My discussion of it will, however, help me to explain what I am trying to do in this lecture.

I shall imagine my audience to be made up entirely of men like Farey. I know that most of them are very much better mathematicians, but I shall not assume so; I shall assume only that they possess the common school knowledge of arithmetic and algebra. But I shall also assume

that, like Farey, they are curious about the properties of integral numbers; one need after all be no Ramanujan for that.

Let us then imagine such a man playing about with numbers (as so many retired officers in England do) and puzzling himself about the curious properties which they seem to possess. What odd properties would strike him? What are the first questions he would ask? We must not try to be very systematic; if we do, we shall make no progress in an hour. We must aim merely at a rough preliminary survey of the ground. If in the course of our survey, we find the opportunity for any illuminating remark, we may delay to make it, as I have already delayed over Mr. Farey, even if it does not seem to fall in quite its proper logical place. Then, if time permits, we may return to examine a little more closely any important difficulties which our preliminary survey has revealed.

3. *Congruences to a Modulus.* There is no doubt that the first general idea which we should have to explain is that of a *congruence.* Two numbers a and b are *congruent to modulus* m if they leave the same remainder when divided by m, that is, if m is a divisor of $a-b$. We write

$$a \equiv b \pmod{m}, \quad m \mid a - b.$$

It is obvious that congruences are of immense practical importance. Ordinary life is governed by them; railway time tables and lists of lectures are tables of congruences. The absolute values of numbers are comparatively unimportant; we want to know what time it is, not how many minutes have passed since the creation.

A great many problems both of arithmetic and of common life depend upon the solution of congruences involving an unknown x, such as

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n \equiv 0 \pmod{m}.$$

Such congruences may be classified like algebraical equations, as linear, quadratic, $\cdots$, according to the value of n.

Our first instinct in dealing with congruences is to follow up the analogy with algebra. In algebra a linear equation has one root, a quadratic two, and so on. We find at once that there are obvious and striking contrasts; even the linear congruence suggests a whole series of problems, and a full discussion of quadratic congruences involves quite an imposing body of general ideas.

Let us take the simplest case, the linear congruence, and suppose first that we are concerned only with one particular modulus, such as 7 or 24. We have then an example of a genuinely finite mathematics. Congruent numbers have exactly the same properties and cannot be distinguished, and our mathematics contains only *a finite number of things.* In such a mathematics any problem can be solved by enumeration; we can solve $2x \equiv 5 \pmod 7$ by trying all possible values of x, and we find there is a unique solution, $x \equiv 6$. If we try to solve $2x \equiv 5 \pmod{24}$, we find that there is no solution; if I lecture every other day, I shall sooner or later lecture on Thursday, but if I lecture every other hour, I may never lecture at 5 P.M.

The difference is of course accounted for by the fact that 7 is *prime* and 24 is not. Here we encounter the notion of a prime, a number without factors, and all kinds of speculations suggest themselves. Can we tell, by any method short of trial of all possible divisors, whether any given number is prime or not? Are there formulas for primes? Are the primes infinite in number, and if so, what is the law of their distribution?

Again, it appears that all numbers are composed of primes, that primes are the ultimate material out of which the world of numbers is built up. We are bound to ask *how;* and here we meet our first big theorem, the "fundamental theorem of arithmetic," the theorem that factorization is unique. But we shall probably be wise to allow our enquirer to take this theorem for granted until he has acquired a little of the sophistication which comes with wider knowledge.

We may observe, however, before passing on, that the con-

trast between arithmetic and algebra becomes much more marked as soon as we consider congruences of higher degree. An equation of the fourth degree has, with appropriate conventions, just four roots. But

$$x^4 \equiv 1 \pmod{13}$$

has 4 roots, 1, 5, 8, and 12;

$$x^4 \equiv 1 \pmod{16}$$

has 8 roots, 1, 3, 5, 7, 9, 11, 13, and 15; and

$$x^4 \equiv 2 \pmod{16}$$

has none.

4. *Regarding Decimals.* I pass to another subject that has an irresistible fascination for amateurs, the subject of *decimals*. Some decimals are finite and some recurring, but it is easy to write down decimals, such as

(a) $0.10100100010\cdots$ (b) $0.11010001000\cdots$

which are neither. Here (a) the number of 0's increases by one at each stage, (b) the ranks of the 1's are $1, 2, 4, 8, \cdots$. More amusing examples are

(c) $0.01101010001010\cdots$

(in which the 1's have prime rank) and

(d) $0.23571113171923\cdots$

(formed by writing down the prime numbers in order). The proof for (c) demands the knowledge that there is an infinity of primes, and that for (d) rather more.*

The answer to some of the obvious questions is immediate. A finite decimal represents a rational fraction $p/(2^\alpha 5^\beta)$, a pure recurring decimal a fraction p/q, where q is not divisible by 2 or 5, and a mixed recurring decimal a fraction in which q is divisible by 2 or 5 and also by some other number. The converses of these theorems are also true, but the proof demands a little genuinely arithmetical reasoning. I shall state the proof in the simplest case, since it depends upon

* See Pólya and Szegö's *Aufgaben aus der Analysis*, vol. 2, pp. 160, 383.

the logical principle which is perhaps our most effective weapon in the elementary parts of the theory, where we are dealing with so simple a subject matter that our choice of arguments is naturally very restricted.

Suppose $p<q$ and q prime to 10. If we divide all powers 10^r by q, there are only q possible remainders, and one at least must be repeated. It follows that there are a ν_1 and a $\nu_2>\nu_1$ such that

$$10^{\nu_2} \equiv 10^{\nu_1}, \quad 10^{\nu_1}(10^{\nu_2-\nu_1} - 1) \equiv 0$$

to modulus q. It follows that, if we write $\nu_2-\nu_1=N$, we have $10^N\equiv 1$, so that $q \mid 10^N-1$ and

$$\frac{p}{q} = \frac{P}{10^N - 1} = P\cdot 10^{-N} + P\cdot 10^{-2N} + \cdots .$$

Since $P<10^N$, this is a pure recurring decimal with a period of at most N. The principles which we have used are (a) that *if there are more than q things of at most q kinds, there must be two of them of the same kind*; (b) that if 10^rQ is divisible by q, and q is prime to 10, then Q is divisible by q. In the second we are of course appealing to the "fundamental theorem". The first is the general logical principle to which I referred just now.

Let us take a slightly more complicated variant of this principle. *If there are two sets of objects*

$$a_1, a_2, \cdots, a_m, \quad b_1, b_2, \cdots, b_m,$$

no two of either set being the same; and if every b is equal to an a; then the b's are the a's arranged in a different order. We may apply this principle to obtain further information about the period of our recurring decimal. I suppose now that q is prime. If q and a are given, and a is not a multiple of q, it is impossible that

$$ra \equiv sa \pmod{q}$$

unless $r\equiv s$. If (ra) is the remainder when ra is divided by q, the two sets

$$r, \quad (ra) \qquad (r = 1, 2, \cdots, q-1)$$

satisfy the conditions of our principle and are therefore the same except in order. It follows that

$$(q-1)!\,a^{q-1} \equiv \prod(ra) \equiv \prod r = (q-1)! \pmod{q},$$

and therefore that

$$a^{q-1} \equiv 1 \pmod{q};$$

Fermat's Theorem. In the particular case in which we are interested, a is 10, and Fermat's Theorem shows that we may take $N=q-1$, so that the period of p/q cannot exceed $q-1$ figures. Observe that we have appealed to the fundamental theorem twice in the proof.

It is familiar to everyone that $\frac{1}{7}$ has 6 figures, the maximum number. We are bound to ask what other primes q possess this property; the values of q less than 50 are in fact 7, 17, 19, 23, 29, and 47, but here we begin to get into deeper water. I cannot stop to discuss this question now, but before passing on I must mention another familiar text-book theorem which I shall have to quote later. This is Wilson's Theorem, that

$$(q-1)! + 1 \equiv 0 \pmod{q}$$

if and only if q is prime. Of the mass of proofs catalogued by Dickson, that of Dirichlet depends most directly on principles which we have used already. It is an immediate consequence of these principles that, if x is any one of the set $1, 2, \cdots, q-1$, there is just one other, y, such that $xy \equiv 1 \pmod{q}$; we call y the *associate* of x. It is plain that 1 and $q-1$ are associated with themselves; and no other number can be, since $x_1^2 \equiv x_2^2$ implies $x_1 \equiv x_2$ or $x_1 \equiv q - x_2$. It follows that the numbers $2, 3, \cdots, q-2$ are composed of $\frac{1}{2}(q-3)$ distinct pairs the product of each of which is congruent to 1. Hence

$$2 \cdot 3 \cdots (q-2) \equiv 1^{(q-3)/2} = 1,$$

$$(q-1)! \equiv q - 1 \equiv -1,$$

which is one half of Wilson's Theorem. The converse half is practically obvious, since $(q-1)!$ would be divisible by any factor of q.

5. *Algebraic and Transcendental Numbers.* The study of decimals leads directly to problems concerning *rationality and irrationality.* Our decimals such as $0.1010010001\cdots$ must represent irrational numbers. What criteria are there for deciding whether a given number is rational or irrational? To ask this question is to go a little outside the theory of numbers proper, which is concerned first with integers, and then with rationals or irrationals of special forms, such as the form $a+b\sqrt{2}$, and not with irrationals as a whole or general criteria for irrationality. The problem is, however, one about which an amateur will certainly demand information.

The famous argument of Pythagoras shows that $\sqrt{2}$ is irrational; if a/b is in its lowest terms and $a^2=2b^2$, then a and b must both be even, a contradiction. It is obvious to us now that the Pythagorean argument extends at once to $\sqrt{3}$, $\sqrt{5}$, $\cdots$, $2^{1/3}$, $\cdots$, and generally to $N^{1/m}$, where N is any number which is not a perfect mth power. There is a curious and very instructive historical puzzle connected with this argument. There is a passage in Plato's *Theaetetus*, discussed at length by Heath in his *History of Greek Mathematics*, about the attempt of Theodorus to generalize Pythagoras's proof. Theodorus, working some 50 years after Pythagoras, proved the irrationality of $\sqrt{N}$ for all values of N (except square values) up to 17 inclusive. Why, ask the historians did he stop? Why in any case should it have taken mathematicians like the Greeks 50 years to make so obvious an extension? Zeuthen in particular expended a great deal of ingenuity upon this question, but I think that the ingenuity was misplaced, and that the answer is obvious.

Theodorus *did not know the fundamental theorem of arithmetic;* there is something of a puzzle about the history of that theorem, but it cannot have been known to the Greeks before Euclid's time. The triviality of the generalization to us is due entirely to our knowledge of this theorem. Suppose, for example, we wish to prove that

$$a^2 = 60b^2,$$

where a and b are integers without common factor, is impossible. We argue that a^2 cannot be divisible by 3 unless a is divisible by 3; hence $a = 3c$, $a^2 = 9c^2$, $3c^2 = 20b^2$, and a repetition of the argument shows that b also is divisible by 3. We can prove that $3 \mid a^2$ implies $3 \mid a$ *without the fundamental theorem*, by enumeration of possible cases, considering separately the cases in which $a \equiv 0, 1, 2 \pmod 3$. If it were 17 instead of 3, the process would be a little tedious; and in any case such a classification of numbers would have been very novel in Theodorus's time. I am so far from being puzzled by the limitations of his work that I regard what he did as a very remarkable achievement.

There are very few types of numbers which present themselves at all naturally in analysis and which can be proved to be irrational. It is obvious that a number like $\log_{10} 2$ is irrational, for a power of 2 cannot be a power of 10. The proof for e, from the exponential series, is quite easy, and that for e^2 not very much more difficult. That for π is decidedly more so, and when we come to numbers like e^3 and π^2, it ceases to be worth while to worry about elementary proofs; we may as well go the whole way and prove e and π are transcendental. The most famous constant in analysis, after e and π, is Euler's constant γ; and the proof of the irrationality of γ is one of the classical unsolved problems of mathematics. It has never been proved that $2^{\sqrt{2}}$, $3^{\sqrt{2}}$, and similar numbers are irrational; no plausible method for attacking such problems has even been suggested. I am inclined to think that the number which holds out the best hopes for new discovery is the number e^{π}, which presents itself so naturally in the formulas of elliptic functions.

I said just now that e and π were "transcendental". I must not stop to talk at length about this famous theorem of Lindemann,* which contains the final proof that the

* See for example Hobson's *Trigonometry*, third edition, p. 305, or the same author's *Squaring the Circle*.

quadrature of the circle, in the classical sense, is impossible; but the *statement* of the theorem introduces a notion that we shall require, that of an *algebraic number*. An algebraic number is the root of an equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

where the a's are integers. An *algebraic integer* is an algebraic number whose characteristic equation has unity for its leading coefficient. Thus $\sqrt{2}$ and $1+\sqrt{(-5)}$ are algebraic integers. A *transcendental* number is a number which is not algebraic; and Lindemann's Theorem is that π *is transcendental*. It is easy to show that all lengths which can be constructed by euclidean methods are algebraic, and indeed algebraic numbers of a quite special kind. If follows that the quadrature of the circle by any euclidean construction is impossible.

There is another direction in which we may be tempted to digress at this point, the theory of the approximation of irrationals by rationals, what is now called "diophantine approximation". There is just one theorem in this field that I shall mention, because it is connected so directly with what I have just been saying, and because it depends upon another of the stock arguments of number theory, the principle that *an integer numerically less than* 1 *is* 0. This is Liouville's theorem, that *there are transcendental numbers*. It is naturally much easier to prove this than to prove that a given number such as π is transcendental.

Liouville proves first that *it is impossible to approximate rationally to an algebraic number with more than a certain accuracy*. It is quite easy to see why. Suppose that ξ is an algebraic number defined by

$$f(\xi) = a_0\xi^n + a_1\xi^{n-1} + \cdots + a_n = 0.$$

We may suppose that the equation is irreducible, that is to say that $f(\xi)$ cannot be resolved into simpler algebraic factors of similar form; in this case we say that ξ is *of degree* n. We can obviously find a number M, depending only on ξ,

such that

$$|f'(x)| < M$$

for x near ξ. Suppose now that p/q is a rational, near ξ. Then

$$f\left(\frac{p}{q}\right) = \frac{N}{q^n},$$

where N is an integer not zero. It follows from our general principle that $|N| \geqq 1$ and

$$\left| f\left(\frac{p}{q}\right) \right| \geqq \frac{1}{q^n}.$$

But

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(\xi) = \left(\frac{p}{q} - \xi\right) f'(\eta),$$

where η lies between p/q and ξ. Hence, for all q,

$$\left| \frac{p}{q} - \xi \right| = \left| \frac{f(p/q)}{f'(\eta)} \right| > \frac{1}{Mq^n}.$$

It is impossible to approximate rationally to an algebraic number of degree n with an order of accuracy higher than q^{-n}.

On the other hand it is easy to write down numbers which have rational approximations of much higher accuracy than this; we have only to take a decimal of 0's and 1's in which the 1's are spaced out sufficiently widely. Thus

$$\xi = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \cdots = .11000100000\cdots$$

is approximated by its first k terms, that is, by a fraction

$$\frac{p}{q} = \frac{p}{10^{k!}}$$

with an error of order $10^{-(k+1)!} = q^{-k-1}$. Hence it is not an algebraic number of degree k and, since k is arbitrary, it must be transcendental. Obviously Liouville's argument enables us to *construct* transcendental numbers as freely as we please.

6. *Arithmetic. Forms.* The theory of irrationals starts from Pythagoras, and there is another great branch of the theory of numbers which also starts from him and about which I must now say something. This is the theory of *forms*.

Our interest in the theory of forms begins when we observe that there are Pythagorean triangles with integral sides; thus $3^2+4^2=5^2$. The first problem which suggests itself is that of determining all such triangles, and the solution given in substance by Diophantus, is easy. All the integral solutions of

$$x^2 + y^2 = z^2$$

are given by

$$x = \lambda(\xi^2 - \eta^2), \qquad y = 2\lambda\xi\eta, \qquad z = \lambda(\xi^2 + \eta^2),$$

where the letters are integers and ξ and η are coprime and of opposite parity. This problem is trivial, but it suggests an infinity of others.

It is natural to begin by a generalization of the problem. Let us discard the hypothesis that the hypotenuse z is integral; then

$$n = z^2 = x^2 + y^2$$

is the sum of two squares, and we are led to ask what numbers n possess this property. This is the first and simplest problem in the theory of *quadratic forms*, and the answer to it shows that no such problem can be quite easy. Even linear forms are not quite trivial; the solution of $ax+by=n$ in integers is a quite interesting elementary problem. When we consider quadratic forms, we come up against difficulties of a different order.

The first theorem in the subject is another theorem of Fermat, that *$x^2+y^2=n$ is soluble when n is a prime $p=4m+1$* and, apart from trivial variations of the sign and order of x and y, uniquely. It is to be observed that the equation is plainly insoluble when n is $4m+3$, since any square is congruent to 0 or 1 to modulus 4. This theorem is one of the most famous in the theory of numbers, and very rightly so, since it was the first really difficult theorem in the subject

proved by any mathematician. There is no really simple proof, and the most natural, that which depends on the Gaussian numbers $a+bi$, introduces a whole series of ideas of revolutionary importance.

The first stage of the proof consists in proving that *there is a number x such that*

$$x^2 \equiv -1 \pmod{p},$$

or $p|1+x^2$. Let us go back for a moment to the proof I sketched of Wilson's Theorem. Let us associate the numbers $x=1, 2, \cdots, p-1$ in pairs x, y not, as then, so that $xy \equiv 1$, but so that

$$xy \equiv -1 \pmod{p}.$$

If any x is associated with itself, our proposition is established. If not, we have arranged the numbers from 1 to $p-1$ in $\frac{1}{2}(p-1)$ pairs of different numbers each satisfying the condition. Hence

$$(p-1)! \equiv \prod xy \equiv (-1)^{(p-1)/2} = 1;$$

which is false, since, by Wilson's Theorem,

$$(p-1)! \equiv -1.$$

We thus obtain our proposition by reductio ad absurdum.

The second stage of the proof depends on much more novel ideas. We are concerned with the simplest case of an *algebraic field*. The field $K(i)$ is the aggregate of numbers

$$\xi = r + si = r + s\sqrt{(-1)},$$

where r and s are rational. This number satisfies the equation

$$\xi^2 - 2r\xi + r^2 + s^2 = 0,$$

and is an algebraic integer, in the sense I defined before, when $2r$ and r^2+s^2 are integers, that is, when r and s are integers. We may denote by $K^*(i)$ the aggregate of all the integers

$$\alpha = a + bi$$

of $K(i)$; a and b are ordinary integers. The numbers of $K^*(i)$ reproduce themselves by addition and multiplication, and we can define division in this field just as we define it in ordinary arithmetic. We can also define a *prime* of $K^*(i)$, and factorization of numbers into primes. There are four numbers, ± 1 and $\pm i$, which play a part in the new arithmetic similar to that of 1 and -1 in ordinary arithmetic. These are the "unities" or divisors of 1. If we define the *norm* of $\alpha = a + bi$ as

$$N(\alpha) = a^2 + b^2,$$

then the unities are characterized by the fact that their norm is 1. We do not count them as primes, just as, in the ordinary theory, we do not count 1 as a prime.

We now make an assumption, namely that *the analog of the fundamental theorem holds in the field* $K^*(i)$, that is to say that, apart from any trivial complications which may be introduced by the unities, *the factorization of a number of* $K^*(i)$ *into primes is unique.* This assumption is in fact correct. Returning now to the first stage of our proof, there is an x such that

$$p \mid 1 + x^2 = (1 + ix)(1 - ix).$$

It is obvious that p does not divide $1+ix$ or $1-ix$, so that *p divides the product of two numbers without dividing either of them.* Hence p cannot be a prime in $K^*(i)$. We may therefore write

$$p = \pi\lambda,$$

where $N(\pi) > 1$ and $N(\lambda) > 1$. But

$$N(\pi)N(\lambda) = N(p) = p^2,$$

so that $N(\pi)$ and $N(\lambda)$ must each be p. If we write

$$\pi = a + ib,$$

it follows that

$$p = N(\pi) = a^2 + b^2,$$

which is Fermat's theorem.

We may be tempted by our success to further efforts in the same direction. It is easy to satisfy ourselves, by considering particular cases, that *any prime* $p=20m+1$ *is of the form* a^2+5b^2: thus $61=4^2+5\cdot 3^2$. Let us try to prove this theorem by a similar method. We must evidently consider now the field $K^*[\sqrt{(-5)}]$ formed of the algebraic integers of the form

$$\alpha = a + b\sqrt{(-5)} ;$$

it is easy to show that such a number is an algebraic integer if and only if a and b are ordinary integers. There is no difficulty in defining divisibility and primality in this field also.

The first step in our proof must plainly be to prove the existence of an x for which $p \mid 1+5x^2$. This is not difficult, but it demands a little more knowledge of quadratic congruences than I can assume, and I must take it for granted.

We define the norm $N(\alpha)$ of a number of this field as a^2+5b^2. We then argue as before; we have

$$p \mid 1 + 5x^2 = (1 + x\sqrt{(-5)})(1 - x\sqrt{(-5)}),$$

so that p divides a product without dividing either factor and is therefore not a prime. Hence, as before $p=\pi\lambda$, where $N(\pi)>1$ and $N(\lambda)>1$, and $N(\pi)$ and $N(\lambda)$ must each be p. It follows that

$$p = N(\pi) = a^2 + 5b^2,$$

the theorem we set out to prove.

At this point, however, there is a shock in store for us; we find that we can prove *too much*. The number

$$q = (2 + \sqrt{(-5)})(2 - \sqrt{(-5)})$$

is divisible by 3, while neither factor is so. Hence 3 is not a prime. Hence

$$3=\pi\lambda, \quad 9=N(\pi)N(\lambda),$$

and $N(\pi)$ and $N(\lambda)$ are each 3. It follows that

$$3 = N(\pi) = a^2 + 5b^2.$$

Similarly we can prove that

$$7 = a_2 + 5b_2;$$

and both of these theorems are obviously false.

There must therefore be a mistake somewhere in our argument, and if you examine it, and are prepared to believe that I have not been misleading you wilfully, you will see that there is only one step which can be questioned. In all three cases I concluded the argument by an appeal to the same theorem; *a number which divides the product of two numbers without dividing either of them cannot be prime.* This is true in ordinary arithmetic, because of the fundamental theorem; if 7 were a divisor of $15 = 3 \cdot 5$, 15 would be factorable into primes in two distinct manners. It follows that *the analog of the fundamental theorem in the field* $K^*[\sqrt{(-5)}]$ *must be false*; and this is easily verified when once our suspicions have been excited; thus

$$2 \cdot 3 = (1 + \sqrt{(-5)})(1 - \sqrt{(-5)}),$$

$$3 \cdot 7 = (1 + 2\sqrt{(-5)})(1 - 2\sqrt{(-5)}),$$

and all of these numbers are prime in $K^*[\sqrt{(-5)}]$. The proof which I gave of the theorem concerning primes $20m+1$ was therefore fallacious, although the theorem is true. The proof of Fermat's theorem, on the other hand, was correct, since factorization *is* unique in $K^*(i)$.

7. *Further Problems.* It is clear that we must go back to the beginning and study the theory of primes a little more closely; but before I do this I should like to call your attention to a series of further problems suggested by Fermat's theorem. We know now when a *prime* is the sum of two squares, and we have to consider the same problem for general n. Here in fact there are three different problems.

The first and most obvious problem is that of determining the necessary and sufficient conditions that n should be representable. This problem may be solved quite easily with the aid of the Gaussian numbers; n must be $2^\alpha M^2 N$, where α is 0 or 1 and N contains prime factors of the form

$4m+1$ only. We are then led naturally to the corresponding problem for other forms, first for the general binary quadratic form

$$ax^2 + bxy + cy^2,$$

then for quadratic forms in a larger number of variables, such as

$$x^2 + y^2 + z^2, \quad x^2 + y^2 + z^2 + t^2,$$

and then for forms of higher degree, such as x^3+y^3 and x^4+y^4. There is a highly developed theory of the general quadratic form; the most famous theorem is perhaps Lagrange's theorem, that *every number is the sum of four squares*. But as soon as we begin to consider cubic or higher forms we find ourselves on the boundary of knowedge. There is for example no criterion analogous to Fermat's by which we can decide whether a given number is the sum of two cubes.

The second problem about the form x^2+y^2 suggested by Fermat's theorem is that of determining the *number of representations*. This problem may be interpreted in two different ways. We may want an exact formula, in terms of the factors of n, and in this case the Gaussian theory again gives what we want; $r(n)$, the number of representations, is given by the formula

$$r(n) = 4\{d_1(n) - d_3(n)\},$$

where $d_1(n)$ and $d_3(n)$ are the numbers of divisors of n of the forms $4m+1$ and $4m+3$ respectively. This is, however, not the most interesting interpretation of the problem. We may want, not a formula like this, but information concerning the *order of magnitude* of $r(n)$, whether $r(n)$ is generally large when n is large, whether numbers are usually representable freely or with difficulty. In this case our formula gives us very little help, and the solution of the problem requires quite different methods.

It is here that we come into contact for the first time with a new branch of the theory, the modern "analytic" theory.

This theory has two special characteristics. The first is one of method; it uses, besides the methods of the classical theory, the methods of the modern theory of functions of a complex variable. The second is that it is concerned primarily with problems of order of magnitude and asymptotic distribution. The distinction is not a perfectly sharp one; there are "exact", "finite" theorems which have only been solved by "analytic" methods. For example, *every number greater than* $10^{10^{10}}$ *is expressible as the sum of* 8 *cubes*; this theorem includes no reference to "order of magnitude", and is a "finite" theorem in just the same sense as Fermat's theorem about the squares, but the only known proof is analytic. On the whole, however, it is the problems of asymptotic distribution which dominate the theory.

The answer given by the analytic theory to the special question which I raised is roughly as follows. The average value of $r(n)$ is π. It must be observed that representations which differ only trivially, that is, in the sign or order of x and y, are reckoned as distinct. If we allow for this, the average number of representations is rather less than a half; this is explained by the fact that, as we shall see, *most* numbers are not representable. On the other hand $r(n)$ tends to infinity with n with tolerable rapidity for numbers of appropriate forms, more rapidly for example than any power of $\log n$. The corresponding problems for cubes or higher powers present difficulties which are at present quite insuperable, and all that I can do is to mention a few curiosities. The smallest number representable by two cubes in two really distinct ways is

$$1729 = 1^3 + 12^3 = 9^3 + 10^3,$$

and the smallest representable in three ways is probably

$$175959000 = 70^3 + 560^3 = 198^3 + 552^3 = 315^3 + 525^3.$$

It can be proved that there exist numbers with as many different representations as we please. A. E. Western has carried out very heavy computations concerning representa-

tions by cubes; he has for example found 6 numbers, of which the smallest is 1,259,712, representable as the sum of *three* cubes in *six* different ways. The smallest number doubly representable by two fourth powers is probably

$$635318657 = 59^4 + 158^4 = 133^4 + 134^4 ;$$

there is, so far as I know, no known example of a number with three such representations, nor any proof that such a number exists.

The nature of the problems of the analytic theory becomes clearer when we consider the third problem suggested by Fermat's theorem. This is the problem of determining the *distribution* of the representable numbers. We want to know *how many numbers are representable*, or, to put it more precisely, how many numbers less than a large assigned number x are representable. If $Q(x)$ is the number of such numbers, what is the order of magnitude of $Q(x)$? Are nearly all numbers representable, or just a majority, or only a few? The answer is in fact that $Q(x)$ is approximately

$$\frac{Ax}{(\log x)^{1/2}},$$

where A is a constant; to put it roughly, quite a lot of numbers are representable, but strictly an infinitesimal proportion of the whole. This explains why the average number of representations turned out to be less than one.

This problem about $Q(x)$ is a very interesting one, but there is another of the same kind which is obviously still more interesting and much more fundamental. This is the problem of the distribution of the primes themselves; *how many primes are there less than* x? I shall say something about this problem in a moment; it is in any case time for us to return to the theory of primes, since all our enquiries have ended in questions about them, and it is obviously impossible to make serious progress until we know more both of their elementary properties and of the laws which govern their distribution.

Part II

8. *The Fundamental Theorem.* The *fundamental theorem* of arithmetic is the beginning of the theory of numbers, and it is plain that our first task must be to make this theorem secure.

There is another historical puzzle about the fundamental theorem. Who first stated the theorem, explicitly and generally? The natural answer is *Euclid*, since the *Elements* contain all the materials for the proof. Everything rests on Euclid's famous algorithm for the greatest common divisor. Given two numbers a, b, of which a is the greater, we form the table

$$a = bc + b_1, \qquad b = b_1c_1 + b_2, \qquad b_1 = b_2c_2 + b_3, \cdots$$

where $b_1, b_2, \cdots$, are the remainders in the ordinary sense of elementary arithmetic. Since

$$b > b_1 > b_2 > \cdots,$$

b_n must sooner or later be zero. The last positive remainder δ has the properties implied by the words *greatest common divisor*, and it follows from the process by which δ is formed that any number which divides both a and b divides δ.

Let us note in passing that there is an analogous process in $K^*(i)$, but that the analogy fails in $K^*[\sqrt{(-5)}]$. In ordinary arithmetic, given a and b, we can find a number congruent to a mod b and less than b. There is a similar theorem for the Gaussian numbers. Here there is no strict order of magnitude between different numbers, and we have to use the order of magnitude of their norms. Given α and β, there is a number, congruent to α mod β, whose norm is less than that of β. There is no such theorem in $K^*[\sqrt{(-5)}]$, and the process analogous to Euclid's fails.

When the existence of δ is once established, the proof of the fundamental theorem is easy. We write

$$\delta = (a, b)$$

and we say that a is prime to b when $(a, b) = 1$. The crucial lemma is that *if* $(a, b) = \lambda$ *and* $b \mid ac$, *then* $b \mid c$; in particular, *a prime cannot divide a product without dividing one or other of*

the factors. This once granted, anybody can construct the proof of the fundamental theorem for himself; and you will remember that it was just this proposition which led to our troubles in $K^*[\sqrt{(-5)}]$.

The lemma itself may be proved as follows. We construct the euclidean algorithm for a and b, with the final remainder 1. If we multiply it throughout by c, we have the algorithm for ac and bc, and the final remainder is c. It follows that

$$(ac, bc) = c.$$

Since b divides ac, by hypothesis, and also bc, it divides c.

This is Euclid's own argument, and with it he had proved what is essential in the fundamental theorem. It is a very singular thing that he should then omit to state the magnificent theorem that he has proved. He is over the line and free, but apparently disdains the formality of touching down. I do not know of any formal statement of the theorem earlier than Gauss. The substance of the theorem, however, is in the *Elements*; it was plainly unknown, as I explained before, to the Greeks from 50 to 100 years before Euclid's time; and I see no particular reason for questioning the obvious view that it is Euclid's own.

As soon as we have proved the fundamental theorem our elementary knowledge falls into line. The theory of linear congruences, the theorems of Fermat and Wilson and all their consequences, the elementary theory of decimals and of the divisors of numbers, may be developed straightforwardly and without the introduction of essentially new ideas. I can now say something about the more modern side of the theory of primes.

9. *Problems Concerning Primes*. What are the most natural questions to ask about primes? I say deliberately the most *natural*; we must remember that a natural question does not always seem, on fuller reflection, to have been a *reasonable* one. It is natural to an engineer to ask us for a finite formula for

$$\int e^{-x^2} dx,$$

or for a solution of some simple looking differential equation in finite terms. If we fail to satisfy him, it is not because of our stupidity, but because the world does not happen to have been made that way.

So, if any one asks us (1) *to give a general formula for the nth prime* p_n, a formula in the sense in which

$$p_n = n^2, \quad p_n = n^2 + 1, \quad p_n = [e^n],$$

where $[x]$ denotes the integral part of x, would be a formula, I can only reply that it is not a reasonable question. It is, I will not say demonstrably impossible, but wildly improbable, that any such formula exists. The distribution of the primes is not like what it would have to be on any such hypothesis. I should make the same reply to a good many other questions which an amateur might be likely to ask, for example if he asked me (2) *to give a rule for finding the prime which immediately follows a given prime.* It would of course be perfectly reasonable that he should press me for the reasons why I gave so purely a negative a reply. On the other hand the problem (3) *to find the number of primes below a given limit* is, if interpreted properly, an entirely reasonable and a soluble problem. The problems (4) *to prove that there are infinitely many pairs of primes differing by* 2, and (5) *to prove that there are infinitely many primes of the form* n^2+1, are also entirely reasonable, and if (as is the case) we cannot solve them, it is quite reasonable to condemn our lack of ingenuity.

10. *The Distribution of Primes.* If we wish to classify these problems and to decide which of them are reasonable and which are not, the first essential is to understand broadly the present state of knowledge about the distribution, the distribution *in the large* or *asymptotic* distribution, of the primes. It is this theory which gives the solution of problem (3).

We denote by $\pi(x)$ the number of primes not exceeding

x. The first step is to prove that (a) *the number of primes is infinite*; $\pi(x)$ *tends to infinity with* x. This is another of Euclid's great contributions to knowledge, and Euclid's proof is perhaps the classical example of proof by reductio ad absurdum. If the theorem is false, we may denote the primes by $2, 3, 5, \cdots, P$, and all numbers are divisible by one of these. On the other hand the number

$$(2 \cdot 3 \cdot 5 \cdots P) + 1$$

is obviously not divisible by any of $2, 3, \cdots, P$, and this is a contradiction.

Another very interesting proof is due to Pólya.* It is easy to see that any two of the numbers

$$2 + 1, 2^2 + 1, 2^4 + 1, \cdots, u_n = 2^{2^n} + 1$$

are prime to each other. For suppose that p is an odd prime and that $p \mid u_n$, $p \mid u_{n+k}$. Then also

$$p \mid 2^{2^{n+k}} - 1 = u_{n+k} - 2,$$

since

$$x^{2^k m} - 1$$

is algebraically divisible by $x^m + 1$, and therefore

$$p \mid u_{n+k} - (u_{n+k} - 2) = 2,$$

which is absurd. It follows that the number of primes less than u_n is at least n, and therefore that the number of primes is infinite. In fact the argument shows not merely that $\pi(x) \to \infty$ but that

$$\pi(x) > A \log \log x,$$

where A is a constant. Something in this direction, though a little less, can be proved by a refinement of Euclid's argument.

There is a third line of argument which is a little less elementary but may be made to prove a good deal more.†

* See Pólya and Szegö, loc. cit., pp. 133, 342.

† See Dickson's *History*, vol. 1, p. 414, where the proof is attributed to Auric.

If 2, 3, 5, $\cdots$, P were the only primes, then every number would be of the form

$$2^a 3^b 5^c \cdots P^k.$$

If this number is less than x, then a fortiori 2^a is less than x, so that a is less than a constant multiple of $\log x$, and the same argument applies to b, c, $\cdots$, k. The number of possible choices of a, b, $\cdots$, k is therefore less than a multiple of $(\log x)^\pi$, where π is the total number of primes. In other words the number of numbers less than x is less than

$$A(\log x)^\pi,$$

where A is a constant, and this is impossible, since x tends to infinity more rapidly than any power of $\log x$. A refinement of the argument leads to the inequality

$$\pi(x) > A\frac{\log x}{\log\log x};$$

and the underlying principle may be stated roughly thus, that *if the number of primes were finite, there would not be enough numbers to go round.*

We are still a very long way from the ultimate truth. It is in fact possible to prove, and by comparatively elementary methods, that *the order of magnitude of* $\pi(x)$ *is* $x(\log x)^{-1}$. This theorem, conjectured by Legendre and Gauss, was first proved by Tchebycheff in 1848.

There are two much earlier theorems of Euler which point in this direction. The first is the theorem that (b) *the series*

$$\sum\frac{1}{p}$$

extended over all prime numbers p, *is divergent.* The proof of this theorem depends upon an identity, also due to Euler, upon which the whole of the modern theory of primes is founded. The identity is

$$1^{-s} + 2^{-s} + 3^{-s} + \cdots = \sum n^{-s}$$

$$= \frac{1}{(1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s}) \cdots}$$

$$= \Pi\left(\frac{1}{1 - p^{-s}}\right)$$

and is valid for $s > 1$; it is at bottom merely the analytical expression of the fundamental theorem, and its importance arises from the fact that it asserts the equivalence of two expressions of which one contains the primes explicitly while the other does not. From Euler's identity we deduce (b) roughly as follows: if $\sum p^{-1}$ were convergent, then

$$\Pi\left(\frac{1}{1 - p^{-1}}\right)$$

would be convergent, and therfore $\sum n^{-1}$ would be convergent, which is false. Of course the proof really needs a rather more careful statement.

Euler's second theorem is (c) *the quotient of* $\pi(x)$ *by* x *tends to zero*; or in symbols

$$\frac{\pi(x)}{x} \to 0 ,$$

or, as we write it now

$$\pi(x) = o(x) .$$

The proportion of primes is ultimately infinitesimal, "*almost all*" *numbers are composite.* The theorem is a quite simple corollary of (b); roughly, if we remove from the numbers less than x all multiples of the primes $2, 3, \cdots, p$, other than these primes themselves, we are left something like

$$x\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{p}\right)$$

numbers. The product multiplying x tends to zero when

$p \to \infty$, because of (b), and from this we can deduce Euler's second theorem.

It is rather curious that, although Euler's second theorem is a corollary of the first, the lessons which we learn from the two theorems concerning the distribution of the primes have exactly opposite tendencies. The second theorem tells us that the number of primes below a given limit is *not too great*, that the primes are in the end rather liberally spaced out; it is in fact exactly equivalent to the theorem that (d) *the nth prime p_n has an order of magnitude greater than n*, or

$$\frac{p_n}{n} \to \infty .$$

If on the other hand the order of magnitude of p_n were *much* greater than n, if it were for example n^2 or $n^{10/9}$ or $n(\log n)^2$, then the series $\sum p_n^{-1}$ would be *convergent*, which is just what Euler's first theorem denies. What we learn from the two theorems together is something like this. If, as we hope, the true order of magnitude of p_n can be measured by some simple function $\phi(n)$, then that function must be of order higher than n, but somewhere near the boundary of convergence of the series

$$\sum \frac{1}{\phi(n)} .$$

The most obvious function which satisfies these requirements is $n \log n$, and to say that p_n is of order $n \log n$ is the same thing as to say that $\pi(x)$ is of order $x(\log x)^{-1}$. This is just what is asserted by Tchebycheff's theorem.

11. *Tchebycheff's Theorem.* The formal statement of Tchebycheff's theorem is (e) *the order of magnitude of $\pi(x)$ is $x(\log x)^{-1}$; there are constants A and B such that*

$$\frac{Ax}{\log x} < \pi(x) < \frac{Bx}{\log x} .$$

This theorem is precisely equivalent to (f) *the order of magnitude of p_n is $n \log n$; there are constants A and B such that*

$$An \log n < p_n < Bn \log n.$$

The proofs of these theorems given by Tchebycheff have been simplified a good deal by Landau, and I can give you a sketch of one half of the proof which should enable you to understand without much difficulty the general character of the whole.

We begin by replacing $\pi(x)$ by another function. We can write $\pi(x)$ in the form

$$\pi(x) = \sum_{p \leq x} 1 ;$$

count one for every prime up to x. A more convenient and really a more natural function is

$$\theta(x) = \sum_{p \leq x} \log p,$$

the logarithm of the product of all primes up to x. This function seems at first sight a more complicated function, but it is easy enough to see why it is more convenient to work with. The most natural operation to perform on primes is *multiplication*, and this is the operation which we employ in forming $\theta(x)$. It is because it is natural to multiply primes and not to add or subtract them that problems like the problem of the prime pairs $(p, p+2)$, or Goldbach's problem of expressing numbers as sums of primes, turn out to be so terribly difficult.

Since $x/x^{1-\delta}$ tends to infinity, for any positive value of δ, we may expect that nearly all the primes which contribute to $\theta(x)$ will lie in the interval $(x^{1-\delta}, x)$, so that their logarithms lie between $(1-\delta) \log x$ and $\log x$. Hence we may expect $\theta(x)$ to be very much the same function as $\pi(x) \log x$, and in fact there is no difficulty in proving that

$$\theta(x) \sim \pi(x) \log x,$$

that is, that the ratio of the two functions tends to 1. It follows that the inequalities in (e) are equivalent to

$$Ax < \theta(x) < Bx.$$

I shall sketch the proof of the second inequality, which is rather the simpler.

Suppose that x is a power of 2, say 2^m. The primes between $x/2$ and x divide $x!$ but not $(x/2)!$, so that

$$\prod_{x/2<p\leqq x} p \,\Big|\, \frac{x!}{(x/2)!(x/2)!}.$$

The expression on the right is *one term* in the binomial expansion of $(1+1)^x=2^x$, and therefore

$$\prod_{x/2<p\leqq x} p \leqq 2^x.$$

Replacing x by

$$x/2,\ x/4,\ x/8,\cdots$$

and multiplying the results, we find that

$$\prod_{p\leqq x} p \leqq 2^{x+x/2+x/4+\cdots} \leqq 2^{2x}.$$

and

$$\theta(x) \leqq 2\log 2\cdot x.$$

This proves the theorem when $x=2^m$. If

$$2^m<x<2^{m+1}$$

we have

$$\theta(x) \leqq \theta(2^{m+1}) \leqq 4\log 2\cdot 2^m < 4\log 2\cdot x.$$

Hence we may take $B=4\log 2$. The proof of the second inequality is, as I said, not quite so simple, but does not involve essentially more difficult ideas. We have thus determined the order of magnitude of $\pi(x)$ and of p_n, and it is perhaps a little astonishing that a problem which sounds so abstruse should have so comparatively simple a solution.

12. *The Prime Number Theorem.* Tchebycheff's solution of the problem is, however, one with which it is impossible to remain content for long, since the whole trend of our discussion has been to suggest that much more is true than we have proved. In fact Tchebycheff's work, fine as it is, is the record of a failure; it is what survives of an unsuccessful attempt to prove what is now called the *Prime Number Theorem.*

This is the theorem that (g) $\pi(x)$ *and* $x(\log x)^{-1}$ *are asymptotically equivalent*; *the ratio of the two functions tends to unity.* We express this by writing

$$\pi(x) \sim \frac{x}{\log x}.$$

The Prime Number Theorem is equivalent to

$$p_n \sim n \log n,$$

and we may express it very roughly by saying that *the odds are* $\log x$ *to* 1 *that a large number* x *is not prime.*

The Prime Number Theorem, the central theorem of the analytic theory of numbers, was proved independently by Hadamard and by de la Vallée-Poussin in 1896. The empirical evidence for its truth had for long been overwhelming, and I suppose that every number-theorist since Legendre had tried to prove it. The theorem differs from all those which I have discussed so far in that it is apparently impossible to prove it by properly elementary methods; there is no proof known which does not depend essentially on complex function theory. I do not mean to imply that there is any terrible difficulty in the proof; there are considerable difficulties of detail, but the fundamental ideas on which it depends are tolerably straightforward. They are, however, quite unlike any of those of which I have spoken, and I should require a whole lecture to explain them even to a strictly mathematical audience. Actually, a good deal more is known; it can be proved that $\pi(x)$ is approximated still more closely by the "logarithm-integral" of x,

$$\operatorname{Li} x = \int_2^x \frac{dt}{\log t},$$

that in fact

$$\pi(x) = \operatorname{Li} x + O\left\{\frac{x}{(\log x)^k}\right\}$$

for every k, the error being of lower order than the quotient of x by *any* power of $\log x$; and it is probable, though not yet proved, that the order of the error does not very materially exceed that of $\sqrt{x}$.

13. *Formulas for Primes.* I return now for a moment to a question which I discussed shortly before, the question whether it was reasonable to expect an "elementary formula" for the nth prime p_n. Let us imagine that my questioner was obstinate in his desire for such a formula; how could I refute his successive suggestions? If he suggested

$$p_n = n \log n,$$

I should have the obvious reply that $n \log n$ is not an integer. Suppose then that he modified his formula to

$$p_n = [n \log n].$$

I should reply that his formula did not agree with the known facts of the asymptotic theory. It agrees with $p_n \sim n \log n$, the first and most obvious deduction from the Prime Number Theorem itself; but the theory carries us much further; it enables us, for example, to show that

$$p_n = n \log n + n \log \log n + O(n),$$

which contradicts the formula. If, becoming more cautious, he asked me what ground I had for denying that p_n might be *some* elementary combination of

$$n, \log n, \log \log n, \cdots,$$

I should naturally find it harder to refute him, but I could advance three arguments which are enough in the aggregate

to make up a tolerably convincing case. (i) Since $\mathrm{Li}\,x$ is a very good approximation to $\pi(x)$, the inverse function $\mathrm{Li}^{-1}n$ must be a very good approximation to p_n. Now it is demonstrable that neither the logarithm integral nor its inverse* is an elementary function. It is therefore very unlikely that there should be an elementary formula for p_n. (ii) If the "elementary formula" does not involve the symbol $[\cdots]$ of the "integral part", the function which it defines will generally not be integral for integral n. If it does, it loses all its simplicity and all its plausibility. (iii) An elementary function may be expected to behave with tolerable regularity at infinity, and so may all its *differences*. Now extremely little is known about the difference $p_{n+1}-p_n$ of two successive primes, but everything that is known, or seems probable from the evidence of the tables, suggests *extreme irregularity* in its behavior. The Prime Number Theorem shows that the *average* value of $p_{n+1}-p_n$ must be $\log n$, and tend to infinity with n. On the other hand there is overwhelming evidence that the smallest possible values of $p_{n+1}-p_n$, namely, 2, 4, 6, $\cdots$, recur indefinitely. It seems practically certain, not merely that there are infinitely many prime pairs $(p, p+2)$ but that there are infinitely many triplets $(p, p+2, p+6)$, and so with any combination of successive primes that is arithmetically possible; such a combination as $(p, p+2, p+4)$ is naturally not possible, since one of these numbers must be divisible by 3. All this seems hopelessly inconsistent with the existence of such a formula as was suggested, and it is clear that speculation in this direction is a waste of time.

There are, however, questions which have a somewhat similar tendency and which cannot be dismissed so summarily. There is one, for example, mentioned in Carmichael's little book. The problem, as he states it, is "*to find a prime greater than a given prime*," which might be interpreted as

* This may be deduced from general theorems proved recently by J. F. Ritt.

meaning either "*to find an elementary function* $\phi(n)$ *such that* $\phi(n)\to\infty$ *and* $\phi(n)$ *is prime for every* n, *or for all* n *beyond a certain limit*" or as meaning "*to find an elementary function* $\phi(p)$ *such that* $\phi(p)>p$ *and* $\phi(p)$ *is prime whenever* p *is prime.*" With either interpretation, it is a reasonable challenge, and the problem has not been solved.

Let us take the first form of the problem, which is perhaps the more natural, and let us begin by demanding *less*, namely that $\phi(n)$ shall be prime only for *an infinity of values of* n. In this case the problem becomes trivial, since n is a solution, by Euclid's theorem. It is, however, very interesting to observe that even then n, and certain simple linear functions such as $4n-1$ and $6n-1$, are the *only* trivial solutions. Dirichlet proved that *any* linear function $an+b$ has the property required, provided only that b is prime to a, or in other words that *every arithmetical progression* (subject to the last reservation) *contains an infinity of primes*. This theorem is quite difficult, except in a few special cases such as those which I mentioned, and it exhausts our knowledge in this particular direction. No one has ever proved that any of the functions

$$n^2+1, \quad 2^n-1, \quad 2^n+1$$

is prime for an infinity of values of n. With functions of *two* variables we can progress a good deal farther; we know for example that every quadratic form $am^2+bmn+cn^2$ contains an infinity of primes, provided of course that a, b, c have no common factor and that $b^2\neq 4ac$, and we can study the law of their distribution.

To find a $\phi(n)$ prime for *every* n is naturally still more difficult. Here linear functions are obviously useless, and no solution of any kind is known. Fermat conjectured that

$$2^{2^n}+1,$$

is always prime, but Euler proved that this is false, since

$$2^{32}+1 = 4294967297 = 641\cdot 6700417.$$

So far as I know, no one else has ever advanced any other suggestion which is even plausible.

In view of the apparently insuperable difficulties of this problem, there is a certain interest in *negative* results. It is plain, first, that $an+b$ cannot be prime for all n, or all large n. More generally, no polynomial

$$f(n) = a_0n^k + a_1n^{k-1} + \cdots + a_k$$

can be prime for all or all large n; for if $f(m) = M$ then $f(rn+m)$ is divisible by M for all r. There are entertaining curiosities in this field; thus

$$n^2 - n + 41$$

is prime for the first 41, and

$$n^2 - 79n + 1601$$

for the first 80 values of n. It is obvious that forms like

$$a^n - 1, \quad a^n + 1$$

cannot be prime for all large n, since, for example, $a^{3m}-1$ is divisible by a^m-1, and it is natural to suppose that the same is true for

$$P(n, 2^n, 3^n, 4^n, \cdots, k^n),$$

where P is any polynomial with integral coefficients.†

14. *The Fundamental Theorem in an Algebraic Field.* I must not allow myself to succumb to the temptation of talking too long about the theory of the distribution of primes, which is after all only one chapter in arithmetic. There are other topics about which our imaginary enquirer will certainly demand more information, and of these I think one stands out; it is certain that he will want fuller explanations about the field $K^*[\sqrt{(-5)}]$ and the other algebraic fields in which the analog of the fundamental theorem fails. All ordinary arithmetic depends, it seems, upon the fundamental theorem; how then can there *be* an arithmetic in a field in

† Morgan Ward of Pasadena has found a very simple proof of this theorem.

which it is false? It would seem that the arithmetic of such a field can bear no real resemblance to ordinary arithmetic. I shall spend the rest of my time in an attempt to explain, in the very broadest outline, how order is restored.

I shall begin by quoting a remark of Hilbert which is trivial in itself but which shows us at once the direction in which we must look for a solution. Consider the numbers

$$1,5,9,13,17,21,\cdots$$

of the form $4m+1$. These numbers form a group for multiplication (though naturally not for addition), and we can define divisibility and primality in the group. The "primes" are the numbers

$$5,9,13,17,21,29,33,37,41,49,\cdots$$

which are greater than 1, of the form $4m+1$, and not decomposable into factors of this form. Thus 21, 57, 77, and 209 are "primes"; but

$$4389 = 21\cdot 209 = 57\cdot 77,$$

so that a number of the group may be resolved into "prime" factors in different ways.

In this case the solution of the mystery is obvious. The "fundamental theorem" fails *because of the absence from the group of the numbers* $4m+3$ *of ordinary arithmetic.* In fact

$$21 = 3\cdot 7, \quad 57 = 3\cdot 19, \quad 77 = 7\cdot 11, \quad 209 = 11\cdot 19$$

and

$$21\cdot 209 = (3\cdot 7)(11\cdot 19) = (3\cdot 19)(7\cdot 11) = 57\cdot 77.$$

We cannot give a proper account of the properties of the numbers $4m+1$ so long as we insist on excluding the numbers $4m+3$; *the numbers* $4m+1$ *do not form by themselves an adequate basis for arithmetic.* This observation has of course no intrinsic interest, since no reasonable person would expect that they would do so. It is trivial in itself, but it is not at all trivial in its suggestion, since it suggests that the troubles of

the field $K^*[\sqrt{(-5)}]$ may be remedied *by considering the field as part of some larger field.*

This is in fact the solution found by Kummer. We consider the field $L[\sqrt{(-5)}]$ of numbers

$$\xi = \sqrt{(a + b\sqrt{(-5)})},$$

where a and b are ordinary integers. This is only an approximate statement; we do not actually consider all such numbers, but only those satisfying certain further conditions; the greatest common divisor of a and b must be a square or five times a square, and a^2+5b^2 must be a square. The field L includes K^*. The numbers of L form a group for multiplication, and we can define divisibility and primality in the field. Finally, the analog of the fundamental theorem is valid; *factorization is unique in L.* The proof of this is quite simple, but requires a little attention to detail, and I must refer you for the details to Mordell's tract on *Fermat's Last Theorem.*

We can now give a simple account of the equations in $K^*(\sqrt{(-5)})$ which puzzled us before. Consider for example the equation

$$3\cdot 7 = (1 + 2\sqrt{(-5)})(1 - 2\sqrt{(-5)}).$$

It is easily verified that

$$3^2=(2+\sqrt{(-5)})(2-\sqrt{(-5)}),$$

$$7^2=(2+3\sqrt{(-5)})(2-3\sqrt{(-5)}),$$

$$(1+2\sqrt{(-5)})^2=-19+4\sqrt{(-5)}=-(2-\sqrt{(-5)})(2+3\sqrt{(-5)}),$$

$$(1-2\sqrt{(-5)})^2=-19-4\sqrt{(-5)}=-(2+\sqrt{(-5)})(2-3\sqrt{(-5)}).$$

Hence, if we write

$$\alpha = \sqrt{(2 + \sqrt{(-5)})}, \qquad \alpha' = \sqrt{(2 - \sqrt{(-5)})},$$

$$\beta = \sqrt{(2 + 3\sqrt{(-5)})}, \qquad \beta' = \sqrt{(2 - 3\sqrt{(-5)})},$$

we have

$$3 = \alpha\alpha', 7 = \beta\beta', 1 + 2\sqrt{(-5)} = -\alpha'\beta, 1 - 2\sqrt{(-5)} = -\alpha\beta',$$

$$3 \cdot 7 = \alpha\alpha' \cdot \beta\beta' = \alpha'\beta \cdot \alpha\beta' = (1 + 2\sqrt{(-5)})(1 - 2\sqrt{(-5)});$$

and all of these equations are entirely natural. *In order to obtain a satisfactory theorem of factorization in K^*, we must conceive K^* as immersed in the larger field L.* The logic of the solution is exactly the same as that of the solution of the corresponding, but trivial, problem for the numbers $4m+1$.

On the other hand there is an obvious contrast between the two solutions. It is *natural* to think of the field "$4m+1$" as part of the field "m"; "m" is the more obvious and simpler field. It is not natural to think of K^* as part of L; K^* is a much simpler and more natural field than L, and we should like to do without the reference to the latter if we could. It will be very tiresome if, whenever we consider an algebraic field, we are to be compelled to construct some more elaborate field of which it is a part. We should prefer to tidy up the house without going out of doors.

We may look for a hint once more in the numbers $4m+1$. Some of these numbers are divisible by 7, a number outside the field; and these numbers stand in certain specific relations to one another inside the field. Could we give a rational account of these relations without explicit reference to the number 7? It is a very unnatural thing to try to do, since what is *important* about the numbers is precisely that they *are* divisible by 7, but we could do it; we could define the class

$$21, 49, 77, 105, \cdots,$$

of numbers $4m+1$ divisible by 7 in terms of the field $4m+1$ itself. For example, we could take the first two numbers 21 and 49, and say "the class in question is the class which begins with these two numbers and whose members recur at regular intervals in the field." It is of course an artificial definition, and it is impossible to conceal from ourselves what we are really doing.

It is often a very profitable exercise for a mathematician to force himself to solve some simple problem without the weapon obviously appropriate to the occasion, to throw away the key of the front door and insist on forcing himself in somehow through the window. The forced and unnatural solution of one problem will often turn out to contain the germ of a quite natural solution of another. So it proves in this case; it is natural to try to define the numbers of K^* divisible by ξ without going outside K^*; it is natural, and possible, and it gives us the key to what is, in the general case, the established method of constructing a satisfactory arithmetic.

It is obvious that, if α and β belong to K^*, and $\xi \mid \alpha$ and $\xi \mid \beta$, then $\xi \mid \lambda\alpha+\mu\beta$, where λ and μ are any numbers of K^*. The converse proposition is not true; it is not true that if I is any set of numbers of the field K^* which has the property "if α and β belong to I, then $\lambda\alpha+\mu\beta$ belongs to I, for every λ and μ of the field", then there exists a number ξ, belonging to K^* which divides every number of I. What *is* true is that every number of I is divisible by a ξ which belongs to L but not in general to K^*. The set I is identical with the set of numbers of K^* divisible by ξ. Such a set I, or the more general set based on any finite number of numbers $\alpha, \beta, \gamma, \cdots$, of K^*, is called an *ideal*, the numbers ξ, underlying K^* but not belonging to it, having been described by Kummer as "ideal numbers". In ordinary arithmetic ideals are simply the sets of numbers divisible by some special number such as 3, and there is nothing in particular to be gained by their introduction. In an algebraic field they are not, in general, the sets of numbers divisible by a number *of the field*, and their introduction is essential before arithmetic can get properly started. We can define multiplication and division of ideals, prime ideals, and so on, and when we have done this we find that the arithmetic of ideals has all the properties of ordinary multiplicative arithmetic. In particular, *every ideal can be resolved uniquely into prime ideals*; the fundamental theorem is true when stated in terms of ideals.

The proof of the fundamental theorem is not particularly difficult; Landau presents it, with all the preliminary definitions, in about a dozen pages of quite simple reasoning. But I would not commit the impertinence, even if I had the time, of assuming the airs of an expert in the algebraic theory of numbers, a subject which I admire only at a distance and in which I have never worked. It is ordinary rational arithmetic which attracts the ordinary man, and I have digressed outside it only because there is a good deal in it which it is impossible to appreciate properly without a little knowledge of the larger theory. It is impossible, for example, to appreciate Euclid's arithmetical achievements until we realize that there are arithmetics in which the most obvious analogs of his theorems are false.

15. *Conclusion. Pedagogy.* There are few things in the world for which I have less taste than I have for mathematical pedagogics, but I cannot resist the temptation of concluding with one pedagogic lesson. There was, and I fear still is, a popular English text book of algebra which I used at school and which contained a chapter on the theory of numbers. It might be expected that such a chapter would be among the most instructive in the book; we might suppose, for example, that Euclid's algorithm, with its elegance, its simplicity, and its far reaching consequences, would be an ideal text for the instruction of a bright young mathematician. In fact the algorithm was never mentioned; one was to find the highest common factor of 12091 and 14803, I suppose, by "trial"; and all that the authors had to say of the fundamental theorem was that "it is so evident that it may be regarded as a necessary law of thought." It is possible of course that all this may have been expunged from later editions. It is certain, however, that chapters on number theory in textbooks of algebra are usually quite intolerably bad, and it is conceivable that Oxford University may have been right in erasing the subject altogether from its more elementary examination schedule.

The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar; the processes of reasoning which it employs are simple, general and few; and it is unique among the mathematical sciences in its appeal to natural human curiosity. A month's intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and at least ten times as entertaining as the same amount of "calculus for engineers". It is after all only a minority of us who are going to spend our lives in engineering workshops, and there is no particular reason why most of us should feel any overpowering interest in machines; nor is it in the least likely that, on those occasions when machines are of real importance to us, we shall require the power of dealing with them by methods more elaborate than the simplest rule of thumb. It is not engineering mathematics that is wanted for the understanding of modern physics, and still less is it wanted by most of us for the ordinary needs of life; we do not actually drive cars by solving differential equations. There may be a case for subordinating mathematics to the linguistic and literary studies which are so much more obviously useful to ordinary men, but there is none for sacrificing a splendid subject to meet a quite imaginary need.

PRINCETON UNIVERSITY

NOTE ON A CERTAIN TYPE OF PARABOLA*

BY W. C. GRAUSTEIN

In the development of the geometry of the complex domain, there has been a tendency to pass over problems of a relatively elementary character, despite their central position and importance. It is the purpose of this note to call attention to one of these problems, namely that of the classification of complex conics with respect to the group of complex rigid motions. A systematic study of this problem brings to light two *special* types of non-degenerate conics: the non-degenerate central conics which contain just one circular point at infinity, and the non-degenerate parabolas which are tangent to the line at infinity at a circular point. These special conics have already been considered, from a different point of view, in this Bulletin.†

It is not our intention to go into detail, either in connection with the general problem of classification or in consideration of the special conics. We propose merely to discuss certain particularly striking properties of the special parabolas.

It is evident that the complex conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

A, B, C not all zero, is a special parabola if and only if

$$B^2 - 4AC = 0, \quad A + C = 0, \quad \Delta \neq 0, \tag{2}$$

where Δ is the discriminant of (1).

Since a special parabola has neither focus nor directrix, the question of its eccentricity cannot be handled directly. We may, however, establish an equation determining the

* Presented to the Society, April 6, 1928.

† E. V. Huntington and J. K. Whittemore, *Some curious properties of conics touching the line at infinity at one of the circular points*, vol. 8 (1901–02), pp. 122–124.

eccentricities of the general conic (1) and extend its validity to all conics. An equation with this property, whose roots are the *squares* of the eccentricities, is

$$(B^2 - 4AC)s^2 - 4[(A + C)^2 + B^2 - 4AC](s - 1) = 0.$$

For a special parabola the coefficients in this equation are all zero. Hence the homogeneous functions of degree zero in A, B, C, which for the general conic are the squares of the eccentricities, are, for a special parabola, undefined.

THEOREM 1. *A special parabola has no eccentricity* and is the only conic with this property.*

Consider a special parabola tangent to the line at infinity at $J:(1,-i,0)$. Let the parabola be referred to the point of contact, O, of the isotropic tangent through $I:(1,i,0)$ as origin of a system of rectangular Cartesian coordinates. Then regardless of the choice of axes of coordinates, the equation of the parabola takes the form

$$(x - iy)^2 = 2k(x + iy), \qquad k \neq 0,$$

or, in terms of isotropic coordinates, the form

$$\bar{z}^2 = 2kz, \qquad k \neq 0. \tag{3}$$

The rotation about O through the angle θ:

$$z' = e^{\theta i}z, \quad \bar{z}' = e^{-\theta i}\bar{z}, \tag{4}$$

carries the parabola (3) into $\bar{z}'^2 = 2ke^{-3\theta i}z'$ and hence, when $3\theta \equiv 0 \pmod{2\pi}$, into itself. Herewith, we have the most bizarre of peculiarities imaginable for a conic.

THEOREM 2. *A special parabola has rotational symmetry about the point of contact O of its unique isotropic tangent: there exists a group of three rotations about O which carry it*

* It is evident that a variable conic may be made to approach a special parabola as a limit in such a way that the square of one of its eccentricities approaches any preassigned value, 0 *not* excluded. Herein we have, for example, the explanation of the presence of two special parabolas in a pencil of equilateral hyperbolas. See Huntington and Whittemore, loc. cit.

into itself, namely the rotations through 0, $2\pi/3$, *and* $4\pi/3$ *radians.*

It follows that there is an infinity of equilateral triangles inscribed in the parabola, all of which have the point O as median or central point. Every point of the parabola, except O and the point at infinity, is a vertex of just one of the triangles.

By means of equations (4), interpreted as a rotation of the axes of coordinates, equation (3) of our special parabola may be reduced to the normal form

$$\bar{z}^2 = 2z. \tag{5}$$

It suffices to choose θ so that $e^{3\theta i} = k$. Hence:

THEOREM 3. *There are only two types of special parabolas: every two special parabolas with the same point at infinity are equivalent with respect to the group of complex rigid motions.*

Here, again, the special parabolas differ from the ordinary ones. The ordinary nondegenerate parabolas tangent to the line at infinity at a given (noncircular) point are of an infinity of types depending on one complex parameter. The parameter may be taken as the square of the semi latus rectum, whose value for an ordinary nondegenerate parabola (1) is given by an expression, $\Delta/(A+C)^3$, which is undefined (infinite) for every special parabola.

An instructive side-light on Theorem 2 is obtained by studying its projective generalization. Consider a complex collineation of general type—of which a rotation is a special metric case—and write its equations in the canonical forms

$$x_1' = a_1 x_1, \qquad x_2' = a_2 x_2, \qquad x_3' = a_3 x_3, \tag{6}$$

where a_1, a_2, a_3 are complex numbers, no two equal and no one zero. If the collineation leaves fixed a nondegenerate conic, it establishes on the conic a projectivity which is readily shown to be nonparabolic. The conic must, therefore, be tangent to two fixed lines of the collineation at two fixed

points; that is, belong to one of the three pencils of conics

$$(7)\quad k_1x_1^2 = 2l_1x_2x_3, \quad k_2x_2^2 = 2l_2x_3x_1, \quad k_3x_3^2 = 2l_3x_1x_2.$$

The degenerate conics of these pencils are obviously invariant with respect to the collineation. Hence, if one of the non-degenerate conics of a pencil is invariant, all the conics of the pencil are invariant. The conditions for invariance in the three cases are respectively

$$a_1^2 = a_2a_3, \qquad a_2^2 = a_3a_1, \qquad a_3^2 = a_1a_2.$$

Any one of these relations is a consequence of the other two. The collineations (6) determined by them are

$$(8)\quad \begin{aligned} &x_1' = \omega^2x_1, \ x_2' = \omega x_2, \ x_3' = x_3\ ; \\ &x_1' = \omega x_1, \ \ x_2' = \omega^2x_2, \ x_3' = x_3, \end{aligned}$$

where ω is a primitive cube root of unity.

THEOREM 4. *The only collineations of general type* (6) *which leave fixed each of the conics of two, and hence of all three, of the pencils of conics* (7) *are the collineations* (8).

If two of the three fixed points of the collineation (6) are the circular points at infinity, two of the pencils (7) are those of the special parabolas having the odd fixed point as the point of contact of their unique isotropic tangents, the third pencil is that of the circles with the odd fixed point as center, and the two collineations (8) are the rotations about the odd fixed point through $2\pi/3$ and $4\pi/3$ radians.

One last remark. A real nondegenerate conic (1) is a circle if and only if

$$(A + C)^2 + B^2 - 4AC = 0 \text{ or } (A - C)^2 + B^2 = 0.$$

But the complex nondegenerate conics (1) defined by this relation include, besides the circles, the special parabolas and the special central conics. Moreover, both eccentricities of a special central conic, as well as both those of a circle, are zero.

HARVARD UNIVERSITY

ON ORDINARY RESTRICTED EXTREMA IN CONNECTION WITH POINT TRANSFORMATIONS

BY JACQUES HADAMARD

1. *Restricted Minima in a Plane.* The question which I shall examine concerns the maximum or minimum of $f(x, y)$, when the variables x and y are not independent, but subjected to the condition

$$(g) \qquad g(x, y) = 0,$$

so that the point (x, y) must describe the curve represented by the preceding equation.

This is what was called, formerly, a *relative* extremum, but what I proposed to call a *restricted** extremum, because the term "relative" is used with another meaning. Indeed, the extrema which we consider, as every extremum treated by the methods of differential calculus, are *relative*, that is, they are extrema only in comparison with neighboring values of f.

The first part of the question, namely, the investigation of stationary values of f on the curve (g), is quite classic. It is usually solved, with the help of the Lagrange multiplier l, by means of the simultaneous equations

$$(1) \qquad p + lp_1 = 0, \qquad q + lq_1 = 0,$$

where p and q are the partial derivatives of f, and p_1 and q_1 are those of g. However, we shall use here the condition with l eliminated, that is,

$$(j) \qquad j = pq_1 - qp_1 = 0,$$

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* The author informs the editors that he, together with David Hilbert and others, had previously agreed that a change of nomenclature is desirable, to replace the older term relative extrema. The agreement is to use "*extremum libre ou lié*" in French, and "*freies oder gebundenes Extremum*" in German. The author had used "*bound*" in English, and the editors have suggested the word "*restricted*." THE EDITORS.

so that j is the jacobian of f and g. A point m (x_0, y_0) of (g) at which f is stationary on (g) necessarily belongs to the curve defined by (j). We shall suppose, throughout this paper, that m is an ordinary point of (g) and of (j). At m, the curve $f(x, y) = f(x_0, y_0)$ is tangent to (g).

This settles the question of stationary values. But there remains the question whether such a value, when once found, is a maximum or a minimum. This, of course, can be answered easily if we begin by imagining that x and y, the coordinates of an arbitrary point of (g), are expressed in terms of a variable parameter u. A maximum or a minimum will occur, according to the sign of d^2f/du^2. Then, expanding the latter derivative and taking account of the fact that

$$dg = p_1 dx + q_1 dy = 0, \quad d^2g = 0,$$

since the point (x, y) must describe the curve (g), it is easily found that the sign involved is that of

$$(2) \qquad (r + lr_1)q_1^2 - 2(s + ls_1)p_1q_1 + (t + lt_1)p_1^2,$$

where r, s, t denote second derivatives of f, and r_1, s_1, t_1 denote second derivatives of g.

But it is worth while to notice that an equivalent, completely different solution may be obtained by the study of the point transformation

$$(T) \qquad X = f(x, y), \qquad Y = g(x, y),$$

in the neighborhood of m.

At m itself, the jacobian j of that transformation vanishes, and the curve (j) divides the plane (or, more exactly, the region surrounding m) into two parts σ and σ', the first of which will correspond to $j > 0$, the second to $j < 0$. Now, we know* that, in the XY plane, these two regions will have images located on *one and the same side* of the line (J) which corresponds to (j): we can say that, in the vicinity of the point M, which is the image of m, the images of σ and σ' only give one area R, limited by (J).

* See my *Cours d'Analyse*, vol. I.

In order to solve the given question, we need only inquire whether a direction issuing from M and having α, β for its directing cosines, is directed towards the inside of R.

Let us call *direction parameters* of a direction, any two quantities proportional to the direction cosines of that direction, *with a positive coefficient of proportionality*. In particular, the normal to (j) at m, directed towards the inside of σ, will admit of direction parameters equal to the partial derivatives of j,

$$P = \frac{\partial j}{\partial x}, \quad Q = \frac{\partial j}{\partial y}.$$

Then let us consider the direction tangent to (j), directed with respect to σ. It will admit of the direction parameters $Q, -P$, so that an infinitesimal displacement (dx, dy) of m on (g) in the direct sense will satisfy the relations

$$\frac{dx}{Q} = \frac{dy}{-P} > 0.$$

Hence the corresponding displacement in the XY plane will be such that

$$\frac{dx}{pQ - qP} = \frac{dy}{p_1Q - q_1P} > 0. \tag{3}$$

The latter displacement will be tangent to (J), in the direct sense with respect to R.

Therefore a necessary and sufficient condition that a direction issuing from M and not tangent to (J) be directed towards the inside of R, is that it shall determine, with the direction defined by (3), an angle with a positive sine. If α and β denote the direction cosines of such a direction, this condition may be written in the form

$$\beta(pQ - qP) - \alpha(p_1Q - q_1P) > 0. \tag{4}$$

The application to restricted maxima or minima of f on the line (g) is obvious. Let us consider an arc of the latter line, which shall contain m and be sufficiently small. Its image in

the XY plane will be a portion of a parallel to the X axis, *which must lie inside* R. If we take its origin at M, its direction cosines, which are $+1$ and 0, must therefore, satisfy the condition (4). We see, then, that the proper sign is plus if we have

$$p_1Q - q_1P < 0, \tag{5}$$

and minus if

$$p_1Q - q_1P > 0. \tag{5'}$$

In the first case, X_0 negative, the value of f at m, which is the abscissa of M, will be a restricted *minimum* of f under the given conditions; in the second case, a *maximum*.

Similarly, the curve $f(x, y) = f(x_0, y_0)$, which goes through m, will have for its image a portion of the straight line $X = X_0$, and the direction from M thus defined will be parallel to the positive Y axis or to the negative Y axis, according to the sign of the quantity

$$pQ - qP.$$

This quantity and the quantity $p_1Q - q_1P$ have the ratio $-l$, on account of (1). The fact that a restricted maximum of f when g is constant coincides with a maximum or with a minimum of g when f is constant, according to the sign of l, is classic.

It is easy to verify by direct computation that the condition thus obtained is the same as that which would result from the expression (2). The remarkable thing is that the latter can be expressed in terms of the *first* derivatives of f, g, and j.

2. *The Case of a Curve in Space.* Let us consider the curves defined by the equations

$$g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0,$$

and a stationary value, let us say at the (ordinary) point $m(x_0, y_0, z_0)$, of $f(x, y, z)$ on that curve.

At m, the jacobian

$$j = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{vmatrix} = \begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}$$

must vanish. Again, we shall suppose that m is an ordinary point of the surface $j=0$, so that not all the three derivatives

$$P = \frac{\partial j}{\partial x}, \quad Q = \frac{\partial j}{\partial y}, \quad R = \frac{\partial j}{\partial z}$$

are zero. Such a surface (j) will divide the space near m into two regions ω, ω', the first of which will correspond to $j>0$. Let us consider the point transformation defined by

$$X = f(x,y,z), \quad Y = g_1, \quad Z = g_2.$$

Around M (the image of m), ω and ω' will have a common image R, on one side of the surface (J) which corresponds to j. Let us find a necessary and sufficient condition that the direction from M with the direction cosines α, β, γ be directed towards the inside of R. Let (dx, dy, dz), $(\delta x, \delta y, \delta z)$ be two directions from m, tangent to (j), so that

$$(6) \qquad \frac{dy\,\delta z - dz\,\delta y}{P} = \frac{dz\,\delta x - dx\,\delta z}{Q} = \frac{dx\,\delta y - dy\,\delta x}{R}.$$

These two directions, in the above order, will define a direct orientation on s, at m, if the common value of the above ratios is positive. In the XYZ space, the corresponding directions will be

$$(7) \qquad \begin{aligned} dX &= p dx + q dy + r dz, \quad dY = p_1 dx + q_1 dy + r_1 dz, \\ dZ &= p_2 dx + q_2 dy + r_2 dz, \\ \delta X &= p\delta x + q\delta y + r\delta z, \quad \delta Y = p_1\delta x + q_1\delta y + r_1\delta z, \\ \delta Z &= p_2\delta x + q_2\delta y + r_2\delta z, \end{aligned}$$

so that the tangent line to s at M will have direction cosines A, B, C, such that

$$\frac{A}{dY\delta Z - dZ\delta Y} = \frac{B}{dZ\delta X - dX\delta Z} = \frac{C}{dX\delta Y - dY\delta X}.$$

Now, our direction (α, β, γ) will be directed towards R if it gives with (7) a direct trihedron, that is, if

$$\begin{vmatrix} dX & dY & dZ \\ \delta X & \delta Y & \delta Z \\ \alpha & \beta & \gamma \end{vmatrix} > 0.$$

Such will be the case for a parallel to the positive X axis if

$$dY\delta Z - dZ\delta Y > 0,$$

or, on account of (6) and (7),

$$P(q_1r_2 - r_1q_2) + Q(r_1p_2 - p_1r_2) + R(p_1q_2 - q_1p_2) > 0.$$

An analogous solution for extrema on a surface would be interesting to find.

3. *Conclusion.* Let us come back to the case of the plane. The above argument rests on the properties of the transformation at a point where its jacobian vanishes, which is especially interesting to study from the geometric point of view, as will be done in a paper by Professor Julia to appear shortly in the Bulletin des Sciences Mathématiques. The fundamental fact is that, at such a point m, any two curves of the xy plane will, in general, have images which will be tangent to each other; but there is an exception, concerning a certain peculiar direction at m. Any curves tangent to that special direction will have ordinarily in the XY plane, an image that has a cusp, or sometimes a terminal point.* The latter case (a particular case of the former) occurs precisely for our curves $f(x, y) = f(x_0, y_0)$, $g(x, y) = 0$.

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* The words *terminal point* are suggested by the editors as a translation of the French term *point d'arrêt.*

A CERTAIN BIRATIONAL TRANSFORMATION OF ORDER r^2 BETWEEN TWO r-SPACES IN AN $(r+1)$-SPACE

BY B. C. WONG

In this paper we present a birational transformation of order r^2 between two r-spaces S_r and S_r' in an $(r+1)$-space S_{r+1}. If $r=2$, we have the well known quartic birational transformation between two planes of an S_3 obtained by intersecting with the two planes the bisecants of a twisted cubic curve in S_3. A point of one plane and a point of the other are said to be corresponding points if they lie on the same bisecant of the cubic curve. It is our object to generalize this construction to hyperspace.

For this purpose we let two r-ic hypersurfaces (necessarily ruled) intersect in a composite manifold composed of an M_{r-1}^{n} of order $n=r(r+1)/2$ and an $M_{r-1}^{n'}$ of order $n'=r(r-1)/2$. The former component manifold, M_{r-1}^{n}, will have just one apparent r-fold point* if it meets the latter, M_{r-1}, in an $(r-2)$-dimensional variety of order $r(r^2-1)/3$. This implies that M_{r-1}^{n} and $M_{r-1}^{n'}$ are such that a general S_3 meets them in two curves C^n and $C^{n'}$, respectively, having $r(r^2-1)/3$ points in common and that C^n has $r(r^2-1)\cdot(3r-2)/24$ and $C^{n'}$ has $r(r-1)(r-2)(3r-5)/24$ apparent double points. It also implies that a general S_4 meets them in two surfaces having respectively $r(r+1)(r-1)^2(r-2)^2/48$ and $r(r-1)(r-2)^2(r-3)^2/48$ apparent triple points. The number of apparent $(k-1)$-fold points on their intersections with an S_k can also be found.†

As we are going to make use of M_{r-1}^{n}‡ to obtain an r^2-ic

* By an apparent r-fold point of a V_{r-1} in S_{r-1} we mean a line passing through a general point of S_{r+1} and meeting V_{r-1} r times.

† See B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35, No. 3, pp. 339–343.

‡ There are in S_{r+1} many $(r-1)$-dimensional varieties of order m $[2r-1\leqq m\leqq r(r+1)/2]$ that have just one apparent r-fold point but the M_{r-1}^{n} here described is the only one that offers the desired transformation.

birational transformation between S_r and S_r', we make a few remarks concerning the manifold. It can be shown that this manifold can be represented upon an S_{r-1} by means of an ∞^{r+1}-system of $(r+1)$-ic $(r-2)$-dimensional varieties passing through an $(r-3)$-dimensional variety of order $(r+1)(r+2)/2$ in S_{r-1}. The case $r=3$ is well known: the points of M_2^6 are in a one-to-one correspondence with the points of a plane, the fundamental curves of representation being quartic curves through 10 points in the plane.

M_{r-1}^n is also the locus of points whose polar r-spaces with respect to r hyperquadrics $Q_r^{(i)}$ $[i=1, 2, \cdots, r]$ in S_{r+1} meet in lines all lying in an S_r. The polar r-spaces of points of S_r with respect to $Q_r^{(i)}$ meet in lines r-uply secant to M_{r-1}^n. Again, an S_r meets M_{r-1}^n in a V_{r-2}^n which is the Jacobian variety of the r quadric varieties in which S_r meets $Q_r^{(i)}$. The polar $(r-1)$-spaces of the points of V_{r-2}^n with respect to the same quadric varieties of S_r meet in lines intersecting V_{r-2}^n r times and forming a $V_{r-1}^{r^2-1}$ of order r^2-1 on which V_{r-2}^n lies r-uply.

If we replace r by $r-1$ in n, the M_{r-1}^n of S_{r+1} becomes an $M_{r-2}^{n'}$ of S_r which is of the same nature as the variety in which S_r meets $M_{r-1}^{n'}$. Now $M_{r-1}^{n'}$ is such that a general S_{r-1}-section of it is the Jacobian variety of $r-1$ $(r-2)$-dimensional quadric varieties in S_{r-1}. It is to be noticed that $M_{r-1}^{n'}$ is a ruled manifold composed of ∞^{r-2} lines all meeting M_{r-1}^n r times. These lines or the points of the variety in which an S_r meets $M_{r-1}^{n'}$ can be set in a one-to-one correspondence with the points of an S_{r-2}.

Now let two r-spaces S_r and S_r' be given in S_{r+1} and let S_r meet M_{r-1}^n in a V_{r-2}^n and S_r' meet M_{r-1}^n in a $V_{r-2}^{'n}$. These two varieties, V_{r-2}^n and $V_{r-2}^{'n}$, have in common a V_{r-3}^n which lies in the R_{r-1} of intersection of S_r and S_r'. The r-fold secants of V_{r-2}^n form a $V_{r-1}^{r^2-1}$ and those of $V_{r-2}^{'n}$ form a $V_{r-1}^{'r^2-1}$. These two r-fold secant varieties both intersect R_{r-1} in the same $V_{r-2}^{r^2-1}$ which contains V_{r-3}^n r-uply.

Since M_{r-1}^n has just one apparent r-fold point, from a point P of S_r we draw the r-fold secant to M_{r-1}^n meeting S_r' in

a point P' which is said to correspond to P. As the construction is reversible, the correspondence is birational. To show that the correspondence is of order r^2, that is, to an $(r-1)$-space of one of the given r-spaces corresponds a $V_{r-1}^{r^2}$ of order r^2 of the other, we notice that the ∞^{r-1} r-uple secant lines of M_{r-1}^n that meet an S_{r-1} form a $V_r^{r^2}$, for an S_r containing S_{r-1} meets it in a composite variety composed of S_{r-1} and a $V_{r-1}^{r^2-2}$. Hence if P describes an S_{r-1} in S_r, the corresponding r-fold secant of M_{r-1}^n describes a $V_r^{r^2}$ which is met by S_r' in a $V'^{r^2}_{r-1}$. Similarly, there is a $V_{r-1}^{r^2}$ in S_r corresponding to an S_{r-1}' in S_r'.

Now $V_{r-1}^{r^2}$ passes through the variety V_{r-2}^n of intersection of M_{r-1}^n with S_r, r times; and through the $V_{r-2}^{r^2-1}$ in R_{r-1}, once. Since, from the very nature of the transformation, every point of R_{r-1} not on $V_{r-2}^{r^2-2}$ is its own image, $V_{r-1}^{r^2}$ has in common with R_{r-1}, besides the $V_{r-2}^{r^2}$, an S_{r-2} in which R_{r-1} meets the S_{r-1}' to which $V_{r-1}^{r^2}$ corresponds.

Now to show that to a line in one r-space corresponds an r^2-ic curve in the other. Let P describe a line l in S_r. The line l meets the variety $V_{r-1}^{r^2-1}$ of lines r-uply secant to the V_{r-2}^n in which S_r meets M_{r-1}^n in r^2-1 points, i.e., meets r^2-1 of the lines of $V_{r-1}^{r^2-1}$. Hence the ∞^1 r-fold secants of M_{r-1}^n that meet l form a ruled surface F^{r^2} of order r^2, for it is met by S_r in a composite curve made up of r^2 lines one of which is l. Now F^{r^2} is met by S_r' in a curve C'^{r^2} which is the image of l in S_r.

Similarly, it can be shown that, if a point describes a k-space in the one r-space, the corresponding point in the other r-space describes a k-dimensional variety of order $\binom{r}{k}^2$.

If we project S_r' upon S_r, we have an involutorial transformation of order r^2 in S_r. Attention is here called to the fact that this involutorial correspondence is the product of two involutorial r-ic correspondences. One such can be set up by means of r of the $r(r-1)$-dimensional quadric varieties $Q_{r-1}^{(i)}$ in S_r. To a point P we make correspond the point P' of intersection of the r polar $(r-1)$-spaces with respect to $Q_{r-1}^{(i)}$. If P describes an S_k, P' describes a V_k^μ, where $\mu=\binom{r}{k}$. By setting up in S_r another involutorial r-ic transformation in

the same manner, we obtain as the image of V_k^{μ} a $V_k^{\mu^2}$ which is of the same nature as that obtained by means of the r^2-ic transformation as the image of an S_{r-1}.

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ON SOME FUNCTIONS CONNECTED WITH $\phi(n)$

BY S. SIVASANKARANARAYANA PILLAI*

Let $\phi(n)$ denote, as usual, the number of numbers not greater than and prime to n. Let $N(x)$ be the number of distinct numbers less than x, which can be the ϕ function of some number; and let $R(n)$ be the number of solutions of the equation

$$n = \phi(x),$$

n being given. The object of this note is to prove some results concerning the magnitude of $N(n)$ and to apply them to prove that

$$\overline{\lim_{n=\infty}} R(n) = \infty .$$

Since there is no reference to such results in Dickson's *History of the Theory of Numbers*, I believe that the last result in particular is new.

THEOREM I. *We have*

$$N(n) > \frac{a \cdot n}{\log n},$$

where a is a constant.

PROOF. For each prime p, $\phi(p) = p - 1$; hence, if we denote by $\pi(n)$ the number of primes not exceeding n, then

$$N(n) \geqq \pi(n).$$

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* I take this opportunity to express my deep gratitude to K. Ananda Rao for his invaluable guidance and encouragement.

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But by elementary methods it has been proved that*

$$\pi(n) > \frac{a \cdot n}{\log n}$$

where $a > 0$ is a constant. Therefore

$$N(n) > \frac{a \cdot n}{\log n} .$$

THEOREM II. *We have*

$$N(n) = O\left\{\frac{n}{(\log n)^t}\right\} ,$$

where $t = (\log 2)/e$.

PROOF. If p is an odd prime, $\phi(p^\alpha)$ is even; and $\phi(m \cdot n) = \phi(m)\phi(n)$ when m and n are prime to each other. Therefore, if any number m is composed of more than r different odd prime factors, then $\phi(m)$ is divisible by 2^{r+1}. So, if a number of the form $2^r \cdot h$ (where h is odd) should be a $\phi(m)$, then m may contain at most r different odd prime factors. Consequently, in the set $2^s \cdot h_s$, where s takes the values $0, 1, 2, \cdots, r$, and h_s runs through all odd numbers not exceeding $n/2^s$, the number of numbers which can be the ϕ of some numbers, is not greater than

$$\pi_1(n) + \pi_2(n) + \cdots + \pi_{r+1}(n),$$

where $\pi_r(x)$ is the number of numbers not exceeding x, which are composed of r different prime factors. But the number of numbers in the set $2^s \cdot h_s$ considered above, is

$$= \sum_{0 \leqq s \leqq r} \left[\frac{n}{2^{s+1}}\right] = \sum_{0 \leqq s \leqq r} \frac{n}{2^{s+1}} + O(r) = n - \frac{n}{2^{r+1}} + O(r).$$

Hence, of the numbers $\leqq n$, at least

$$n - \frac{n}{2^{r+1}} + O(r) - \sum_{1 \leqq s \leqq r+1} \pi_s(n)$$

numbers cannot be the ϕ function of any number. Therefore

* Ramanujan's *Collected Papers*, pp. 208–209. Landau, *Vorlesungen über Zahlentheorie*, vol. 1; Theorem 112. etc.

$$N(n) \leqq n - \left\{ n - \frac{n}{2^{r+1}} + O(r) - \sum_{1 \leqq s \leqq r+1} \pi_s(n) \right\}$$

$$= \frac{n}{2^{r+1}} + O(r) + \sum_{1 \leqq s \leqq r+1} \pi_s(n).$$

By elementary methods, Hardy and Ramanujan have proved that*

$$\pi^{\gamma}(n) < \frac{kn(\log \log n + c)^{\gamma-1}}{(\gamma - 1)! \log n},$$

where k and c are constants.

Hence, if $r+1 < \log \log n$,

$$\sum_{1 \leqq s \leqq r+1} \pi_s(n) = O\left(\sum_{1 \leqq s \leqq r+1} \frac{n(\log \log n + c)^{s-1}}{(s-1)! \log n} \right)$$

$$= O\left(\sum_{1 \leqq s \leqq r+1} \frac{n(\log \log n)^{s-1}}{(s-1)! \log n} \right),$$

for

$$\left(1 + \frac{c}{\log \log n}\right)^{s-1} \leqq \left(1 + \frac{c}{\log \log n}\right)^{\log \log n}$$

$$= e^{\log \log n \log (1 + c/\log \log n)} \leqq e^{c} = O(1).$$

But, since $r+1 < \log \log n$,

$$\frac{(\log \log n)^{s-1}}{(s-1)!} < \frac{(\log \log n)^{s}}{s!}.$$

Therefore, if $r+1 < \log \log n$,

$$\sum_{1 \leqq s \leqq r+1} \pi_s(n) = O\left(\frac{r \cdot n}{\log n} \frac{(\log \log n)^{r}}{r!} \right).$$

Therefore, if $r+1 < \log \log n$,

$$N(n) = O\left(\frac{n}{2^{r+1}}\right) + O(r) + O\left(\frac{n}{\log n} \frac{(\log \log n)^{r}}{(r-1)!} \right)$$

$$= O(s_1) + O(s_2) + O(s_3), \text{ say.}$$

Put

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* Ramanujan's *Collected Papers*. Paper No. 35, 2.2, Lemma A.

$$r = \left[\frac{\log \log n}{e}\right].$$

Then, by Stirling's theorem,

$$\begin{aligned}
\log s_3 &= \log n - \log\log n + r\log\log\log n - (r-1)\log(r-1) \\
&\quad - \frac{1}{2}\log(r-1) + r - 1 + O(1) \\
&\leqq \log\log n - \log\log n + \frac{\log\log n \log\log\log n}{e} \\
&\quad - \frac{\log\log n}{e}\log\frac{\log\log n}{e} - \frac{1}{2}\log\frac{\log\log n}{e} \\
&\quad + \frac{\log\log n}{e} + O(1) \\
&= \log n - \log\log n\left(1 - \frac{1}{e} - \frac{1}{e}\right) - \frac{1}{2}\log\log\log n \\
&\quad + O(1) \leqq \log n - \log\log n\left(\frac{e-2}{e}\right) + O(1) \\
&\leqq \log n - \frac{\log 2}{e}\log\log n + O(1) \\
&= \log n - t\log\log n + O(1).
\end{aligned}$$

Therefore

$$s_3 = O\left(\frac{n}{(\log n)^t}\right),$$

$$s_2 = O(r) = O(\log\log n) = O\left(\frac{n}{\log^t n}\right),$$

$$s_1 = O\left(\frac{n}{2^{\log\log n/e}}\right) = O\left(\frac{n}{\log^t n}\right).$$

Therefore

$$\begin{aligned}
N(n) &= O(s_1 + s_2 + s_3) \\
&= O\left(\frac{n}{\log^t n}\right).
\end{aligned}$$

Now we shall apply the above result to prove the following theorem.

THEOREM III. *We have*

$$R(n) \neq o(\log^t n)$$

and in particular,

$$\overline{\lim_{n=\infty}} R(n) = \infty .$$

PROOF. Let

$$s(n) = \sum_{1 \leqq m \leqq n} R(n) .$$

If possible, let

$$R(m) = o(\log m)^t .$$

Then

$$\begin{aligned} s(n) &\leqq N(n)\Big(\operatorname*{Max}_{1 \leqq m \leqq n} R(m)\Big) \\ &= N(n)\left\{o(\log^t n)\right\} \\ &= o\left(\frac{n}{\log^t n}\right)\left\{O(\log^t n)\right\} \\ &= o(n) , \end{aligned}$$

from Theorem II.

Now, $S(n)$ is the number of numbers whose ϕ functions are $\leqq n$. But, since $\phi(m) < m$, $S(n) \geqq n$, which contradicts $S(n) = o(n)$. The theorem is therefore proved.

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ON A FUNCTION CONNECTED WITH $\phi(n)$

BY S. SIVASANKARANARAYANA PILLAI

Let $\phi(n)$ be the number of numbers not greater than and prime to n. Further, let

$$\begin{aligned}\phi_1(n) &= \phi(n),\\ \phi_2(n) &= \phi\{\phi_1(n)\},\\ \phi_3(n) &= \phi\{\phi_2(n)\},\\ &\cdots\cdots\cdots\\ \phi_{r+1}(n) &= \phi\{\phi_r(n)\}.\end{aligned}$$

It is obvious that at one stage

$$\phi_r(1) = 1.$$

Hence, with an integer n, there is associated another $R(n)=r$, such that r is the least integer for which

$$\phi_r(n) = 1.$$

This short note is about this function $R(n)$.*

THEOREM I.

$$R(n) \leqq \left[\frac{\log n}{\log 2}\right] + 1.$$

PROOF. When $n>2$, $\phi(n)$ is always even. If x and y are even, and y contains at least one odd prime factor and x does not, then

$$\frac{\phi(x)}{x} > \frac{\phi(y)}{y}.$$

Hence, when n is even, $R(n)$ is maximum only if n is a power of 2.

If n is a power of 2,

* This problem was suggested by R. Vaidyanathaswami.

$$\frac{\phi(n)}{n} = \frac{1}{2},$$

and $\phi(n)$ is again a power of 2. Hence,

$$2^{R(n)} = n.$$

Therefore,

$$R(n) = \frac{\log n}{\log 2}.$$

But, by (1), $\phi(n)$ cannot be odd when $n > 2$.

Hence, the maximum is attained when

$$n = p_1 p_2 \cdots p_s,$$

where $p_1, p_2, \cdots, p_s$ are primes of the form $2^l + 1$. Hence, if there are infinitely many primes of the form $2^l + 1$, then there are infinitely many n's for which

$$R(n) > \frac{\log n}{\log 2} + 1 - \epsilon$$

for every positive ϵ.

But even in this case,

$$R(n) \leqq \frac{\log n}{\log 2} + 1.$$

But there are infinitely many values of n for each of which $R(n) = \log n / \log 2$, namely, $n = 2^r$.

THEOREM II. *If n is such that $R(n) < R(n')$ for every $n' > n$, let us define n as H. Then*

$$H = 2 \cdot 3^r.$$

To prove this, we want two lemmas.

LEMMA 1. *If x is even,*

$$R(2x) = R(x) + 1.$$

The proof is obvious.

LEMMA 2. *Whether x is odd or even, a multiple of 3, or not,*

$$R(3x) = R(x) + 1.$$

provided that $x \geqq 2$.

Since $\phi(3) = 2$, this easily follows from Lemma 1.

Now we are in a position to prove Theorem II. Put

$$2m = m_1, \qquad m_1 p = n_1,$$

where p is an odd prime greater than 3. As long as $\phi_{r-1}(m_1)$ is divisible by p,

$$\frac{\phi_r(m_1)}{m_1} = \frac{\phi_r(n_1)}{n_1}.$$

Let $\phi_{r-1}(m_1)$ be divisible by p, and $\phi_r(m_1)$ not. Then, obviously,

$$\phi_r(n_1) = \phi_r(m_1) \times p$$

and

$$\phi_{r+1}(n_1) = \phi_{r+1}(m_1) \times (p-1) = 2\phi_{r+1}(m_1) \cdot \frac{p-1}{2} = 2 \cdot u \text{ (say)}.$$

Since $\phi_{r-1}(m_1)$ is divisible by the odd prime p which is >3, and $\phi_{r-1}(m_1)$ is even, $\phi_r(m_1)$ is even and is greater than 2. Hence, $\phi_{r+1}(m_1)$ is even. So, whether $(p-1)/2$ is odd or even, u is even. Therefore, by Lemma 1,

$$R(\phi_{r+1}(n_1)) = R(2u) = R(u) + 1.$$

But

$$\frac{\phi_{r+1}(n_1)}{\phi_r(n_1)} = \frac{\phi_{r+1}(m_1) \times (p-1)}{\phi_r(m_1) \times p} = \frac{2\phi_{r+1}(m_1)}{\phi_r(m_1)} \times \frac{p-1}{2p};$$

and

$$\frac{p-1}{2p} \geqq \frac{5-1}{2 \cdot 5} = \frac{4}{10} > \frac{1}{3}.$$

Hence, before the rate of decrease of $\phi_r(n_1)$ becomes 3 times that of $\phi_r(m_1)$, the value of $R(n_1)$ will be increased at least by one. This is the case at every stage where the rate of decrease is increased due to an odd prime in $\phi_r(n_1)$, which is not in $\phi_r(m_1)$. Hence, if

$$\frac{\phi_r(n_1)}{n_1} \leqq \frac{\phi_r(m_1)}{m_1} \times \frac{1}{3^i},$$

then

$$R(\phi_r(n_1)) \geqq R(\phi_r(m_1)) + l.$$

Hence we may write

$$\begin{aligned} R(n_1) &\geqq R(m_1) + \left[\frac{\log p}{\log 3}\right] + 1 \\ &= R(m_1) + k \text{ (say)} \\ &= R(3^k m_1), \end{aligned}$$

by Lemma 2. However, we have

$$3^k m_1 > n_1.$$

Therefore, n_1 cannot be an H.

Therefore, in order that n may be an H, n should be divisible by no odd prime greater than 3. Hence, if n is an H,

$$n = 2^s \cdot 3^t.$$

But

$$\phi_1(2^s \cdot 3^t) = 2^s \cdot 3^{t-1},$$

when $s \geqq 1$,

. . . . . .

and

$$\phi_t(2^s \cdot 3^t) = 2^s.$$

Therefore

$$R(2^s \cdot 3^t) = t + s.$$

Hence

$$R(2^s \cdot 3^t) = R(2 \cdot 3^{t+s-1}).$$

But

$$2 \cdot 3^{t+s-1} > 2^s \cdot 3^t.$$

Hence, unless $s < 2$, n is not an H. So, $n = 2 \cdot 3^t$ or 3^t. But

$$R(2 \cdot 3^t) = t + 1 = R(3^t).$$

Hence

$$n = 2 \cdot 3^t.$$

THEOREM III.

$$R(n) \geqq \left[\frac{\log n - \log 2}{\log 3}\right] + 1.$$

PROOF. By Theorem II, $R(n)$ is a minimum, when $n = 2 \cdot 3^t$, and then $R(n) = t + 1$.

But

$$t = \left[\frac{\log n - \log 2}{\log 3}\right].$$

Therefore

$$R(n) \geqq \left[\frac{\log n - \log 2}{\log 3}\right] + 1.$$

Hence

$$\left[\frac{\log n}{\log 2}\right] \geqq R(n) - 1 \geqq \left[\frac{\log \frac{n}{2}}{\log 3}\right].$$

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MULTIPLE POINTS OF ALGEBRAIC CURVES*

BY T. R. HOLLCROFT

1. *Introduction.* Limits to the number of multiple points of algebraic curves were first found by Cramer.† He found and tabulated the maximum numbers of multiple points of all possible orders for curves of orders up to and including eight. Plücker‡ obtained the general expression $(n-1)(n-2)/2$ for the maximum number of double points of an algebraic curve of order n.

Except for individual curves, the maximum number of multiple points of higher order than two for a curve of given order has not been found. A general expression for the maximum number of compound singularities or singularities of different orders is not practicable. When, however, the curve possesses only multiple points or sets of multiple points of the same order, serviceable limits for the maximum number of such singularities can be found.

The purpose of this paper is to determine the maximum number of distinct multiple points of given order and con-

* Presented to the Society, June 20, 1929.

† G. Cramer, *Introduction à l'analyse des lignes courbes algébriques*, Geneva, 1750, pp. 455–459.

‡ J. Plücker, *Theorie der Algebraischen Curven*, Bonn, 1839, p. 215.

secutive multiple points of given order that may belong to an algebraic curve of given order.

2. *Invariants Associated with Multiple Points.* With each singularity of a curve f there are associated certain functions of the coefficients of the equation of f, invariant under a projective transformation, whose vanishing is necessary and sufficient for f to possess this singularity. Such functions will be called invariants. Also, all curves herein considered are proper curves.

A necessary and sufficient condition for f to have a multiple point of order r at P is that all the partial derivatives of f up to and including the $(r-1)$st vanish at P. These partial derivatives constitute $r(r+1)/2$ linear functions of the coefficients of f. The postulation of an ordinary r-fold point on f is, therefore, $r(r+1)/2$. The two independent coordinates of P may be eliminated from the $r(r+1)/2$ relations among the coefficients in $r(r+1)/2-2$ independent ways giving rise to $r(r+1)/2-2$ invariants associated with each ordinary r-fold point.

Consecutive multiple points of f are multiple points that coincide at a point P in such a way that all the branches of f common to two or more of these multiple points have a common tangent at P. Enriques* has proved that the postulation of s consecutive multiple points of orders r_i on f is the same as the total postulation of the s multiple points considered as distinct on f, that is

$$\frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1).$$

Assume that s points are consecutive on a curve. If one of the points P is fixed, one condition determines the direction of approach to P of each of the remaining $s-1$ points. The two parameters defining P and the $s-1$ parameters determining the $s-1$ directions through P total $s+1$ parameters.

* F. Enriques, *Lezioni sulla Teoria Geometrica delle Equazione e delle Funzioni Algebriche*, vol. 2, pp. 404–408.

Therefore the number of independent parameters involved in the location of s consecutive points on any plane curve is $s+1$.

The postulation of a singularity is the total number of conditions necessary and sufficient to determine both the nature of the singularity and its position. The $\sum r_i(r_i+1)/2$ relations among the coefficients of f then involve the $s+1$ parameters which determine the positions of the s consecutive points. From these $\sum r_i(r_i+1)/2$ relations the $s+1$ parameters may be eliminated in $\sum r_i(r_i+1)/2-s-1$ independent ways, each eliminant being an invariant associated with the singularity. There are, then,

$$\frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1) - s - 1$$

invariants associated with s consecutive multiple points of orders r_i. These invariants determine the nature of this singularity of f and only that.

3. *The Maximum Number of Consecutive Multiple Points.* It has been proved by Lefschetz* that there can be no more that $n(n+3)/2-8$ invariants among the coefficients of the equation of an algebraic curve of order n and genus $p>1$. In the preceding section it has been found that $sr(r+1)/2-s-1$ invariants are associated with s consecutive r-fold points. A limit to the number of consecutive r-fold points that may occur on a curve of order n and genus $p>1$ is therefore determined by the inequality

$$\tfrac{1}{2}sr(r+1) - s - 1 \leqq \tfrac{1}{2}n(n+3) - 8$$

which, when solved for s, becomes

$$s \leqq \frac{n^2+3n-14}{(r-1)(r+2)}. \tag{1}$$

* S. Lefschetz, *On the existence of loci with given singularities*, Transactions of this Society, vol. 14 (1913), pp. 23–24.

For $r=2$, this is the limit to the number of consecutive nodes of a curve of order n found by Sharpe and Craig.*

Another limit to the number of consecutive r-fold points of a curve of order n is found from the fact that the total number of double points cannot exceed $(n-1)(n-2)/2$. Since an r-fold point contains $r(r-1)/2$ nodes, this gives the limit

$$s \leqq \frac{(n-1)(n-2)}{r(r-1)}. \tag{2}$$

For $r=2, 3, 4$, limit (1) holds for $n \geqq 6, 10, 13$, respectively, otherwise limit (2). Both limits give $s=10$ for $r=2$ and $n=6$. For $n \leqq 6$, therefore, all the nodes of an algebraic curve of any genus may be consecutive.

Limits (1) and (2) are equal for

$$r = \frac{(n-1)(n-2)}{3n-8}.$$

Limit (1) holds for all values of r less than or equal to the above value and limit (2) holds for all values of r greater than or equal to this value. However, there are certain exceptions to limit (2) now to be discussed.

Since limit (2) is determined by the maximum number of double points of the curve, this limit would permit a curve of order n to have more than one point of order $r \geqq (n+1)/2$. This is impossible, since a line joining two such points would intersect the curve in more than n points. The range of r for limit (2) is, therefore, restricted to $(n-1)(n-2)/(3n-8) \leqq r \leqq n/2$.

Within the above range for r, limit (2) holds except when the value of s determined by limit (2) is of the form $n'(n'+3)/2$, where n' is any positive integer. A curve of order n' through the s consecutive r-fold points of f cannot intersect f in more than nn' points. In case the value of s is not the exact number necessary to determine a curve of order

* Sharpe and Craig, *Plane curves with consecutive double points*, Annals of Mathematics, vol. 16 (1914–15), p. 19.

n', then additional simple points of f must be chosen to determine the curve of order n' through all the s consecutive r-fold points of f. In this case, the number of intersections of the two curves is always less than nn' when limit (2) is satisfied.

There remains, then, an exception to limit (2) for $r \leqq n/2$ only when limit (2) gives the value $s = n'(n'+3)/2$ and then only when $nn' < sr$. We shall now prove that in all cases when $n > 4$, $s = n'(n'+3)/2$ and $nn' < sr$, the inequality $sr - nn' < r$ holds.

Assume $s = n'(n'+3)/2 = (n-1)(n-2)/[r(r-1)]$ and eliminate n' from this equation and the inequality $sr - nn' < r$ to be proved. This gives

$$3nr^2 - r(n^2+6n-2) - (n-1)(n-2) < 0.$$

This inequality is evidently satisfied except for large values of r. The largest value of r occurs when $s = 5$. If we eliminate r from the equality $5r(r-1) = (n-1)(n-2)$ and the above inequality, there results an inequality in n which is satisfied for all values of $n > 4$.

Therefore, when the value of s defined by limit (2) is of the form $n'(n'+3)/2$ and $nn' < sr$, since also $sr - nn' < r$, the maximum number of consecutive r-fold points for that order n is one less than the value given by limit (2).

The least value of r for which limit (2) holds is $(n-1)(n-2)/(3n-8)$. Substitute this value of r in limit (2) and there results $s = 9$ for $n > 6$. Substituting the largest value $n/2$ of r in limit (2), we obtain $s = 3$. The entire range of values of s given by limit (2) for $n > 6$ is, therefore, $9 \geqq s \geqq 3$.

When $s = 9$, a cubic can be passed through the nine multiple points, that is, $n' = 3$. Since $(n-1)(n-2)/(3n-8) - n/3$ is less than $1/3$ for all values of $n > 6$, the largest value of r in proportion to n for which $s = 9$ occurs when n is a multiple of 3 and $r = n/3$. In this case, a cubic through the nine multiple points intersects f in $3n$ points, that is, all the intersections of f and the cubic occur at the multiple points. The only

restriction on the positions of the nine multiple points of order $n/3$ is that they can not all lie at the nine intersections of two cubics.

The only case in which $nn' < sr$, therefore, occurs when $n' = 2$ and $s = 5$. If we set $s = 5$ in limit (2), there results a value of r in terms of n such that five times this value of r is always greater than $2n$. Then, in accordance with the above proof, when $r > 2n/5$ and limit (2) gives $s = 5$, the maximum number of consecutive multiple points of this multiplicity is four. This is the only exception to limit (2) within the range of the values of r for which it holds.

Since limit (2) is defined by the maximum number of double points of a curve, this limit as well as the above discussion of it applies equally well to distinct multiple points and to consecutive multiple points.

When r has its maximum value for a given n, if n is even, $r = n/2$, $s = 3$ and if n is odd, $r = (n-1)/2$, $s = 4$ for $n > 5$. These multiple points may be either distinct or consecutive.

If k cusps, $0 \leqq k \leqq r-1$, replace the same number of nodes in s consecutive r-fold points, the limit for s obtained by considering the number of invariants associated with the singularity is

$$s \leqq \frac{n^2 + 3n - 14 - 2k}{(r-1)(r+2)}. \tag{3}$$

Limit (2) is, however, unchanged. For given values of r, k and n, the smaller of the two limits (2) and (3) now determines the maximum value of s. In case the inequality $k \geqq 3n - 8 - (n-1)(n-2)/r$ is satisfied, limit (3) holds, otherwise limit (2). The same restrictions on limit (2) obtained above for $k = 0$, hold for all values of k since in limit (2) and in determining the number of intersections of two curves, a cusp counts merely as a double point.

If a curve f of order n has j singularities each consisting of s consecutive nodes, the following limits are found:

(a) $$j \leqq (n^2 + 3n - 16)/(4s - 2),$$

(b) $$j \leqq (n-1)(n-2)/(2s).$$

For $s \leqq (n-1)(n-2)/[(n-4)(n-5)]$ limit (a) holds, otherwise limit (b).

For $s=2$, solve $(n-1)(n-2)/[(n-4)(n-5)] \geqq 2$ for n and we obtain $n \leqq 11$. For $n=12$, however, 54 of its possible 55 nodes may form 27 tacnodes and limit (a) still be satisfied. Then for $n \leqq 12$, a curve of any genus may have all its nodes consecutive in pairs to form tacnodes when the number of nodes is even or all but one when the number is odd. Similarly, for $n \leqq 9$ a curve of any genus may have as many oscnodes as the integral number of times three is contained in the number of its nodes.

If each of the singularities consists of $s-1$ nodes and one cusp, all consecutive, each singularity accounts for $2s$ invariants. Then the maximum number j of such singularities must satisfy limit (b) and also the limit

(a′) $$j \leqq (n^2 + 3n - 16)/(4s).$$

Limit (a′) is less than limit (b) for $n \geqq 6$ and all values of s. For example, a quintic may have three ramphoid cusps or two tacnode-cusps. A sextic may have three tacnode-cusps, but not five ramphoid cusps nor two sets each containing four nodes and one cusp.

More generally, in order that a curve of order n have j singularities each consisting of s consecutive r-fold points, the following inequality must be satisfied:

$$j \leqq (n^2 + 3n - 16)/[s(r-1)(r+2) - 2],$$

and also the total number of nodes in the j singularities must not exceed $(n-1)(n-2)/2$.

In order to show that a curve of order n can possess a singularity consisting of s consecutive multiple points of orders $r_1, r_2, \cdots, r_s$, both of the following inequalities must be satisfied:

$$\sum r_i(r_i+1) - 2s \leqq n(n+3) - 14,$$
$$\sum r_i(r_i-1) \leqq (n-1)(n-2),$$

and also it must be shown that the sum of the $n'(n'+3)/2$

largest values of r_i does not exceed nn' for any value of n' such that $s \geqq n'(n'+3)/2$.

None of the above limits hold when more than two multiple points are collinearly consecutive. Since a line can not intersect a curve of order n in more than n points, the limit to the number of collinearly consecutive r-fold points is $s \leqq n/r$, or in case the multiple points are of different orders, the inequality $\sum r_i \leqq n$ must be satisfied.

4. *Limits for Distinct Multiple Points.* Since each r-fold point involves $r(r+1)/2-2$ invariants, the following limit defines the maximum number s of distinct r-fold points:

$$(1') \qquad s \leqq (n^2 + 3n - 16)/(r^2 + r - 4).$$

It was noted in the preceding section that limit (2) applies to distinct as well as to consecutive multiple points. Then limits (1′) and (2) apply to distinct multiple points. The discussion of limit (2) in the preceding section also applies equally well to distinct multiple points.

Limit (1′) holds for

$$r \leqq \{n^2 - 7 + [(n^2 - 7)^2 - 24(n-1)(n-2)(n-3)]^{1/2}\}/[6(n-3)],$$

otherwise limit (2). The radicand is negative for $n \leqq 16$, but for a given value of r the least value of n for which limit (1′) is less than limit (2) is 18.

For $r=2$, limit (1′) is always greater than limit (2). For $r=3$ or 4, limit (2) holds for $n \leqq 17$ and limit (1′) for $n \geqq 18$. As r increases, this discriminating value of n increases, for example, when $r=5$ or 6, limit (1′) holds for $n \geqq 20$, 23 respectively, otherwise limit (2).

We shall now prove that neither limit (1) for the maximum number of consecutive multiple points of order r nor limit (1′) for the maximum number of distinct multiple points of order r is subject to any restriction due to intersections of f with a curve through its multiple points.

For $r[(4n^2+12n-47)^{1/2}-1]/2$, limit (1′) $\geqq$ limit (1). For

$n \geqq 4$, this value of r is greater than the upper limiting value of r for which limit (1) holds. Then limit (1′) is larger than limit (1) for all values of n and r for which limit (1) holds. Then to prove that limit (1′) is never so large as to allow a curve of order n' through the s multiple points to have more than nn' intersections with the curve of order n possessing the multiple points will also establish the same property for limit (1).

Let $s = (n^2+3n-16)/(r^2+r-4) = n'(n'+3)/2$, that is, assume the largest possible value of s given by limit (1′) and assume further that this is the exact number of points necessary to determine a curve of order n' through the s distinct multiple points of order r. These are the conditions under which a curve of order n' through the s multiple points intersects the curve possessing the multiple points in the greatest number of points.

Together with the above equality, assume $nn' \geqq sr$. Eliminate n' from this inequality and the above equality and there results

$$r^2(n^2 + 3n - 16) - n(2n - 3r)(r^2 + r - 4) \leqq 0.$$

Substitute in this inequality the maximum value of r in terms of n for which formula (1′) holds and the resulting inequality in n is satisfied by all values of $n \geqq 5$.

Then the maximum number of multiple points of order r as defined by limit (1′) when distinct, or by limit (1) when consecutive, satisfies the criterion that no proper curve of order n' can be described through them intersecting the curve of order n possessing the multiple points in more than nn' points.

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CONTINUOUS CURVES IN WHICH EVERY ARC MAY BE EXTENDED*

BY W. L. AYRES†

1. *Introduction.* An arc α belonging to a point set K is said to be a *maximal arc* of K if there is no arc which belongs to K and contains the arc α as a proper subset. S. Mazurkiewicz‡ proved in 1921 that every arc of a bounded acyclic continuous curve§ M is a subset of a maximal arc of M. However there are many types of continuous curves in which every arc of a continuous curve is a subset of a maximal arc and which are not acyclic; and in a recent paper‖ we have given a necessary and sufficient condition in order that a continuous curve have this property. In this note we shall treat the opposite problem of finding under what conditions no arc is a subset of a maximal arc. We shall say that an arc α of a point set K is *extendible* in K if there exists an arc β of K of which α is a proper subset. If X and Y are the end points of α and Y is an interior point of β, then we say that α is *extendible in K in the direction XY*. And if there exists an arc β of K containing α and such that both end points of α are interior points of β, then we say α is *extendible in K in both directions*. In this note we shall give two necessary and suffi-

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‡ *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), pp. 119–130. See Lemma 13, p. 129.

§ A *continuous curve* is a closed connected, and connected im kleinen point set. A continuous curve is *acyclic* if it contains no simple closed curve. It must be noticed that under these definitions a single point is a continuous curve and is acyclic. However, to avoid trivial cases, we will assume that the continuous curve M, which is mentioned repeatedly, contains more than one point.

‖ W. L. Ayres, *Conditions under which every arc of a continuous curve is a subset of a maximal arc of the curve*, Mathematische Annalen, vol. 101 (1929), pp. 194–209.

cient conditions that every arc of a continuous curve M be extendible in M in both directions. The first of these conditions is that M shall not contain a certain type of point. From this it may be seen that in order that every arc of M be merely extendible in M it is necessary and sufficient that M contain at most one point of this type.

All of the point sets mentioned in this paper are assumed to lie in a two-dimensional euclidean space E_2.

2. *Preliminary Lemmas.* P. Urysohn* has shown that every point of a continuous curve may be ϵ-separated,† for any positive number ϵ, by a set consisting of a finite number of continua. By a result due to G. T. Whyburn and the author,‡ each of these continua may be replaced by a continuous curve so that we still have an ϵ-separation. Hence in characterizing types of points of a continuous curve through their ϵ-separating sets we may consider only the sets of continuous curves which form an ϵ-separating set. In the plane we may show that we need consider only arcs§ and simple closed curves.

LEMMA A. *If P is a point of a continuous curve M and ϵ is a positive number, there exists a simple closed curve J of*

* *Ueber im kleinen zusammenhängende Kontinua*, Mathematische Annalen, vol. 98 (1927), p. 297.

† A set B is said to be an *ϵ-separation* of the point P in the set M if $M-B=A+C$, where A and C are mutually separated sets and A contains the point P, and further the diameter of the set $A+B$ is less than ϵ. See P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), p. 65.

‡ *On continuous curves in n dimensions*, this Bulletin, vol. 34 (1928), pp. 349–360, Theorem 1. For two quite similar theorems published at approximately the same time, see P. Alexandroff and L. Tumarkin, *Beweis des Satzes, dass jede abgeschlossene Menge positiver Dimension in einem lokal zusammenhängenden Kontinuum von derselben Dimension topologisch enthalten ist*, Fundamenta Mathematicae, vol. 11 (1928), pp. 141–144; and W. Stepanoff and L. Tumarkin, *Ueber eine Erweiterung abgeschlossener Mengen zu Jordanschen Kontinuen derselben Dimension*, Fundamenta Mathematicae, vol. 12 (1928), pp. 43–46. However for the present application we require the property (2) of the theorem by Whyburn and myself. This property is stated in neither of the latter two theorems.

§ We consider a point as a special case of an arc.

diameter less than ϵ, enclosing P and such that $J \cdot M$ consists of a finite number of components, that is, every point of M may be ϵ-separated by a simple closed curve or a finite set of mutually exclusive arcs. If there exists a positive number r and a positive integer n such that, for every circle C with center P and radius less than or equal to r, the point P belongs to the boundary of exactly n $(n>0)$ complementary domains of $M>C$, then P may be ϵ-separated in M by a set of n arcs, for any positive number ϵ.

If P belongs to the boundary of no complementary domain of M, then M contains a simple closed curve which encloses P and is of diameter less than ϵ^*. If P belongs to the boundary of a complementary domain D of M, let C be a circle with center P and radius less than $\epsilon/2$ and such that C contains a point of D. Now using a slight modification of the proof of Theorem 1 of our paper, *On the separation of points of a continuous curve by arcs and simple closed curves*,† we may complete our proof.

LEMMA B. *If the point P of the continuous curve M may be ϵ-separated, for any positive number ϵ, by a set S of n mutually exclusive continua such that P is not contained in a bounded complementary domain of any one, but not by any set S of less than n, then P may be ϵ-separated by a set of n mutually exclusive arcs.*

For each positive integer k, let C_k be a circle with center P and radius $1/k$ and let π_k be the number of complementary domains of $M+C_k$ that have P and a point of C_k on their boundaries. Evidently $\pi_{k+1} \geqq \pi_k$ for every k. If the number π_k increases indefinitely with k, then it is easy to see that the number of components of any ϵ-separating set must increase indefinitely as ϵ approaches zero. But by hypothesis, n components suffice. Hence there is an integer s such that for $t \geqq s$, $\pi_t = \pi_s$. First let us suppose that $\pi_s > 0$.

* G. T. Whyburn, *Concerning continua in the plane*, Transactions of this Society, vol. 29 (1927), p. 381, Theorem 7.

† Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 201–206.

It follows from Lemma A that P may be ϵ-separated by a set of π_s mutually exclusive arcs for any $\epsilon>0$. By our hypothesis, $\pi_s \geqq n$. And it is easy to see that for $\epsilon<1/s$, no ϵ-separating set has less than π_s components. But P may be ϵ-separated by a set of n continua for each ϵ. Therefore $\pi_s \leqq n$. Hence $\pi_s = n$ and our lemma is proved for the case $\pi_s > 0$.

Now consider $\pi_s = 0$. There are two cases according as P is or is not a limit point of $E_2 - M$. If P is not a limit point of $E_2 - M$, there exists a positive number r such that every point of the interior of a circle with center P and radius r belongs to M. Then for $\epsilon < r/2$ there is no ϵ-separating set S such that P is not contained in a bounded complementary domain of some one component of S. Hence this case is impossible. In the remaining case we shall show that n must be equal to one and complete our proof by establishing the following lemma.

LEMMA C. *If B denotes the point set consisting of the sum of the boundaries of the complementary domains of a continuous curve M, and P is a point of $\overline{B} - B$, then P may be ϵ-separated in M by a single arc, for any positive number ϵ.*

There exists a simple closed curve J_1 which belongs to M, is of diameter less than ϵ and contains P in its interior. As P belongs to $\overline{B} - B$ there exists a complementary domain D of M in the interior of J_1 and a positive number ϵ' such that no point of $J_1 + D$ is within a distance ϵ' of P. There exists a simple closed curve J_2 which belongs to M, is of diameter less than ϵ', and contains P in its interior.* Then J_2 lies in the interior of J_1 and D lies between J_1 and J_2. Now let J_3 be any simple closed curve which lies between J_1 and J_2 and contains a point of D. Let $G_1, G_2, G_3, \cdots$ be the set of all components of $J_3 \cdot (E_2 - M)$. This set contains at least one component and is countable. Each component G_i lies in a complementary domain D_i of M. Let B_i be the boundary of D_i and let

$$G = (J_3 - \sum G_i) + \sum B_i.$$

* G. T. Whyburn, *Concerning continua in the plane*, loc. cit.

Now (1) G lies entirely between J_1 and J_2 except for points on J_1 and J_2, (2) M contains G, (3) G is a continuous curve, (4) P lies in a bounded complementary domain of G.* The outer boundary of this domain is a simple closed curve J_4† which belongs to M, contains P in its interior, and has two points X and Y on the boundary of some complementary domain D_i. Let XUY be an arc from X to Y which lies in D_i except for X and Y. The point P lies in the interior of J_4 and thus it lies in the interior of one of the two simple closed curves formed by the arc XUY plus one of the arcs of J_4 from X to Y. Call this arc XVY. Then XVY is the desired ϵ-separating arc, for P lies in the interior of the simple closed curve $XUY+XVY$ and this simple closed curve lies in the set consisting of J_1 plus its interior.

3. THEOREM 1. *In order that every arc of a continuous curve M be extendible in M in both directions it is necessary and sufficient that M contain no point which can be ϵ-separated in M by an acyclic continuous curve for every positive number ϵ.*

The condition is sufficient. Every "point" arc may be extended because every point of a continuous curve, except end points, is interior to an arc of M. But an end point may be ϵ-separated by a point for every ϵ, and there exists no such point by our condition. Now let α be any true arc of M and let P and Q be the end points of α. We shall show that α is extendible in M in the direction QP. Suppose this is not the case. Let us consider the components of $M-\alpha$. No one of these has P as a limit point for then α would be extendible in M in the direction QP, contrary to our supposition. Let ϵ be any positive number. There exists a point X of α so that the diameter of the subarc XP of α is less than ϵ, and there exists a point Y $(X \neq Y)$ of the subarc XP so that

* For detailed proof of these and the following statements, see page 204 of my paper, *On the separation of points of a continuous curve by arcs and simple closed curves*, loc. cit.

† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 4.

the diameter of the subarc YP of α is less than $\epsilon/2$. Since only a finite number of the components of $M-\alpha$ are of diameter greater than any given positive number,* there exists a point $Z(Z \neq Y)$ on the subarc YP of α so that (1) no component of $M-\alpha$ of diameter greater than $\epsilon/2$ has a limit point on ZP and (2) no component of $M-\alpha$ has a limit point on both ZP and XQ. Then the subarc XZ is an ϵ-separating set of P in M. But this is contrary to the hypothesis. Hence α is extendible in M in the direction QP, and similarly in the direction PQ.

The condition is necessary. Suppose M contains a point P which can be ϵ-separated by an acyclic continuous curve for every ϵ. We shall show that this leads to a contradiction by constructing in M an arc which cannot be extended in M in one direction. By Lemma B, P may be ϵ-separated in M by an arc. There exists a sequence $J_1, J_2, J_3, \cdots$ of simple closed curves such that (1) the interior of J_i contains J_{i+1} and P, (2) the diameter of J_i approaches zero as i increases indefinitely, (3) $J_i \cdot M = A_iB_i$ consists of a single arc. When A_iB_i is not a point, then A_i and B_i are the end points of A_iB_i. In the order from A_i to B_i let C_i and D_i denote the first and last limit points of the component of $M-A_iB_i$ that contains P. Let E_i and F_i $(i>1)$ denote the first and last limit points on A_iB_i of that component of $M-A_iB_i$ that contains $A_{i-1}B_{i-1}$. Each of the subarcs C_iD_i and E_iF_i of A_iB_i is a cutting of M. The component of $M-C_iD_i$ containing P lies in the interior of J_i, and the component of $M-E_iF_i$ containing P lies in the interior of J_{i-1}. There exists an arc α with end points D_1 and D_2 and lying except for D_1 entirely in the component of $M-A_1B_1$ that contains P.† In the order from D_1 to D_2 there is a first point G_1 of A_2B_2 on α. Consider the two arcs C_3D_3 and E_3F_3. Either one end point of E_3F_3 is not an interior point of C_3D_3 or both end

* W. L. Ayres, *Note on a theorem concerning continuous curves*, Annals of Mathematics, vol. 28 (1927), pp. 501–2.

† R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 342.

points of E_3F_3 are interior points of C_3D_3. In the first case let H_1 be this end point. And in this case there is an end point H_2 of C_3D_3 so that the subarc H_1H_2 of A_3B_3 contains the entire arc C_3D_3. In the latter case let H_1 be either end point of E_3F_3 and let H_2 be that end point of C_3D_3 so that the subarc H_1H_2 of A_3B_3 contains the entire arc E_3F_3. In either case, then, we have a subarc H_1H_2 of A_3B_3 so that H_1H_2 contains either C_3D_3 or E_3F_3. There exists* an arc G_2H_1 of M which lies, except for one end point G_2 on J_2 and one end point H_1 on J_3, entirely between J_2 and J_3. In general, the arc $A_{2n+1}B_{2n+1}$ $(n=1, 2, 3, \cdots)$ contains an arc $H_{2n-1}H_{2n}$ which contains one of the two arcs $C_{2n+1}D_{2n+1}$ or $E_{2n+1}F_{2n+1}$, and is such that H_{2n-1} is either E_{2n+1} or F_{2n+1} and H_{2n} is either C_{2n+1} or D_{2n+1}. There exist two arcs $G_{2n}H_{2n-1}$ and $H_{2n}G_{2n+1}$ of M such that G_{2n} is a point of J_{2n}, G_{2n+1} is a point of J_{2n+2}, and these two arcs lie except for their end points entirely between J_{2n} and J_{2n+1} and J_{2n+1} and J_{2n+2} respectively. Let

$$\beta = P + C_1D_1 + \text{subarc } D_1G_1 \text{ of } \alpha + \sum_{n=1}^{\infty} (\text{subarc } G_{2n-1}G_{2n} \text{ of } A_{2n}B_{2n}) + \sum_{i=1}^{\infty} H_{i+1}G_{i+1} + \sum_{n=1}^{\infty} (\text{subarc } H_{2n-1}H_{2n} \text{ of } A_{2n+1}B_{2n+1}).$$

Evidently β is an arc of M from C_1 to P. And further we may show that this arc may not be extended in M in the direction C_1P for each of the arcs $H_{2n-1}H_{2n}$ contains a cutting of M and these cuttings of M are such that the diameter of the component of $M - H_{2n-1}H_{2n}$ containing P approaches zero as n increases. Hence any arc of M with P as an end point contains a point of infinitely many of the arcs $H_{2n-1}H_{2n}$. Therefore β cannot be extended in M in the direction C_1P.

4. Theorem 2. *In order that every arc of the continuous curve M be extendible in M in both directions it is necessary and sufficient that* (1) *the sum B of the boundaries of all the complementary domains of M be a closed point set,* (2) *every point of B be either a cut point of M or belong to the boundaries of at least two complementary domains of M.*

* See R. L. Wilder's paper cited in the footnote on p. 855.

The conditions are necessary. Suppose $\overline{B}-B$ contains a point P. By Lemma C, P may be ϵ-separated in M by an arc for every positive ϵ. But by Theorem 1, M contains no such point P. Therefore $\overline{B}-B$ is vacuous and the first condition is necessary. Now suppose P is any point of B. If C is any circle with center P, then P is accessible from at least one complementary domain of $M+C$ since B contains P. If for every circle C, P is accessible from just one complementary domain of $M+C$, then P may be ϵ-separated, for any $\epsilon>0$, by an arc, from Lemma A. But this is impossible by Theorem 1. Hence there is some circle C so that P is accessible from at least two complementary domains D_1 and D_2 of $M+C$. If D_1 and D_2 belong to different complementary domains of M, we have the latter part of condition (2). And if D_1 and D_2 belong to the same complementary domain of M, it may be shown that P is a cut point of M, which is the first part of condition (2).

The conditions are sufficient. Let P be a point of M. If P is not a point of B, it is not a point of $\overline{B}$ by condition (1) and there exists a circle with center P and radius r such that every point of the interior of the circle belongs to M. Then P cannot be ϵ-separated by an acyclic continuous curve for $\epsilon<r$. If P belongs to the boundary of just one complementary domain of M, P is a cut point of M and for ϵ sufficiently small, no cut point may be ϵ-separated in M by a set consisting of just one component. And if P belongs to the boundaries of two complementary domains of M, we may show that any ϵ-separating set, for ϵ sufficiently small, must contain more than one component. Hence M contains no point which can be ϵ-separated in M, for every positive ϵ, by an acyclic continuous curve. By Theorem 1, every arc of M is extendible in M in both directions.

5. Theorem 3. *If every arc of the continuous curve M is extendible in M in both directions, every point of M lies on the boundary of some complementary domain of M* $(M\equiv B)$.

By Theorem 2, the set B, the sum of the boundaries of the

complementary domains of M, is a closed point set. Now suppose there is a point P of $M-B$. Every complementary domain of M is a complementary domain of the point set B and in addition there is at least one complementary domain of B, all of whose points belong to M. Let D denote the complementary domain of B containing P. We shall assume that D is bounded for we may always produce this by an inversion. Let B_1 denote the outer boundary of D.* Then B_1 is the common boundary of two domains, one the exterior of B_1 and the other a bounded domain containing D.† Then no point of B_1 disconnects B_1.‡ Then by a theorem due to R. L. Moore,§ only a countable number of the points of B_1 are cut points of M. Hence, by Theorem 2, uncountably many points of B_1 are accessible from two complementary domains of M. Each of these points is also a limit point of D. There are only a countable number of complementary domains of M and thus only a countable number of pairs of complementary domains of M. And by a theorem due to G. T. Whyburn,¶ at most two points can be accessible from each of the two domains of a pair and at the same time be limit points of D. Thus in all only a countable number of points of B_1 can be accessible from two complementary domains of M. Therefore our assumption that $M-B$ is non-vacuous is false.

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* R. L. Moore, loc. cit., p. 256.

† Ibid., Theorem 3, p. 258.

‡ R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213, Theorem 2.

§ *Concerning the cut-points of continuous curves and of other closed and connected point-sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106, Theorem B*.

¶ *Concerning plane closed point sets which are accessible from certain subsets of their complements*, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 657–666, Theorem 1 and lines 1–3 of page 659.

CLASSES OF DIOPHANTINE EQUATIONS WHOSE POSITIVE INTEGRAL SOLUTIONS ARE BOUNDED*

BY D. R. CURTISS

1. *Introduction.* If upper bounds have been found for the positive integral solutions of a diophantine equation, the problem of obtaining all such solutions is reduced to making a finite number of trials. It may therefore be of interest to note certain cases where upper bounds are given by simple algegraic processes. Hereafter the term *solution* will always mean a solution in positive integers.

Our starting point is the observation that if $P(t)$ is a polynomial in t, then all positive values of x that satisfy the inequality $P(1/x) \geqq 0$ are bounded if (and only if) the term of lowest degree in $P(t)$ has a negative coefficient.

2. *A Type whose Solutions are always Bounded.* Every algebraic diophantine equation in n variables $x_1, x_2, \cdots, x_n$ can be thrown into a form where the right side is zero and the left side is a polynomial in the reciprocals of the x's. When this has been done, the first type here to be considered is the following:

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) - k = 0, \tag{1}$$

where F is a polynomial all of whose coefficients are positive, while k is a positive constant, and $F(0, 0, \cdots, 0) = 0$.

The positive integral solutions of every equation of type (1) *are bounded.*

To prove this statement, and to show how to obtain bounds for the solutions, let us first consider a solution such that $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. We shall then have

* Presented to the Society, December 28, 1927.

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - k \geqq 0, \tag{2}$$

since each term of F is not decreased when another x is replaced by x_1. The term of lowest degree in (2), $-k$, is negative, hence, from the observation made in the second paragraph of this paper, x_1 must be bounded. To obtain an explicit upper bound we note that, since $1/x_1^n \leqq 1/x_1$ when n and x_1 are positive integers, and since all the terms of F are positive, we have

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) \leqq \frac{1}{x_1} F(1,1, \cdots, 1).$$

Hence when this result is applied to (2) we obtain

$$\frac{1}{x_1} F(1,1, \cdots, 1) - k \geqq 0,$$

or

$$x_1 \leqq \frac{1}{k} F(1,1, \cdots, 1). \tag{3}$$

Usually a lower bound than this can be derived by finding a closer approximation for the (unique) positive root of the equation obtained from (2) by retaining only the sign of equality.

Let us now find a bound for $x_r (r \leqq n)$ when upper bounds X_j have been assigned for each x_j from $j=1$ to $j=r-1$ in a solution where the x's form an ascending sequence from x_1 to x_n. We write (1) in the form

$$\begin{aligned} &F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) \\ &- F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) \\ &= k - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right). \end{aligned} \tag{4}$$

The first two lines of (4) reduce to a polynomial

$$F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_{r+1}}, \cdots, \frac{1}{x_n}\right)$$

each of whose terms is positive and involves at least one of the variables $x_r, \cdots, x_n$. Its value will not be decreased if each of the variables $x_{r+1}, \cdots, x_n$ is replaced by x_r. The last line of (4) is positive for all x's from x_1 to x_{r-1} that belong to solution systems, since the F function that appears here consists only of certain terms of the complete F in equation (1) and must lack some of the terms* of the complete F when $r-1<n$. Hence the inequality derived from (4),

$$F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_r}, \cdots, \frac{1}{x_r}\right)$$
$$-\left[k - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)\right] \geqq 0,$$

is of the type $P(1/x_r) \geqq 0$, with term of lowest degree in $1/x_r$ negative. It follows that x_r is bounded for each set of values of the preceding x's. If these preceding x's are bounded, there must be an upper bound for all the values of x_r that belong to solution systems. Since we have shown that x_1 is bounded, it follows that all the x's are bounded, in solutions where $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. We conclude at once that they are bounded for every order of relative magnitudes.

If the x's are arranged in order of magnitude from x_1 to x_n, an explicit upper bound for x_r, of which that given by (3) for x_1 is a special case, is indicated by the inequality

$$x_r \leqq \frac{1}{m_r} F_r(1, 1, \cdots, 1) = X_r, \tag{5}$$

where m_r is the least positive value of

$$k - F(1/x_1, 1/x_2, \cdots, 1/x_{r-1}, 0, 0, \cdots, 0)$$

for $x_j \leqq X_j$, $(j = 1, 2, \cdots, r-1)$. Since m_r may be difficult to

* We assume that each of the variables $x_1, \cdots, x_n$ is explicitly present in the F function of equation (1).

evaluate, we note that (5) can be replaced by the weaker inequality

$$x_r \leqq D_r F_r(1,1,\cdots,1) = X_r', \tag{6}$$

where D_r is the product of all the denominators of terms of

$$k - F(1/X_1', 1/X_2', \cdots, 1/X_{r-1}', 0, 0, \cdots, 0).$$

As an illustration, consider the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

Here $F(1, 1, \cdots, 1) = n$, and $F_r(1, 1, \cdots, 1) = n - r + 1$. The value of $1/m_r$ is u_r, where $u_1 = 1$, $u_{k+1} = u_k(u_k + 1)$.* Thus from (5) we obtain the upper bounds

$$X_r = (n - r + 1)u_r, \quad (r = 1, 2, \cdots, n).$$

From (6) we have another set of bounds X_r', such that

$$X_r' = (n - r + 1)X_1' X_2' \cdots X_{r-1}'.$$

Hence

$$X_1' = n, \; X_2' = (n-1)n,$$
$$X_r' = (n-r+1)(n-r+2)(n-r+3)^2(n-r+4)^4 \cdots n^{2^{r-2}}, r > 1.$$

3. *More General Types.* Equations of more general type than (1) can be written in the form

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right), \tag{7}$$

where F and G are polynomials all of whose coefficients are positive; we suppose all possible cancellations to have been performed. One or more of the variables may not be present in F, and the same may be true of G, but no variable is to be absent from $F - G$. We now investigate bounds for solutions where $x_1 \leqq x_2 \leqq \cdots \leqq x_n$.

If there is a constant term on either side of (7), x_1 *is bounded.*

* See *On Kellogg's diophantine problem*, American Mathematical Monthly, vol. 29 (1922), p. 380.

For, if, for example, we have $G(0, 0, \cdots, 0) = k \neq 0$, then $F(1/x_1, 1/x_1, \cdots, 1/x_1) \geqq k$, and formula (3) gives a bound for x_1. If (7) has no constant term, we consider the two inequalities derived from (7),

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - G\left(\frac{1}{x_1}, 0, \cdots, 0\right) \geqq 0,$$

$$G\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - F\left(\frac{1}{x_1}, 0, \cdots, 0\right) \geqq 0.$$

If the term of lowest degree in $1/x_1$ on the left of either of these inequalities is negative, we conclude that x_1 is bounded.

Again, using the notation of the earlier part of this paper, we deduce from (7) the inequality

$$(8) \qquad F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_r}, \cdots, \frac{1}{x_r}\right) - \left[G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)\right] \geqq 0,$$

and another in which F_r is replaced by G_r, and F and G are interchanged. Unless the expression in brackets is zero, one of these inequalities is of the type $P(1/x_r) \geqq 0$, with negative constant term. Hence *unless the pair of equations*

$$(9) \qquad F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right),$$

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)$$

has a solution in positive integers $x_1, x_2, \cdots, x_n$, *arranged in that order of magnitude, every solution of* (7) *so ordered has* x_r *bounded if all the preceding* x*'s are bounded.* From this we at

once obtain a sufficient condition that all the x's be bounded.

If the above condition fails we may replace the expression in brackets in (8) by

$$G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_r}, 0, 0, \cdots, 0\right) - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_r}, 0, 0, \cdots, 0\right),$$

and F_r by F_{r+1}. If in the new inequality thus obtained the term of lowest degree in $1/x_r$ is negative, or if it is negative in the companion inequality, we infer again that x_r is bounded if this is true of the preceding x's.

An example to show that the above conditions may not be fulfilled is given by the equation

$$1 + \frac{1}{x_2} = \frac{1}{x_1} + \frac{1}{x_3^2}.$$

Here for $r = 2$ the second equation of system (9) becomes

$$1 = \frac{1}{x_1},$$

and the pair of equations (9) is satisfied by $x_1 = 1$, $x_2 = x_3^2$, where x_2 and x_3 are not bounded. On the other hand, many equations whose solutions are bounded escape these tests; an example is

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_1} - \frac{3}{x_2} = 0,$$

whose only solution in positive integers is $x_1 = x_2 = 1$.

4. *Algebraic Equations with Positive Integral Roots.* A corollary of the first theorem of this paper concerns itself with algebraic equations

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots + (-1)^n a_n = 0,$$

all of whose roots are positive integers. There is, of course, an infinite number of such equations for any given n. But

there is only a finite number whose coefficients satisfy a relation

$$F\left(\frac{a_1}{a_n}, \frac{a_2}{a_n}, \cdots, \frac{a_{n-1}}{a_n}\right) = k,$$

where F is a polynomial with positive coefficients and $k>0$; for F is a polynomial in the reciprocals of the roots, and, when thus expressed, F has no constant term, so that the first theorem of this paper applies. We could obtain upper bounds for the roots, and therefore for the a's, by the methods of this paper. For example, if $a_{n-1}=a_n$, and if $x_1, x_2, \cdots, x_n$ are the roots, the x's must be solutions of the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1,$$

which has been discussed in § 2.

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ERRORS IN KRAITCHIK'S TABLE OF LINEAR FORMS

BY D. H. LEHMER

Tables of the linear forms that belong to a given quadratic residue D, or in other words, the linear divisors of the quadratic form t^2-Du^2 were first published by Legendre.* A list of errors in these fundamental tables has been given by D. N. Lehmer.† Kraitchik‡ has recalculated and extended these tables to the limit $D=\pm 250$. It is of great importance in using the table that every entry be correct. Therefore in constructing his factor stencils, D. N. Lehmer found it advisable to make a new table by means of a more or less graphical method.§ This table which has not been pub-

* *Théorie des Nombres*, 1st. ed., Tables III–VII, 1798.

† This Bulletin, vol. 8 (1902), p. 401. See also the correction in this Bulletin, vol. 31 (1925), p. 228.

‡ *Théorie des Nombres*, vol. 1, p. 164–186, Paris, 1922. *Recherches sur la Théorie des Nombres*, vol. 1, p. 205–215, Paris, 1924.

§ This Bulletin, vol. 31 (1925), pp. 497–498.

lished extends to $D = \pm 300$. The following list of errors results from collating these two tables. Kraitchik's forms for $D = \pm 182$ contain so many errors that a complete list of correct forms is given below. In as much as there were no errors detected in Lehmer's table it is reasonable to expect that the following list is complete.

ERRORS IN KRAITCHIK'S TABLE OF LINEAR DIVISORS

Théorie des Nombres, vol. 1.

D	For	Read
+38	59	53
−38	116	117
−42	55	53
−42	159	157
+69	55	53
−86	89	87
−102	147	145
−103	67	79
−103	177	179
−105	57	67
−106	73	71
−107	191	193
−109	333	103
−110	39	49
−110	207	217
−113	397	171
+122	195	199
−138	163	169
−141	413	415
+146	77	119
−146	77	303
−149	367	365
+151	183	189

Théorie des Nombres, vol. 1.

D	For	Read
+157	107	109
−157	471	529
+165	112	113
−166	473	477
−173	655	309
+174	203	61
−181	359	357
−181	491	461
−181	719	721
−185	661	253
+190	119	197
+191	173	175
+191	271	275
+193	155	129
−193	541	155
−193	617	231
+194	41	47
−194	453	455
−197	191	199
+199	309	257
−199	309	257
−199	insert	371

Recherches, vol. 1.

D	For	Read
+211	287	289
−217	319	317
−218	533	535
+222	99	95
−222	483	485
−226	375	373
−226	385	395
−226	387	397
+227	241	261
−229	197	199

Recherches, vol. 1.

D	For	Read
+230	23	33
−233	915	925
−241	607	357
−241	697	693
−241	731	733
−246	387	389
−247	105	449
−249	197	695
−249	301	799

CORRECT TABLES FOR $D = \pm 182$

$D = +182$

$728n \pm$						
	1	9	15	19	25	33
	37	41	43	51	55	59
	61	69	71	73	81	83
	85	87	89	93	97	101
	103	107	109	113	115	121
	135	141	145	149	151	155
	157	159	171	173	179	181
	187	197	199	201	211	225
	227	233	235	237	239	241
	253	265	269	285	289	297
	307	311	317	319	333	335
	337	341	347	353	359	361

$D = -182$

$728n +$						
	1	3	9	11	23	25
	27	31	33	37	41	47
	61	67	69	73	75	79
	81	85	89	93	95	97
	99	101	109	111	113	121
	123	127	131	139	141	145
	149	157	163	167	173	181
	183	191	197	201	207	215
	219	223	225	233	237	241
	243	251	253	255	263	265
	267	269	271	275	279	283
	285	289	291	295	297	303
	317	323	327	331	333	337
	339	341	353	355	361	363
	369	379	381	383	393	407
	409	415	417	419	421	423
	435	447	451	467	471	479
	489	493	499	501	515	517
	519	523	529	535	541	543
	549	551	557	563	569	573
	575	577	591	593	599	603
	613	621	625	641	645	657
	669	671	673	675	677	683
	685	699	709	711	713	723

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MENGER ON THEORY OF DIMENSIONS

Dimensionstheorie. By Karl Menger. Leipzig and Berlin, B. G. Teubner, 1928. iv+319 pp.

There is an important phase in the development of modern point set theoretical geometry which has been closely associated with the concept of dimensionality,—we refer to the attempt to create precise mathematical meaning for the simple geometric spaces of our intuition in terms of primitive non-arithmetical concepts. That the idea of dimensionality should have come into play and itself have been studied and made precise is indeed natural, since the curves, surfaces, and solids of our experience furnish the very basis for our intuitive ideas of dimensionality. The simple arithmetic definition of dimensionality, however, as the number of parameters required to define a space, while useful in ordinary geometry, was of course entirely inadequate when the more abstract spaces came into consideration, and it became highly desirable that dimensionality be relieved of its arithmetical vesture and be based on the inner structure of space itself.

Consider the spaces of our experience. What non-arithmetic relations among them are intuitively certain? The following immediately suggest themselves: a solid can be separated into several parts by one or more surfaces, a surface by curves and a curve by points. It was Poincaré who in 1912 suggested that precisely this type of phenomenon might lead to a satisfactory non-arithmetic definition of dimensionality,—a definition by recurrence. A space may be called n-dimensional, he suggested, if it can be separated into several parts by means of continua of $n-1$ dimensions.

Although the Poincaré definition was far from satisfactory either in precision or in content, it must be regarded as of the highest historical importance, since it indicated the possibilities of definition by recurrence, and was moreover essentially topological. These virtues were recognized by Brouwer, who in 1913 slightly modified the content of the definition and stated it in terms of the topologically precise notions of separation and connectedness; as a basis for recurrence, a zero-dimensional space was defined to be one which contained no continuum as subset. Brouwer showed that this "natural" definition of dimensionality satisfied the formal requirement of yielding the number n when applied to a cartesian S_n.

The dimensionality definition has since undergone further modification. If dimensionality was to be studied per se, it was of course desirable to arrive at a definition which would give rise to a theory of the highest generality and simplicity. To this end the basis for recurrence was altered, and what was of more significance, the concept of dimensionality as a local property was introduced. In its modern form the definition is as follows: a space is at most n-dimensional if each point is contained in an arbitrarily small neighborhood with a boundary of dimensionality at most $n-1$; (-1)-dimensional spaces are null spaces. A space which fails to be at most $(n-1)$-dimensional is at least n-dimensional. A space is n-dimensional if it

is at most and at least n-dimensional. Finally, an obvious modification yields the definition of dimensionality at a point.

An equivalent to this definition was introduced some eight years ago by the youthful Russian mathematician Paul Urysohn, who developed his theory in one of the most brilliant memoirs of recent years (posthumously published). The definition in the form given above was introduced independently at about the same time by Professor Karl Menger of Vienna and developed in a sequence of shorter papers characterized by their elegance and generality. The essentials of the dimensionality theory, which has by now attained a considerable perfection through the recent writings of Menger, Hurewicz, Alexandroff and others, have been developed with admirable clarity and completeness in a recently published book by Professor Menger. Let us review briefly some of the notable results of the theory as it is presented by the author.

A subset of a given space may itself be regarded as a space. If each of the subsets $M_1, M_2, \cdots$, is n-dimensional, is the set $\sum M_i$ n-dimensional? The answer is yes provided the M_i are closed. That the situation is entirely different in the general case is seen from Urysohn's remarkable theorem that an n-dimensional set is the sum of $n+1$, but not of fewer than $n+1$, 0-dimensional sets. On account of its obvious importance, the summation theorem for closed sets is developed by the author at the very outset. There is included an extremely elegant proof by Hurewicz which yields at the same time the Urysohn theorem. The polish of the modern methods is strikingly revealed when this proof is compared with the original proof of Urysohn.

In 1911 Lebesgue pointed out the following property of euclidean n-space: if a bounded region is covered by a finite number of closed sets of sufficiently small diameter, there must exist points which are common to $n+1$ of the covering sets. This remarkable property of ordinary space holds with suitable modifications for abstract n-dimensional spaces. Certain striking generalizations, together with a sort of converse which states that covering sets with intersection properties analagous to those of the (closed) n-cells of an n-complex, can always be chosen,—are among the author's own achievements. For reasons which we shall later indicate, the importance of this group of theorems can hardly be overestimated.

A significant feature of the theory as it stands at present is the relative unimportance of the assumption of compactness. The extension of the greater part of the theory to the more general separable metric spaces,—made possible by recent improvements in methods, furnishes further justification for the modern definition of dimensionality. The permanence, in this sense, of the theorems of dimensionality is analogous, as the author points out, to the permanence of the laws of arithmetic on the extension of a number domain.

There are, however, certain interesting exceptions. If a space R is n-dimensional, it must certainly be n-dimensional at certain of its points. Now if R^n represents the totality of such points, what is the dimension of R^n? If we assume that R is compact, the answer is that R^n is n-dimensional *at each of its points*. Quite different is the situation if R is not com-

pact. In fact we have actually an example, due to Sierpiński, of a one-dimensional space which is one-dimensional at only a denumerable set of points, and these points constitute a zero-dimensional set by the summation theorem. The author has succeeded in demonstrating that R^n as a space can not be of fewer than $n-1$ dimensions. Whether or not R^n may be less than $(n-1)$-dimensional at certain of its points remains one of the important unsolved problems in the structure analysis of separable spaces.

The cartesian spaces S_n are of course n-dimensional in the sense under consideration; were this not the case, the definition of dimensionality would fail to satisfy the most fundamental requirement. Since dimensionality is clearly a topological invariant, we have here a proof of Brouwer's "classical" theorem concerning the topological non-equivalence of S_p and S_q where $p \neq q$. It is this theorem which imparts validity to the definition of n-dimensional complex in the sense of combinatorial analysis situs.

We see then that one aspect of the problem of determining completely the relations between abstract spaces and number spaces of a given dimensionality is solved. Another aspect, concerning which there is little known as yet, we may characterize as the problem of the introduction of coordinate systems into point-set theoretical geometry. This problem is of extreme importance, since it is just here that a fusion between combinatorial analysis situs and point-set theoretical analysis situs is beginning to take place.

The few facts in this connection which are known are highly interesting and suggestive. We have for example the fact already referred to concerning the complex-like structure of an arbitrary space of finite dimensionality. There is also the recent result due to the author, and destined, it seems, to be of considerable importance, that every n-dimensional space R is homeomorphic to a subspace of a cartesian S_{2n+1}. That this theorem holds when R is a complex is obvious from the remark that two S_n's in S_{2n+1} fail in general to intersect. It is this fact together with the complex-like structure of R, which furnishes a guiding principle for a proof of the general theorem. The question as to whether or not S_{2n+1} is the space of lowest dimensionality in which R can be immersed, is still one of the interesting unsolved problems.

We mention finally the theorem of Alexandroff that if R is a closed n-dimensional space immersed in a euclidean space, there exists an n-dimensional approximating complex into which R can be carried by means of a singular continuous deformation in which each point moves an arbitrarily small distance; such an approximating complex can not be of fewer than n dimensions. Thus the permanence of dimensionality under a distortion depends on the magnitude of the distortion. It is essentially this fact which constitutes the germ of Brouwer's original proof of the "classical" invariance theorem.

It is impossible to compare Menger's book with the original sources without admiring the thoroughness and elegance with which he has embraced the essentials of the dimensionality theory as it stands today into a single volume of some three hundred pages. Nor can it be said that this has necessitated undue condensation. One may complain perhaps that the historical notes at the end of each section, while quite complete, avoid in

their conciseness certain questions of emphasis. But as for the subject matter, there is no feeling of compression; on the contrary the author frequently pauses to suggest problems, explain difficulties and guiding principles, and emphasize values. For some theorems there are developed several distinct proofs, each bringing to light new aspects of the theory and new points of view. The essentials of the point set theory of separable spaces are developed at an early stage so that one may read the entire volume without outside reference. The material is well arranged and the printing is excellent, with exceptionally few errors. The book constitutes, in short, a notable presentation of an important chapter in modern mathematics.

P. A. Smith

TONELLI ON TRIGONOMETRIC SERIES

Serie Trigonometriche. By Leonida Tonelli. Bologna, Nicola Zanichelli, 1928. viii+523 pp.

The extent of the existing literature on the theory of trigonometric series is tremendous and keeps increasing very rapidly. During the quarter of a century after 1900 the theory has made remarkable progress and it would not be an exaggeration to say that now it is of equal and fundamental importance for all branches of the modern mathematics including the theory of numbers on the one hand, and mathematical physics on the other. In spite of, or perhaps because of this, there is practically no place in the literature where an adequate account of the theory is given, except for the second volume of Hobson's *Theory of Functions* and two excellent but short reports by M. Plancherel (L'Enseignement Mathématique, vol. 24 (1925), and by E. Hilb and M. Riesz (Encyklopädie der mathematischen Wissenschaften, vol. II, 3_2, 1924).

Under such circumstances the publication, by a mathematician of Tonelli's rank, of a large volume devoted exclusively to the theory of trigonometric series must be considered as a significant event, even if it does not represent a step toward the solution of the difficult problem of creating an all-inclusive treatise on trigonometric series.

The work under review originated as a course of lectures on trigonometric series delivered at the University of Bologna in 1924–1925 with a view "to expounding in a systematic manner the classical results, together with more recent investigations on these series." The author found it more convenient to abandon the usual order of treatment of trigonometric series. The book begins with a study of general trigonometric series, in order "to reveal at once the properties which are common to all such series," and subsequently passes to the discussion of the special properties of Fourier series. An advantage of such a treatment lies, according to the author's opinion, in the fact that "the theory of the general trigonometric series can be presented in a form essentially elementary in character, while the theory of Fourier series, for its complete development, requires speculations of a more advanced nature." No space is given to the "beautiful investiga-

tions of Denjoy concerning the computation of the coefficients of an everywhere convergent trigonometric series, nor to the important theory of quasi-periodic functions" of H. Bohr. The author expects to devote a separate volume to the applications of the theory to geometry and mathematical physics.

The topics which are most completely treated in the monograph under review are: the absolute convergence of the general trigonometric series, Riemann's method of summation, the representation of functions by trigonometric polynomials and, above all, double Fourier series.

A short historical sketch is followed by Chapter I (General trigonometric series; §1, necessary conditions of convergence; §2, sufficient conditions; §3, necessary and sufficient conditions). §1 contains a detailed analysis of the Cantor theorem and of the problem of absolute convergence of trigonometric series. The reader will find here some important theorems (Steinhaus, Denjoy-Lusin) which are not so well known as they should be. §2 treats primarily of the uniform convergence and of the convergence almost everywhere on $(0, 2\pi)$. The principal tool in treating the former of these two problems is of course the classical Abel transformation which figures here under the name of Brunacci-Abel transformation, with a reference to the *Corso di Matematica Sublime* by V. Brunacci (vol. 1, Firenze, 1804). As to the latter problem, we find here an important theorem due to Kolmogoroff-Seliverstoff and to Plessner: If $\sum(a_n^2 + b_n^2) \log n$ converges, then the corresponding trigonometric series converges almost everywhere. It should be observed that the method of proof used is hardly of an "elementary" nature; on the one hand there is used an unproved lemma (p. 65) in the general theory of infinite series, which, so far as the reviewer is aware, is not stated in the standard treatises, like those of Bromwich or Knopp; on the other hand a reference is made to the Riesz-Fischer theorem, which is proved considerably later. The last §3 of Chapter I is devoted to the fundamental theorem of Riemann.

Chapter II (Representation of functions by means of trigonometric series) contains three parts: §1, conditions for the representation; §2, uniqueness of the expansion in a trigonometric series; §3, Fourier series. §§1 and 2 contain an interesting treatment of Riemann's method of summation and Cantor's uniqueness theorem, while in §3 the definition of Fourier series, together with some elementary examples, is given and the theorem of Du Bois-Reymond-Lebesgue is proved.

Chapter III (Approximate representation by means of trigonometric polynomials) contains four parts: §1, trigonometric interpolation; §2, method of least squares; §3, trigonometric polynomials of Fejér; §4, method of approximation of Tchebychev.

Chapter IV (Fourier successions) is subdivided into three parts: §1, Fourier sequences and series; §2, necessary conditions for Fourier successions; §3, sufficient conditions for Fourier successions. In §1 a short discussion of the uniqueness of determination of a function by means of the sequence of its Fourier coefficients is given. §2 contains the Riemann-Lebesgue theorem, an estimate of the order of magnitude of the Fourier coefficients of a function under various conditions, the Parseval identity,

the Young-Hausdorff theorem, the approximation theorem of Bernstein, the theorem of Szász and, finally, a discussion of convergence of the series $\sum a_n/n, \sum b_n/n, \sum a_n, \sum b_n$. The proof of the Young-Hausdorff theorem proceeds along the lines of the proof of F. Riesz with one deviation, however, which makes the proof more difficult; in establishing the existence of a minimum of a certain integral the reader is referred, not to the classical paper by F. Riesz in Mathematische Annalen, vol. 69, but to a general theorem of the calculus of variations as given in the second volume of the author's *Fondamenti di Calcolo delle Variazioni*. §3 is devoted to the proof of the Riesz-Fischer theorem and its corollaries.

Chapter V (Convergence of Fourier series) consists of six parts: §1, absolute convergence; §2, simple convergence; §3, uniform convergence; §4, partial convergence; §5, conjugate series; §6, degree of approximation. In §1 the theorem of Bernstein and some of the author's results are proved. §2 contains a discussion of the Dirichlet integral and of the classical criteria of convergence. One of these criteria (that of Young) is not mentioned and a new special criterion (due to the author) is added, namely, a "criterion of convergence at singular isolated points." The usefulness of this important paragraph would have been considerably increased if the author had given space to a comparative analysis of various criteria and showed examples of their applications, his own criterion included. In §4 we find an interesting theorem of Kolmogoroff: If $S_n(x)$ is the nth partial summation of the Fourier series of a function $f(x)$, then the sequence $\{S_{n_k}(x)\}$ converges to $f(x)$ almost everywhere, provided there is a constant $\lambda>1$, such that $n_{k+1}/n_k>\lambda$. The treatment of the conjugate series as given in §5 is rather short and hardly gives a right idea of the present situation of this question whose importance becomes more and more obvious.

Chapter VI (Operations on Fourier series) consists of three parts: §1, addition and multiplication; §2, integration; §3, differentiation. A remarkable theorem of Lusin, given in §2, deserves to be mentioned separately: If a trigonometric (not necessarily a Fourier) series converges at all points of (a, b) to a sum $S(x)$ which is integrable on (a, b), then the series can be integrated term by term over any subinterval which is entirely interior to (a, b).

Chapter VII (Singularities of the Fourier series: §1, singularities of the Fourier series of continuous functions; §2, singularities of the Fourier series of discontinuous functions) gives a discussion of singularities of Du Bois-Reymond and Lebesgue, and an account of the Gibbs phenomenon.

Chapter VIII (Classical integrals: §1, Poisson integral; §2, Fourier integral). §1 contains an exposition of classical results concerning the Poisson integral, including some of the results of Fatou. The treatment of the Fourier integral in §2 proceeds along the lines of Jordan-Pringsheim and of the Acta Mathematica paper by Hahn. The end of this paragraph is devoted to a discussion of the Sommerfeld integral.

Chapter IX (Double Fourier series: §1, preliminaries; §2, convergence of the double Fourier series; §3, summation by rows and columns; §4, Fejér trigonometric polynomials in two variables; §5, supplements). This concluding chapter is perhaps one of the most interesting in the volume. It is

based to great extent upon the author's own investigations and contains many results which reach farther than those known before. Considerable progress and simplification were made possible by introducing a new notion of a function of bounded variation in two variables, which contains most of those given before (Hardy, Vitali-Lebesgue-Fréchet, Arzelà): A function $f(x, y)\,[a \leqq x \leqq b,\ a \leqq y \leqq b]$ is of bounded variation in (x, y) if (i) $f(x, y)$ is of bounded variation in y for almost all values of x and of bounded variation in x for almost all values of y; (ii) the corresponding total variations are integrable as functions of x or of y respectively.

The selection of material in a course of lectures which is not to be considered as a treatise is to great extent a matter of taste. It is to be regretted, however, that such important topics as the theory of Fourier transforms, the general Cesàro summability of Fourier series (only C_1 summability is given consideration in the book), the relationship between the Fourier and power series, are not discussed in the volume under review. It is hardly possible to obtain a correct idea of the recent development of the theory of trigonometric series and integrals without touching these topics. But even if we concentrate our attention on the topics treated in the book, we might find several facts and names which perhaps deserve somewhat more than to be merely mentioned in a few footnotes. Such are, in the reviewer's opinion, important investigations of Hardy-Littlewood on the theory of Fourier constants (=Fourier successions), on the so-called strong summability of the Fourier series by the same, the theory of convergence factors by various writers, some investigations by Zygmund, Titchmarsh, Wiener, etc. There are also some names which should be referred to, at least in the footnotes, for instance, the name of Carleman, in connection with the discussion of the exponent of convergence of a Fourier succession of a continuous function, of the strong summability of Fourier series, etc.

Despite these criticisms we are convinced that *Serie Trigonometriche* by Tonelli will be of great value to anyone seriously engaged in studying the immense field of trigonometric series, and we are looking forward to the pleasure of reading the promised volume on the applications of trigonometric series.

J. D. Tamarkin

BOREL ON DIVERGENT SERIES

Leçons sur les Séries Divergentes. Par Émile Borel. Deuxième édition, revue et entièrement remaniée avec le concours de Georges Bouligand. Paris, Gauthier-Villars, 1928. vi+260 pp.

Those who have followed the progress made in the theory of divergent series during the past quarter of a century are well aware of the important rôle which the first edition of Professor Borel's book has played in stimulating the development of this theory. This was due, not only to the importance of the results contained in it, but also to the elegance of the exposition and the many suggestive lines of thought that were indicated.

Throughout the period referred to Professor Borel's book has remained a classic in its field, but in view of the rapid growth of the theory it was inevitable that certain portions of the discussion should seem somewhat lacking in completeness after a lapse of twenty-five years. It was therefore quite appropriate that a second edition should be issued, bringing up to date the sections where later research had thrown fuller light on the questions involved, and giving a succinct resumé of the many important developments in various branches of the theory that have been made since the date of the first edition.

The account of recent work is for the most part contained in an additional chapter of some thirty-five pages at the close of the volume. Considering the extent of the literature involved,* it was obviously somewhat of a task to give a coherent account of it in such brief space. The difficulties of the undertaking have been overcome in admirable fashion, and while necessarily many important studies had to be dismissed with merely a reference, the chapter on the whole gives an excellent bird's-eye view of the research material with which it deals. By selection of certain central lines of thought that have characterized the research in question, it has been possible to unify many separate studies and thus make a considerable gain in the brevity and directness of their treatment.

Theoretical discussions of divergent series can be broadly classified in groups in accordance with their relationship to two fundamental methods of procedure. The methods in question may be designated briefly as follows: (1) summation by means; (2) summation by convergence factors. In many studies one or the other of these methods is used exclusively; others involve both of them, and some deal explicitly with various relationships between the two methods. In the first edition of the *Leçons* summation by means played the dominating rôle, and naturally the same thing is

* In Smail's *History and Synopsis of the Theory of Summable Infinite Processes*, which deals with work through the year 1922, there are listed more than 250 publications since the appearance of the first edition of Borel's book. The synoptic account of these books and memoirs, in most cases quite brief, occupies more than 150 pages.

true of that part of the present edition which constitutes a revision of the first edition. However, in Chapter III, which deals with the general theory of summable series, attention is called at the beginning to these two methods. This is done in connection with an illustrative discussion of the application of the methods of divergent series to the study of Fourier's series. No better introduction to the two points of view could have been selected, since the application to Fourier's series furnishes a simple and illuminating example of the efficiency of the two methods and the interplay between them.

In the added chapter of the present work, Chapter VI, a condensed account is given of a variety of studies which center about the method of convergence factors. As some of the most fundamental work of recent years based on the method of means has already been treated in Chapter III, the latter method is more briefly dealt with here. In the field of applications of divergent series, the account is limited in the main to work in connection with Dirichlet's series, factorial series, and quasi-analytic functions.

Of the chapters that occur in the first edition, Chapter V, dealing with developments in series of polynomials, is carried over unchanged. The reason for this is well stated in a footnote by Professor Bouligand, who remarks that the chapter was not merely of a didactic character, but exhibited the contact of ideas of two great mathematicians.* Of the remaining chapters, those dealing with asymptotic series, continued fractions and the theory of Stieltjes, and analytic extension by the method of Borel summability cover practically the same ground as before. Many sections have been revised and certain amplifications and modifications suggested by later researches have been made, but the subject matter is substantially the same.

The chapter on the theory of summable series contains much new material, being about twice the size of the corresponding chapter in the first edition. Among the new topics introduced, in addition to the discussion of Fourier's series referred to above, may be mentioned a more complete discussion of the means of Cesàro and Hölder, including a proof of their equivalence, and some work of Sannia and Le Roy connected with Borel summability.

The final chapter of the book is followed by an interesting note by Professor Bouligand, demonstrating the impossibility of securing analytic extension by the use of means of certain general types. This note is followed by a list of results culled from various memoirs on divergent series and integrals, stated in the form of exercises.

C. N. MOORE

* Mittag-Leffler and Borel.

KÖNIG AND KRAFFT'S ELLIPTIC FUNCTIONS

Elliptische Funktionen. By R. König and M. Krafft. Leipzig, Walter de Gruyter and Co., 1928. 263 pp.

Many are the methods of approach to the study of elliptic functions. Since Abel's classic memoir (1826), an outgrowth of his researches on the lemniscate definite integral, much attention has been given to the development of the theory of algebraic and of abelian functions and of the chapter in this theory which has to do with the subject of this book. Some, as Riemann (1857) and Weierstrass (1875–6) followed first the lead of Abel in the 1826 paper (anticipated by Gauss as early as 1798) but later redeveloped the theory using the algebraic differential equation of the first order as a cornerstone on which to build. Others, as Jacobi, (1829 and 1834), and Eisenstein, starting with properties of multiply periodic functions, have developed the theory of elliptic functions as an elegant chapter in the theory of functions. The geometric properties and applications interested Riemann, Clebsch, (1864), and Brill and Noether, (1874); while the latest and indeed the least traveled avenue of approach, viz., the number-theoretic, called also arithmetic, was employed by Dedekind and Weber, (1879), by Kronecker, (1881), and by Hensel and Landsberg, (1902).

This book by König and Krafft gives the first detailed exposition of the arithmetic theory as applied specifically to functions having the three fundamental properties which characterize elliptic functions, viz., (1) they form a linear manifold, (2) they form an algebraic field, (3) they are of genus one. The aim has been "not to set up theorems and formulas with lemmas and computations constructed for the purpose but to develop the results organically and genetically" from these three properties of which the first is the simplest and at the same time the most general. Thus it is evident that the scope of the book is considerably more than the development of the arithmetic theory of functions of genus one. As the introduction tells us: The reader becomes familiar with Riemann's doctrine, which lies at the bottom of the whole plan of the book, a doctrine which not only supplies a Riemann surface on which the function is single-valued, but which also enables the reader to understand the classification of elliptic functions as a member of a great organism, Riemann function-pairs as an instance of Riemann function-systems; moreover, according to Riemann and F. Klein, the fundamental law which rules the organism is a law of duality by which every theorem on functions can be restated for differentials. The reviewer thinks the book justified by their claim that "though the material is not new, the contribution is in the painstaking methodical simplifications made even in the things well known." It is a contribution which is welcome. The book can be read with pleasure by one who is familiar with the theory of elliptic functions in any of its phases or ramifications, and it can be used quite satisfactorily as a text with a class which knows the elements of function theory though not without considerable amplification of the

material by the instructor. In particular the great array of notational symbols in the book makes the reading more difficult than the mathematical theory should require. If it could have been presented without this piling of symbol upon symbol it would have been better.

After reading the first three chapters a reader can turn to the general theory of algebraic functions as given in Hensel and Landsberg's treatise and in Chapter 24 of the third volume of Weber's *Lehrbuch der Algebra* and get the extension to an n-valued algebraic function with considerable ease. One should not assume that because the elliptic case is a special instance of the general case it is not worth while to study it in as detailed a fashion as is done here. As a matter of fact the translation of the notions from general (p arbitrary) to special ($p=1$) are far from obvious while the wide applications of the elliptic case make the special case important per se.

The list of chapter headings gives an indication of the scope of the book:

1. Grundlagen.
2. Arithmetischer Teil.
3. Die Elementarfunktionen.
4. Elementarfunktionen und Elementardifferentiale von zwei Veränderlichen.
5. Die Elliptischen Integrale.
6. Funktionen zweiter und dritter Art.
7. Das Abel'sche Theorem.
8. Konforme Abbildung durch elliptische Funktionen und birationale Transformationen.
9. Die elliptischen Funktionen und Integrale im Periodenparallelogramm.
10. Die Funktionen des Ringes.

In the first chapter are given fundamental definitions and notations. An elliptic algebraic function is single-valued and regular save for a finite number of poles on a two sheeted Riemann surface with four branch points. It is a special case of a class (K) of (Riemann) function-pairs on two sheets on which the branch cuts have been made but not joined. All properties of elliptic functions as fast as developed are discussed with respect to elliptic differentials.

Chapter 2 gives the arithmetic part. Using the methods of number theory we can find a basis for the entire class; an ideal basis for the ideal of the divisor, Q; a multiple basis for the divisor, Q, at all finite points; all functions which are multiples save possibly at P_∞ of a given divisor; a normal basis for such functions; and finally a basis which is normal at P_∞.

Chapter 3 treats of the characterization of elliptic functions and differentials. An elementary function with a fixed pole of arbitrary order is the element in terms of which any elliptic function is expressible. This elementary function plays a rôle similar to the fraction $1/(x-a)$ in the decomposition of rational functions into partial fractions. A corresponding elementary differential is defined. The method is extended to functions of two complex variables in Chapter 4, while in Chapters 5 and 6 elliptic integrals on the Riemann surface which has been made simply connected

by $2p$ period cuts are constructed in terms of integrals of the elementary differentials of two complex variables. Here the Weierstrass, Klein, and Prym prime functions are defined, again in terms of the elementary functions.

Chapter 7 gives an elegant exposition of the theory underlying Abel's theorem for elliptic integrals of the three kinds. In these seven chapters the functions have been studied on the two-sheeted Riemann surface.

In Chapter 9 the functions are studied on the period parallelogram which in its turn in Chapter 10 is mapped conformally on a ring. This follows an excellent discussion of conformal mapping by elliptic functions and birational transformations in Chapter 8. The arithmetic method is abandoned now for the more usual methods of analysis. The authors tell us that the shriveling of the theory due to the fact that now functions and differentials become identical gives greater simplicity but with loss in beauty and richness of structure. The Fourier developments of elliptic functions are obtained directly from known properties of integrals on the ring and may then be carried back to the period parallelogram. The simplicity of this exposition will appeal to those who know how Fricke obtained the same results by the application of functional equations.

The work of printer and editor have been well done. The text is almost entirely free from typographical errors.

Mayme I Logsdon

SHORTER NOTICES

Abriss der Logistik. By Rudolf Carnap. Wien, J. Springer, 1929. iv+114 pp.

The first part of this "outline" will be of interest primarily to those who desire a brief introduction to modern symbolic logic and the foundations of mathematics. Dr. Carnap, following, for the most part, the usages, development and analyses of the *Principia Mathematica*, expresses in the Peanesque notation the more important concepts and propositions of propositional logic, the logic of classes, the logic of relations, cardinal arithmetic, series, progressions, continua, etc. New and difficult concepts are frequently illuminated by simple illustrations. A considerable number of new symbols are introduced, and a definite formulation of the principle of substitution, (inexplicably omitted from the *Principia Mathematica*), is provided.

In the second, more important and original part, in terms of a few primitive ideas for each discipline, some of the fundamental concepts and propositions of such diverse studies as the theory of aggregates, geometrical topics, projective geometry, modern physics, causality, epistemology, etc., are represented. This section (together with the appendix, where a number of suggestions are offered for alternative approaches to the same subjects) marks a considerable extension of logistic; though, as the novel applications are not thoroughly exploited, and only obvious deductions and analyses are provided, there is no surety that the chosen primitives are always adequate for the given disciplines.

Logistic developments are never final. They are merely intermediate steps in the derivation of special sciences from logic. As all ideas are expressible in terms of class concepts, relations and associated notions, the "axioms" (Carnap's term) of all exact, logisticized sciences should be expressible as propositions which are completely derivable from the concepts and propositions of logic. Such a procedure will immediately reveal the strength and weakness of the familiar thesis (which Dr. Carnap accepts), that mathematics is a branch of logic. How much caste that thesis will retain when it is realized that, in the same sense in which mathematics is a part of logic, physics, epistemology, biology, grammar, etc. are also parts of it, is a matter for speculation. That realization, however, will bring home the as yet unrecognized necessity for a rediscussion of the nature of necessary truths, inasmuch as there does seem to be a sense in which the propositions of logic are necessary apart from all systems, in contrast to those of mathematics and other essentially systemic sciences.

The first part of this work is an excellent pocket *Principia Mathematica*; the second part should prove of value to those who are interested in extensions of logistic; as a whole it should help provide answers to questions regarding the significance of the logistic and logico-mathematical movement.

Paul Weiss

The Foundations of Euclidean Geometry. By Henry George Forder. xii+349 pp. Cambridge University Press, 1927.

Rarely is the object of a book so tersely expressed on the wrapper as in the case of the foundations of euclidean geometry. The publishers quote these words from the preface: "Although the euclidean geometry is the oldest of the sciences and has been studied critically for over two thousand years, it seems there is no text-book which gives a connected and rigorous account of that doctrine in the light of modern investigations. It is hoped that this book will fill the gap."

There may be a difference of opinion on the question of the completeness of the gap filling, but Mr. Forder has written an interesting book and one which should be welcomed by those interested in seeing the rigor of modern analysis achieved in geometrical studies.

Being anxious to achieve exactness, the author intersperses his definitions and axioms with copious notes explaining the advantages of particular statements. Thus, for example, he uses forty pages to establish his concepts of classes, relations, linear order, non archimedean systems, etc.

With this background he proceeds to set up axioms of order, which he points out were "almost completely ignored by Euclid, but which are fundamental for this work."

Starting with point and order as undefined entities definitions are made of line, plane, and space followed by development of fundamental properties of these.

"We have now laid the foundations of geometry. Our Axioms OI–VIII (order axioms) together with a continuity axiom and a Euclidean parallel axiom enable us to erect a complete Euclidean Geometry. Also—they enable us to introduce algebraic methods."

Following the introduction of angles and their properties, the question of congruence is considered. Here are first introduced the congruence axioms. With the introduction of circle axioms it is shown that the congruence axioms may be greatly weakened.

The introduction of the parallel axiom is deferred to the sixth chapter and is then stated in not fewer than eight different forms. In this chapter is also considered the relations between projective and euclidean spaces, ideal elements are defined and the addition of these to euclidean elements is discussed.

The Desargues and Pappus theorems come in for lengthy discussion from various points of view, while constructions are given a chapter to themselves with special attention to the Poncelet-Steiner and Mascheroni theorems and to the Gaussian theory of the regular polygon.

After chapters on analysis situs of polygons, areas of polygons and volumes of tetrahedra the author says: "We now reach the climax of our investigation. So far we have not assumed that our lines are continua. We shall now secure this property by assuming a continuity axiom." He completes his argument by showing that all euclidean geometry may be logically built upon order axioms, a very restricted parallel axiom and a weakened continuity axiom.

The last two chapters are quite distinct, almost in the nature of an appendix "designed to illustrate one or two interesting points" such as congruences as the sole undefined relation between points, non-euclidean areas, etc.

Mr. Forder has undertaken a difficult task and, while his work may be open to criticisms in some respects, he has made a readable book, the study of which by our teachers of euclidean geometry might go far toward eliminating the inaccuracy of concepts and illogical deductions from the minds of those supposedly engaged in teaching accurate, precise thinking. It is a worth while book which may well serve as a stimulus to others to write into our literature more exact presentations of our basic geometric concepts.

F. W. Owens

Operational Methods in Mathematical Physics. By Harold Jeffreys. London, Cambridge University Press, 1927. vii+101 pp. Price 6 s. 6 d.

This valuable monograph is number 23 of the important series entitled Cambridge Tracts in Mathematics and Mathematical Physics. The author says in the preface: "My own reason for writing the present work is mainly that I have found Heaviside's methods useful in papers already published, and shall probably do so again soon, and think that an accessible account of them may be equally useful to others." As a matter of fact he has filled admirably a gap of some thirty years standing in the literature pertaining to the solution of the differential equations of physics, since Heaviside's own work is not systematically arranged and in places its meaning is rather obscure. Jeffreys also affirms that "· · · it is certain that in a very large class of cases the operational method will give the answer in a page when ordinary methods take five pages, and also that it gives the correct answer when the ordinary methods, through human fallibility, are liable to give a wrong one."

A general idea of the scope of the text may be derived from the chapter headings, which are in order: Fundamental Notions, Complex Theory, Physical Applications: One Independent Variable, Wave Motion in One Dimension, Conduction of Heat in One Dimension, Problems with Spherical or Cylindrical Symmetry, Dispersion, and Bessel Functions. A wealth of illustrative material is compressed within these main divisions. Thus, for example, the third chapter deals with the following topics: (1) variation with time of the electric charge on the condenser plates for a circuit containing a voltaic cell, a condenser, and a coil possessing inductance and resistance; (2) the Wheatstone bridge method of determining inductance; (3) the seismograph, with special reference to the instruments of Galitzin and Milne-Shaw; (4) resonance for a simple pendulum; (5) motion of three particles attached to a stretched string; (6) radioactive disintegration of uranium; and (7) some dynamical applications.

The monograph will be interesting to mathematicians as well as to physicists because it brings out difficulties which require further investigation. For illustration, the first sentence on page 53 is: "A general proof that the results given by the operational method, when applied to the

vibrations of continuous systems, are actually correct, has not yet been constructed." Likewise the comments in section 8.8 are suggestive and illuminating. The volume closes with a note on the notation for the probability integral, with a page on interpretations of the principal operators, with a bibliography of papers in which operational methods are used, and with author and subject indexes.

H. S. UHLER

Nouvelles Tables de Log n! By F.-J. Duarte. Paris, Index Generalis, 1927. xxiv+136 pp.

This book contains in large clear type the common logarithms to 33 decimals of factorial n from $n=1$ to $n=3000$. A preface is given by Professeur M. R. de M. de Ballore, who emphasizes the importance and reliability of the work. The largest previous table by Degen, (1824), contained 18-place logarithms up to log 1200! The aim of Duarte was to allow adequate study of expressions constantly occurring in the theory of probability.

Incidentally the author points out errors in the logarithms of 829, 1087, 1409, 1900 as given by Wolfram and published in Vega's Thesaurus, also in Thoman's values for log 45! and log 55! and Degen's values for log 1093! and log 1180!.

In 1925 Duarte and Ballore published a similar twelve-figure table to $n=1000$.

The present table was constructed by adding successively the logarithms to 39 decimals of numbers from 1 to 3000. To control the table Stirling's formula was used to calculate independently to 36 places log $50k!$ for $k=1, 2, \cdots, 60$. The forcing of a digit in the 33d place is indicated by an asterisk.

If p is a prime beyond 1000, its logarithm may be calculated by means of the remarkable series $\log p=\frac{1}{2}\log(p-1)+\frac{1}{2}\log(p+1)+\Delta_0+\Delta_1+2.8\Delta_2+72.4\Delta_3/7+304.48\Delta_4/7+R_4$, where $4p\Delta_0=\log(p+1)-\log(p-1)$, $6p^2\Delta_k=\Delta_{k-1}$, $k=1, 2, 3, \cdots$, and $R_4<.0^{38}5$. The author used a similar series to compute $\log p$ when $17<p<1000$ by a proper choice of m such that $mp-1$ and $mp+1$ contain prime factors less than p. He accordingly started with 42-place logarithms of 2, 3, 5, 7, 11, 13, 17 and calculated all the other primes by this new method.

Three obvious errors in printing occur on pp. xiv, xvi, and xxii.

In calculating a logarithm the method of Flower is used and illustrated by the evaluation of $\log \pi$ and $500!/40!\times 460!$ The necessary table for Mn, $n=1, 2, \cdots, 100$; and the brief table for $\log 1.0^s k$, $s=0, 1, \cdots, 16$; $k=1, 2, \cdots, 9$, constitute Tables II, III, respectively, the former running to 40 decimals.

The reviewer was pleased to find a use for this new table in computing $\log \pi$ from the formula

$$\pi^{35}E_{17}=2^{36}\times 34!(1-1/3^{35}+1/5^{35}-1/7^{35}+\cdots)$$

correct to 33 places.

C. C. CAMP

The Symmetrical Optical System. By G. C. Steward. London, Cambridge University Press, 1928. viii+102 pp. Price 7 s. 6 d.

This monograph is number 25 of the important series known as Cambridge Tracts in Mathematics and Mathematical Physics. The elementary theory of the first chapter is followed by a discussion of the characteristic function of Hamilton, the "eikonal," and the aberration function. The way is thus prepared for the immediate consideration of coma, distortion, and geometrical aberrations of the first and higher orders. A chapter is next devoted to interesting presentations and proofs of the sine-condition, the cosine-conditions, Herschel's condition, and the very general optical cosine law. The fifth chapter is devoted to the preparation of formulas especially adapted to the computation of optical systems. However, no attempt is made to apply these results to numerical data which have arisen in the designing of actual optical apparatus. The two remaining chapters take into account the finite wave-length of light, and they deal respectively with the diffraction patterns associated with the symmetrical optical system and with three forms of aperture, namely, the annular aperture, the slit aperture, and the semi-circular aperture.

The usefulness of the tract is enhanced by the inclusion of lists of references at the ends of all the chapters. The figures or line diagrams are clear, well chosen, and sufficiently numerous, and the monograph will constitute undoubtedly a valuable reference source both on account of the uniformly clear, logical presentation of the material involved and because of the powerful analytical methods developed and employed.

H. S. Uhler

Théorie des Champs Gravifiques. (Mémorial des Sciences Mathématiques, No. 14.) By Th. de Donder. Paris, Gauthier-Villars, 1926. 60 pp.

This is the second of a sequence of pamphlets on the theory of relativity which de Donder is contributing to the Mémorial des Sciences Mathématiques (for a review of the first of these pamphlets see this Bulletin, vol. 32, p. 563).

In this pamphlet the author first derives the equations of a gravitational field in all possible generality. He then considers the more important special cases, such as those fields due to "ordinary" gravitation associated with the presence of material bodies, those electromagnetic fields due to the presence of charged bodies, and those due to the simultaneous presence of matter and electric charges. As in the other pamphlets of the Mémorial series, we have here a minimum of formal derivation with many references to the literature, the style suggesting that of the Encyklopädie. As a result we have a very good bird's eye view of a very wide domain.

C. N. Reynolds, Jr.

NOTES

In addition to those on the editorial staff of the Bulletin, the following persons have assisted the editors, either by refereeing papers or by advising concerning papers offered for publication in the present volume: A. A. Albert, Harry Bateman, E. T. Bell, B. A. Bernstein, H. F. Blichfeldt, Henry Blumberg, R. D. Carmichael, A. B. Coble, J. L. Coolidge, H. T. Davis, L. E. Dickson, L. L. Dines, G. C. Evans, Peter Field, T. C. Fry, Raymond Garver, Lois W. Griffiths, G. H. Hardy, T. R. Hollcroft, E. V. Huntington, Edward Kasner, J. R. Kline, E. P. Lane, C. H. Langford, C. G. Latimer, W. A. Manning, Marston Morse, F. D. Murnaghan, J. H. Neelley, James Pierpont, J. F. Ritt, J. A. Shohat, Virgil Snyder, E. B. Stouffer, J. D. Tamarkin, H. S. Vandiver, Warren Weaver, J. H. M. Wedderburn, G. T. Whyburn, W. M. Whyburn, D. V. Widder, R. L. Wilder, A. R. Williams, W. A. Wilson, R. M. Winger, B. C. Wong, F. E. Wood, J. W. Young. The editors desire publicly to recognize this service.

On account of the death of Dr. J. W. L. Glaisher, editor of the Quarterly Journal of Mathematics and the Messenger of Mathematics, the former journal will be discontinued, with the completion of its fiftieth volume. Professor G. H. Hardy has consented to edit the current volume (volume 58) of the Messenger of Mathematics, and it is hoped that this journal can be continued.

The first Congress of Roumanian Mathematicians was held at the University of Cluj, May 9-12, 1929. The attendance included one hundred fifty persons, and fifty communications were read.

Professor Florian Cajori, of the University of California, has been elected a vice-president of the Comité International d'Histoire des Sciences, which met at Paris, May 24, 1929. The official organ of this society is Archeion, edited by A. Mieli.

The Swiss Mathematical Society has established a foundation for the advancement of the mathematical sciences in Switzerland. The income of this foundation is to be used for the support of the Commentarii Mathematici Helvetici and other mathematical publications.

At the tenth annual meeting of the American Geophysical Union, Dr. William Bowie was elected chairman of the Union, and Mr. W. D. Lambert chairman of the section of geodesy.

The Ernst Abbe Memorial Prize for mathematics and physics for 1928 has been awarded to Professor Alexander Meissner, of Berlin, "dem Erfinder des Röhrengenerators."

The committee on astronautics of the Astronomical Society of France

has awarded its first annual international prize to Hermann Oberth, of the College of Medinsch, Roumania.

The New York Academy of Sciences offers its A. Cressy Morrison prize "No. I" for 1930 for a paper on solar and stellar energy. Competition is open to all.

On the occasion of its centenary celebration, the Stuttgart Technical School has made Professors A. Stodola of Zurich, and S. Finsterwalder, of Munich "Ehrenbürger," and has conferred an honorary doctorate on Professor E. Trefftz, of Dresden. The Zurich Technical School has conferred honorary degrees on Professors F. Emde and R. Grammel, of Stuttgart.

Professor O. Chisini, of the University of Milan, has been elected correspondent of the Reale Istituto Lombardo.

On the occasion of its seventh centenary celebration, the University of Toulouse conferred an honorary doctorate on Professor Tullio Levi-Civita, of the University of Rome. Professor Levi-Civita has also recently received an honorary doctorate in engineering from the Technical School of Aachen, for his work in mechanics and especially in hydrodynamics.

The University of Pisa has established a foundation for the advancement of mathematics, in honor of the late Professor L. Bianchi, of that university.

The University of Durham has conferred an honory doctorate on Sir Frank Dyson, astronomer royal.

Assistant Professor E. C. Bower, of the department of astronomy and mathematics at Ohio Wesleyan University, has been appointed Martin Kellogg fellow at the Lick Observatory of the University of California, with academic residence at Berkeley.

Dr. Deborah M. Hickey has been awarded an Alice Freeman Palmer fellowship at Wellesley College.

Dr. Alexander Pogo, of the Yerkes Observatory, has been appointed fellow in the history of science by the Carnegie Institution of Washington. He will work at Harvard University.

Dr. E. Esclangon, director of the Strasbourg Observatory, has been appointed director of the Paris Observatory.

Professor F. Gonseth, of the University of Bern, has been called to the professorship of higher mathematics at the Zurich Technical School, as successor to Professor J. Franel, who has retired.

Dr. Hermann Müntz, of the Institut Einstein in Berlin, has been appointed professor of mathematics at the University of Leningrad.

Mr. N. F. Mott has been appointed lecturer in mathematical physics at the University of Manchester.

At Ohio State University, Professor Alfred Landé, of the University of Tübingen, has been appointed visiting professor of theoretical physics for the autumn and winter quarters of 1929-30. Dr. L. H. Thomas, of Trinity College, Cambridge, has been appointed visiting assistant professor of theoretical physics for the autumn, winter, and spring quarters.

Dr. W. L. Ayres has been appointed to an assistant professorship at the University of Michigan.

Professor D. P. Bartlett, of the Massachusetts Institute of Technology, has retired.

Professor R. D. Carmichael has been appointed administrative head of the department of mathematics at the University of Illinois, as successor to Professor E. J. Townsend, who has retired.

Professor W. L. Crum, of Stanford University, has been appointed professor of economics at Harvard.

Assistant Professor P. D. Edwards, of Ball Teachers College, has been promoted to an associate professorship.

Mr. W. I. Foster, of Rochester Junior College, has been appointed assistant professor of mathematics at Northern Montana School.

Dr. B. F. Kimball, of Cornell University, has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor E. E. Libman, of the University of Illinois, has resigned to accept a position in the marine and air craft engineering department of the General Electric Company.

Associate Professor T. A. Pierce, of the University of Nebraska, has been promoted to a professorship.

Dr. Hillel Poritsky, has accepted a position with the General Electric Company.

Assistant Professor S. A. Schelkunoff, of the State College of Washington, has resigned to accept a position in the Bell Telephone Laboratories, New York City.

Mr. C. K. Sherer, of the University of Nebraska, has been appointed head of the department of mathematics at Texas Christian University.

Assistant Professor G. W. Smith, of the University of Kansas, has been promoted to an associate professorship.

Dr. Roxana H. Vivian has been appointed professor of mathematics at Hartwick College, Oneonta.

Mr. E. H. Wells, of Princeton University, has been appointed to an assistant professorship at the University of New Hampshire.

Dr. E. W. Woolard, of the George Washington University, has been promoted to an assistant professorship of mathematics.

Dr. H. A. Zinszer, professor of physics and acting professor of mathematics at Hanover College, has been appointed professor of physics and astronomy at the Kansas State Teachers College.

The following appointments to instructorships are announced:
Connecticut College, Miss Grace Shover;
Newark College of Engineering, Mr. J. H. Fithian;
New York University, Dr. D. A. Flanders, Mr. E. H. Johnson, Mr. A. S. Peters;
Ohio State University, Dr. P. M. Swingle;
Pennsylvania State College, Dr. Leo Zippin.

Mr. G. Coradi, known as a maker of mathematical instruments, especially integraphs and harmonic analyzers, died March 2, 1929, at the age of eighty-one.

Professor Emil Hilb, of the University of Wurzburg, died August 6, 1929.

Professor T. Lalesco, of the University of Bucarest, known for his work in integral equations, died June 15, 1929.

Professor F. Rudio, retired, of the Zurich Technical School, died, in June 1929, at the age of seventy-two.

Associate Professor Otto Schreier, of the University of Rostock, is dead.

Mr. G. R. Kaye, well known as a writer on the early history of Hindu mathematics, died July 1, 1929, at the age of sixty-two. He is best known for his elaborate study of the Bakhshālī Manuscript, appearing in the Archaeological Survey of India, volume 43, Calcutta, 1927. In all his writings, he tended to the belief that India owed her early mathematics only to the Greeks.

Professor F. C. Kent, of the Oregon State Agricultural College, died June 11, 1929. Professor Kent had been a member of this society since 1911.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ABBOT (E. A.). Flächenland, eine Geschichte von den Dimensionen von einem Quadrat (Edwin A. Abbot); aus dem Originalwerk "Flatland" ausgewält und ins Deutsche übertragen von W. Bieck. Leipzig, Teubner, 1929. 40 pp.

BENNY (L. B.). Mathematics for students of technology. Junior course, first year. London, Oxford University Press, 1929. 158 pp.

BERZOLARI (L.). See ENCYKLOPÄDIE.

BIECK (W.). See ABBOT (E. A.).

BJERKNES (C. A.). Niels Henrik Abel. En skildring av hans liv og arbeide. Omarbeidet og forkortet utgave i anledning 100-årsdagen for Abels død. Ved V. Bjerknes. Oslo, Aschehoug, 1929. 155 pp.

BJERKNES (V.). See BJERKNES (C. A.).

BRESLICH (E. R.). Senior mathematics. Book 3. (University of Chicago Mathematical Series, School of Education Texts and Manuals.) Chicago, University of Chicago Press, 1929. 216 pp.

BUCHANAN (S.). Poetry and mathematics. New York, John Day, 1929. 197 pp.

COBLE (A. B.). Algebraic geometry and theta functions. (American Mathematical Society, Colloquium Publications, volume 10.) New York, American Mathematical Society, 1929. 8 + 282 pp.

COSTER (B.). Constructie van ruimteelementen. Groningen, Wolters, 1928. 63 pp.

CRATHORNE (A. R.). See RIETZ (H. L.).

DIJKSTERHUIS (E. J.). De Elementen van Euclides. Deel 1: De ontwikkeling der Grieksche wiskunde voor Euclides. Boek I der Elementen. Groningen, Noordhoff, 1929. 236 pp.

DURELL (C. V.) and SIDDONS (A. W.). Graph-book. An exercise book and text-book. London, Bell, 1929.

ECKHART (L.). Der vierdimensionale Raum. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1929. 54 pp.

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. Band III, Teil 2, Heft 9: K. Rohn und L. Berzolari, Algebraische Raumkurven und abwickelbare Flächen. Leipzig, Teubner, 1928.

FIELDS (J. C.). See INTERNATIONAL MATHEMATICAL CONGRESS.

FORD (L. R.). Automorphic functions. New York, McGraw-Hill, 1929. 12 + 333 pp.

FORREST (S. N.). Calculus for technical students. London, Edward Arnold, 1929. 8 + 231 pp.

GALE (A. S.). See SMITH (P. F.).

GANS (R.). Vektoranalysis mit Anwendungen auf Physik und Technik. 6te, verbesserte Auflage. Leipzig, Teubner, 1929. 112 pp.

Hilb (O.). Einführung in die Algebra. 2 Bände. Leipzig, Akademische Verlagsgesellschaft, 1929. 15 + 367 + 13 + 295 pp.

Hunter (W.). Groundwork of calculus. London, University Tutorial Press, 1929. 6 + 220 pp.

Huygens (C.). Christiaan Huygens, 1629—14 April—1929, zijn geboortedag, 300 jaar geleden, herdacht. Amsterdam, H. J. Paris, 1929. 4 + 61 pp.

International Mathematical Congress. Proceedings of the International Mathematical Congress held in Toronto, August 11-16, 1924. Edited by J. C. Fields. 2 volumes. Toronto, University of Toronto Press, 1928.

Ives (H. C.). Seven place natural trigonometrical functions. New York, Wiley, 1929. 222 pp.

Keyser (C. J.). The pastures of wonder. New York, Columbia University Press, 1929. 12 + 208 pp.

Lainé (E.). Premières leçons de géométrie analytique et de géométrie vectorielle. Paris, Vuibert, 1929. 47 pp.

Longley (W. R.). See Smith (P. F.).

Lotze (A.). Punkt- und Vektor-Rechnung. Berlin, de Gruyter, 1929. 192 pp.

Neelley (J. H.). See Smith (P. F.).

Perron (O.). Die Lehre von den Kettenbruchen. 2te verbesserte Auflage. Leipzig, Teubner, 1929. 12 + 524 pp.

Rietz (H. L.) and Crathorne (A. R.). College algebra. 3d edition. New York, Holt, 1929. 16 + 273 pp.

Rohn (K.). See Encyklopädie.

Schuh (F.). Lessen over de hoogere algebra. Tweede volledig herziene druk. Eerste deel. Groningen, Noordhoff, 1929. 584 pp.

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Smith (P. F.), Gale (A. S.), and Neelley (J. H.). New analytic geometry. Boston, Ginn, 1928. 10 + 326 pp.

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Valentiner (S.). Vektoranalysis. 4te, wenig veränderte Auflage. (Sammlung Göschen.) Berlin, de Gruyter, 1929. 136 pp.

Valiron (G.). Familles normales et quasi-normales de fonctions méromorphes. (Mémorial des Sciences Mathématiques, No. 38.) Paris, Gauthier-Villars, 1929. 56 pp.

de Vries (H.). Beknopte differentiaal- en integraalrekening. Groningen, Noordhoff, 1929.

PART II. APPLIED MATHEMATICS

Ambronn (R.). Elements of geophysics as applied to explorations for minerals, oil, and gas. Translated by M. C. Cobb. London, McGraw-Hill, 1928. 372 pp.

Ames (J. S.) and Murnaghan (F. D.). Theoretical mechanics. An introduction to mathematical physics. Boston, Ginn, 1929. 462 pp.

Bauer (G. N.). Mathematics preparatory to statistics and finance. New York, Macmillan, 1929. 8+337 pp.

Boutaric (A.). Les ondes Hertziennes et la télégraphie sans fil. Paris, Flammarion, 1929. 253 pp.

Bowley (A. L.). Eléments de statistique. Traduits sur la cinquième édition anglaise par L. Suret et G. Lutfalla. Paris, Girard, 1929. 12 + 616 pp.

Bricout (P.). Ondes et électrons. Paris, Armand Colin, 1929. 215 pp.

Bundschu (F.). Angewandte Hydraulik. Berlin, Springer, 1929. 75 pp.

Cabannes (J.). La diffusion moléculaire de la lumière. Paris, Presses Universitaires de France, 1929. 8 + 326 pp.

Carter (C. C.). See Monteith (C. N.).

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Conrady (A. E.). Applied optics and optical design. Part 1. London, Oxford University Press, 1929. 9 + 518 pp.

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Cotter (J. R.). See Preston (T.).

Crowther (J. A.). Ions, electrons and ionizing radiations. 5th edition. London, Arnold, 1929.

Cugnin (L.). Structure de l'atome, tourbillon d'éther et pensées scientifiques indépendantes. Sceaux, Bry, 1929. 184 pp.

Defretin (A.). Cours d'électricité industrielle. Tome 1. Paris, Hermann, 1929. 582 pp.

Detlefs (H.). Darstellende Geometrie. Heft 3. Frankfurt a. M., Diesterweg, 1929.

Drew (C. E.). See Duncan (R. L.).

Dull (C. E.). Modern physics. London, Harrap, 1929. 8+778 pp.

Duncan (R. L.) and Drew (C. E.). Radio telegraphy and telephony. New York, Wiley, 1929. 10+950 pp.

Eberhard (G.), Kohlschütter (A.), und Ludendorff (H.). Handbuch der Astrophysik. Band 4. Berlin, Springer, 1929. 501 pp.

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Eve (A. S.) and Keys (D. A.). Applied geophysics in the search for minerals. Cambridge, University Press, 1929. 10+253 pp.

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Füllmer (M.). See Richter (O.).

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Haag (J.). Cours complet de mathématiques élémentaires. Tome 5: Mécanique. Paris, Gauthier-Villars, 1929. 192 pp.

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Hills (W. D.). The elements of mechanics. London, University of London Press, 1929. 8+140 pp.

Hopmann (J.). Weltallkunde. Arbeitsweise und Ergebnisse der heutigen Astronomie. Berlin, Dümmler, 1929. 231 pp.

Hudson (R. G.). Engineering electricity. 2d edition. New York, Wiley, 1928. 8+214 pp.

Humphreys (W. J.). Physics of the air. 2d edition, revised and enlarged. New York, McGraw-Hill, 1929. 12+654 pp.

Hund (A.). Hochfrequenzmesstechnik. Ihre wissenschaftlichen und praktischen Grundlagen. 2te vermehrte und verbesserte Auflage. Berlin, Springer, 1928. 18+526 pp.

Jellinek (K.). Lehrbuch der physikalischen Chemie. Band 3: Die Lehre von der Statik homogener und heterogener Gasreaktionen. Lieferung 7. Stuttgart, Enke, 1929. 336 pp.

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LICHTENSTEIN (L.). Grundlagen der Hydromechanik. (Grundlehren der Mathematischen Wissenschaften, Band 30.) Berlin, Springer, 1929. 16+507 pp.

—— See GEORGIEWICZ (C.).

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LUDENDORFF (H.). See EBERHARD (G.).

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MONTEITH (C. N.). Simple aerodynamics and the airplane. 3d edition, revised by C. C. Carter. New York, Ronald Press, 1929. 418 pp.

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MURNAGHAN (F. D.). See AMES (J. S.).

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REYNOLDS (J. B.). Analytic mechanics. New York, Prentice-Hall Company, 1929. 10+348 pp.

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RUPERT-JONES (J. A.). Tidal research. The simplification of the prediction of the "time of the tide" problem. Southampton, the author, Southampton Harbor Board, 1929. 24 pp.

SCHERSCHEVSKY (A. B.). Die Rakete für Fahrt und Flug. Berlin, Volckmann, 1928.

SCHLEEDE (A.) und SCHNEIDER (E.). Röntgenspektroskopie und Kristallstrukturanalyse. Band 2. Berlin, de Gruyter, 1929. 4+344 pp.

SCHNEIDER (E.). See SCHLEEDE (A.).

SCHÜLE (W.). Leitfaden der technischen Wärmemechanik. Berlin, Springer, 1928. 8+323 pp.

STREET (R. O.). Examples in applied mathematics. London, Methuen, 1929.

STUART (A. H.). Problems in technical mathematics. London, Oxford University Press, 1929. 72 pp.

SURET (L.). See BOWLEY (A. L.).

THORNTON (W. M.). Notes on the electrical transmission of energy. The physical basis of electrical transmission, its methods and phenomena from the standpoint of the electron. London, Isaac Pitman, 1929. 120 pp.

VALIER (M.). Vorstoss in den Weltenraum. Eine technische Möglichkeit. München, Oldenbourg, 1928. 263 pp.

VERNON (C. G.). Light. An introductory text-book. Cambridge, University Press, 1929.

VÉRON (M.). See ROSZAK (C.).

VON VOSS (R.). See RICHTER (O.).

WALKER (H. M.). Studies in the history of statistical method. Baltimore, Williams and Wilkins, 1929.

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THIRTY-EIGHTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF PUBLICATION

ADAMS, C. R. On the linear ordinary q-difference equation. Read April 6, 1928. *Annals of Mathematics*, (2), vol. 30, No. 2, pp. 195–205; April, 1929.

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